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QUADRUPLED BEST PROXIMITY POINT THEOREMS IN PARTIALLY ORDERED METRIC SPACES

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Abstract: In this paper, we prove some quadrupled best proximity point theorems in partially ordered metric space by using (ψ, ϕ) contraction. Our results generalise the results of Kumam et.al. (Coupled best proximity points in ordered metric spaces, Fixed Point Theory and Application 2014, 2014:107). An example is also given to verify the results obtained.

Keywords: partial ordered set; best proximity point; quadrupled fixed point; quadrupled best proximity point.

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1. INTRODUCTION

In a metric space (X, d) a self mapping T on X is said to posses fixed point if the equation $Tx = x$ has at least one solution. In this case x is said to be the fixed point of T. The existence of fixed point of a mapping may or may not be unique. The study of the existence of fixed point of mapping has been an area of extensive research for the past many years and Banach

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contraction principle has the most important tool in order to obtain fixed point of a mapping. Now, the question arises if the equation $Tx = x$ has no solution. In order to investigate further the study of best proximity started.

Let A and B be non-empty closed subsets of a metric space (X, d) and $T: A \to B$ be a non-self mapping. A point x in A for which $d(x, Tx) = d(A, B)$ is called a best proximity point of T. It can be noted that the best proximity point becomes a fixed point if the underlying mapping is assumed to be self mapping.

The best approximation theorem given by Ky Fan [1] stated as follows:

If K is a non-empty compact convex subset of a Hausdroff locally convex topological vector space E with a continuous seminorm p and $T: K \to E$ is a single valued continuous function, then there exists an element $x \in K$ such that

$$p(x - Tx) = d_p(Tx, K) \coloneqq \inf\{p(Tx - y): y \in K\}$$

The result of Ky Fan was generalised by a large number of authors in various direction (see [2-19] and references there in).

The concept of coupled fixed point was introduced by Guo and Lakshmikantham [20] in the year 1987. Further, Bhaskar and Lakshmikantham [21] introduced the concept of mixed monotone mapping and established some coupled fixed point theorems for mapping satisfying mixed monotone property.

The concept of coupled fixed point was further extended to triple fixed point by Berinde and Borcut [22] and quadrupled fixed point by Karapinar and Luong [23]. These concepts of Tripled and quadrupled fixed points were presented in a more generalised form by Wu and Liu [24]. For more results on coupled, tripled, quadrupled fixed point, one can see the research articles ([25-50] and references there in).

In this note we prove some quadrupled best proximity point theorems in partially ordered metric space by using proximally quadrupled weak (ψ, ϕ) contraction on the line of proximally coupled weak (ψ, ϕ) contractions introduced by Kumam et.al. [34]

Following definition was given in Karapinar and Luong [23].

DEFINITION 1. [23] Let X be a non empty set and $F: X^4 \to X$ be a given mapping. An element $(x, y, z, w) \in X^4$ is called a quadrupled fixed point of the mapping F if

$$x = F(x, y, z, w),\ y = F(y, z, w, x),\ z = F(z, w, x, y) \text{ and } w = F(w, x, y, z).$$

The authors mentioned above also introduced the notion of mixed monotone mapping. If $(X, \preceq)$ is a partially ordered set, the mapping F is said to have the mixed monotone property, if

$$x_{1,}x_2 \in X, x_1 \preceq x_2 \Longrightarrow F(x_1, y, z, w) \preceq F(x_2, y, z, w), y, z, w \in X,$$

$$y_1, y_2 \in X, y_1 \preceq y_2 \Longrightarrow F(x, y_1, z, w) \succeq F(x, y_2, z, w), x, z, w \in X,$$

$$z_1, z_2 \in X, z_1 \preceq z_2 \Longrightarrow F(x, y, z_1, w) \preceq F(x, y, z_2, w), x, y, w \in X,$$

$$w_{1,}w_2 \in X, w_1 \preceq w_2 \Longrightarrow F(w, y, z, w_1) \succeq F(x, y, z, w_2), x, y, z, \in X,$$

Let A and B be non empty subsets of a metric space(X, d). We use the following notions in the sequel:

$$dist(A, B) = \inf\{d(x, y): x \in A \text{ and } y \in B\},$$

$$A_0 = \{x \in A: d(x, y) = dist(A, B), \text{for some } y \in B\}$$

$$B_0 = \{y \in B: d(x, y) = dist(A, B), \text{for some } x \in A\}$$

Now we give the following definition.

DEFINITION 2. Let $(X, d, \preceq)$ be a partially ordered metric space and A, B are nonempty subsets of X. A mapping $F: A \times A \times A \times A \to B$ is said to have proximal mixed monotone property if $F(x, y, z, w)$ is proximally non decreasing in *x* and *z*, and is proximally non increasing in *y* and *w*, that is for all $x, y, z, w \in A$,

$$x_1 \preceq x_2 \preceq x_3 \preceq x_4$$

$$d\left(u_{1,}F\left(x_{1,}y, z, w\right)\right) = dist(A, B)$$

$$d\left(u_{2,}F\left(x_{2,}y, z, w\right)\right) = dist(A, B)$$

$$d\left(u_{3,}F\left(x_{3,}y, z, w\right)\right) = dist(A, B)$$

$$d\left(u_{4,}F\left(x_{4,}y, z, w\right)\right) = dist(A, B)$$

$$\Longrightarrow u_1 \preceq u_2 \preceq u_3 \preceq u_4$$

$$y_1 \preceq y_2 \preceq y_3 \preceq y_4$$

$$d\left(u_{5,} F(x_, y_1, z, w)\right) = dist(A, B)$$

$$d\left(u_{6,} F(x_, y_2, z, w)\right) = dist(A, B)$$

$$d\left(u_{7,} F(x_, y_3, z, w)\right) = dist(A, B)$$

$$d\left(u_{8,} F(x_, y_4, z, w)\right) = dist(A, B)$$

$$u_5 \succeq u_6 \succeq u_7 \succeq u_8$$

$$z_1 \preceq z_2 \preceq z_3 \preceq z_4$$

$$d\left(u_{9,} F(x_, y, z_1, w)\right) = dist(A, B)$$

$$d\left(u_{10,} F(x_, y, z_2, w)\right) = dist(A, B)$$

$$d\left(u_{11,} F(x_, y, z_3, w)\right) = dist(A, B)$$

$$d\left(u_{12,} F(x_, y, z_4, w)\right) = dist(A, B)$$

$$u_9 \preceq u_{10} \preceq u_{11} \preceq u_{12}$$

and

$$w_1 \preceq w_2 \preceq w_3 \preceq w_4$$

$$d\left(u_{13,} F(x_, y, z, w_1)\right) = dist(A, B)$$

$$d\left(u_{14,} F(x_, y, z, w_2)\right) = dist(A, B)$$

$$d\left(u_{15,} F(x_, y, z, w_3)\right) = dist(A, B)$$

$$d\left(u_{16,} F(x_, y, z, w_4)\right) = dist(A, B)$$

$$\Longrightarrow u_{13} \succeq u_{14} \succeq u_{15} \succeq u_{16}$$

where $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4, u_1, u_2, \ldots, u_{16} \in A$

If we take $A = B$ the above definition, the notion of the proximal mixed monotone property becomes the conditions of mixed monotone property. The following lemmas are essential for our results.

LEMMA 1. Let $(X, d, \preceq)$ be a partially ordered metric space and A, B are nonempty subsets of X. Assume that A_0 is nonempty. A mapping $F: A \times A \times A \times A \to B$ has the proximal mixed monotone property with $F(A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0$ whenever $x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3, z_{0,} z_1, z_2, z_3, w_0, w_1, w_2, w_3 \in A$ such that

$$\begin{cases} x_0 \preceq x_1 \preceq x_2 \preceq x_3; y_0 \succeq y_1 \succeq y_2 \succeq y_3; z_0 \preceq z_1 \preceq z_2 \preceq z_3; w_0 \succeq w_1 \succeq w_2 \succeq w_3 \\ d\left(x_1, F\left(x_0, y_0, z_{0,} w_0\right)\right) = dist(A, B). \\ d\left(x_2, F\left(x_1, y_1, z_{1,} w_1\right)\right) = dist(A, B). \\ d\left(x_3, F\left(x_2, y_2, z_{2,} w_2\right)\right) = dist(A, B). \\ d\left(x_4, F\left(x_3, y_3, z_{3,} w_3\right)\right) = dist(A, B). \\ \Longrightarrow x_1 \preceq x_2 \preceq x_3 \preceq x_4 \end{cases} \tag{1}$$

Proof : By hypothesis we have $F(A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0$, therefore $F(x_3, y_0, z_{0,} w_0) \in B_0$. Hence, there exist $x_1^* \in A$ such that

$$d\left(x_1^*, F\left(x_3, y_0, z_{0,} w_0\right)\right) = dist(A, B). \tag{2}$$

Since F is proximal mixed monotone (in particular F is proximally non decreasing in x) from (1) and (2), we get

$$\begin{cases} x_0 \preceq x_1 \preceq x_2 \preceq x_3; \\ d\left(x_1, F\left(x_0, y_0, z_{0,} w_0\right)\right) = dist(A, B). \\ d\left(x_2, F\left(x_1, y_1, z_{1,} w_1\right)\right) = dist(A, B). \\ d\left(x_3, F\left(x_2, y_2, z_{2,} w_2\right)\right) = dist(A, B). \\ d\left(x_1^*, F\left(x_3, y_0, z_{0,} w_0\right)\right) = dist(A, B). \\ \Longrightarrow x_1 \preceq x_2 \preceq x_3 \preceq x_1^* \end{cases} \tag{3}$$

Similarly, using the fact that F is proximal mixed monotone (in particular, F is proximally non increasing in y) from (1) and (2), we get

$$\begin{cases} y_0 \succeq y_1 \succeq y_2 \succeq y_3 \\ d\left(x_1^* , F(x_3, y_0, z_{0,} w_0)\right) = dist(A, B) \\ d\left(x_4 , F(x_3, y_3, z_{3,} w_3)\right) = dist(A, B) \\ \Longrightarrow x_1^* \preceq x_4 \end{cases} \tag{4}$$

From (3) and (4), we have $x_1 \preceq x_2 \preceq x_3 \preceq x_4$. This completes the proof.

LEMMA 2. Let $(X, d, \preceq)$ be a partially ordered metric space and A, B are non empty subsets of X. Assume A_0 is nonempty. A mapping $F: A \times A \times A \times A \to B$ has the proximal mixed monotone property with $F(A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0$ whenever $x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3, z_{0,} z_1, z_2, z_3, w_0, w_1, w_2, w_3 \in A$ such that

$$\begin{cases} x_0 \preceq x_1 \preceq x_2 \preceq x_3; y_0 \succeq y_1 \succeq y_2 \succeq y_3; z_0 \preceq z_1 \preceq z_2 \preceq z_3; w_0 \succeq w_1 \succeq w_2 \succeq w_3 \\ d\left(y_1 , F(y_0, z_{0,} w_0, x_0)\right) = dist(A, B). \\ d\left(y_2 , F(y_1, z_{1,} w_1, x_{1,})\right) = dist(A, B). \\ d\left(y_3 , F(y_2, z_{2,} w_2, x_{2,})\right) = dist(A, B). \\ d\left(y_4 , F(y_3, z_{3,} w_3, x_3)\right) = dist(A, B). \\ \Longrightarrow y_1 \succeq y_2 \succeq y_3 \succeq y_4 \end{cases} \tag{5}$$

LEMMA 3. Let $(X, d, \preceq)$ be a partially ordered metric space and A, B are non empty subsets of X. Assume A_0 is nonempty. A mapping $F: A \times A \times A \times A \to B$ has the proximal mixed monotone property with $F(A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0$ whenever $x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3, z_{0,} z_1, z_2, z_3, w_0, w_1, w_2, w_3 \in A$ such that

$$\begin{cases} x_0 \preceq x_1 \preceq x_2 \preceq x_3; y_0 \succeq y_1 \succeq y_2 \succeq y_3; z_0 \preceq z_1 \preceq z_2 \preceq z_3; w_0 \succeq w_1 \succeq w_2 \succeq w_3 \\ d\left(z_1 , F(z_0, w_{0,} x_0, y_0)\right) = dist(A, B). \\ d\left(z_2 , F(z_1, w_{1,} x_1, y_{1,})\right) = dist(A, B). \\ d\left(z_3 , F(z_2, w_{2,} x_2, y_{2,})\right) = dist(A, B). \\ d\left(z_4 , F(z_3, w_{3,} x_3, y_3)\right) = dist(A, B). \\ \Longrightarrow z_1 \preceq z_2 \preceq z_3 \preceq z_4 \end{cases} \tag{6}$$

LEMMA 4. Let $(X, d, \preceq)$ be a partially ordered metric spaces and A, B are non empty subsets of X. Assume A_0 is nonempty. A mapping $F: A \times A \times A \times A \to B$ has the proximal mixed monotone property with $F(A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0$ whenever $x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3, z_{0,} z_1, z_2, z_3, w_0, w_1, w_2, w_3 \in A$ such that

$$\begin{cases} x_0 \preceq x_1 \preceq x_2 \preceq x_3; y_0 \succeq y_1 \succeq y_2 \succeq y_3; z_0 \preceq z_1 \preceq z_2 \preceq z_3; w_0 \succeq w_1 \succeq w_2 \succeq w_3 \\ d\left(w_1, F\left(w_0, x_{0,} y_0, z_0\right)\right) = dist(A, B). \\ d\left(w_2, F\left(w_1, x_{1,} y_1, z_{1,}\right)\right) = dist(A, B). \\ d\left(w_3, F\left(w_2, x_{2,} y_2, z_{2,}\right)\right) = dist(A, B). \\ d\left(w_4, F\left(w_3, x_{3,} y_3, z_3\right)\right) = dist(A, B). \\ \Rightarrow w_1 \succeq w_2 \succeq w_3 \succeq w_4 \end{cases} \tag{7}$$

Proof of lemma 2, 3, 4 are omitted.

Following definition was given by Luong and Thuan [37]

DEFINITION 3. [37] Let ϕ be the class of all functions $\phi: [0, \infty) \to [0, \infty)$ which satisfy:

1. ϕ is continuous and non decreasing,
2. $\phi(t) = 0$ if and only if $t = 0$,
3. $\phi(t + s) \leq \phi(t) + \phi(s), \forall\, t, s \in [0, \propto)$.

And let ψ be the class of all functions $\psi: [0, \infty) \to [0, \infty)$ which satisfy $\lim_{t \to r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t \to 0^+} \psi(t) = 0$.

Now we give the following definition of proximally quadrupled weak (ψ, ϕ) contraction.

DEFINITION 4. Let $(X, d, \preceq)$ be a partially ordered metric space and A and B are non empty subsets of X. Assume A_0 is non empty. A mapping $F: A \times A \times A \times A \to B$ is said to be proximally quadrupled weak (ψ, ϕ) contraction on A, whenever

$$\begin{cases} x_1 \preceq x_2; y_1 \succeq y_2; z_1 \preceq z_2; w_1 \succeq w_2 \\ d\left(u, F(x_1, y_1, z_1, w_1)\right) = dist(A, B) \\ d\left(v, F(x_2, y_2, z_2, w_2)\right) = dist(A, B) \end{cases}$$

$$\Rightarrow \phi\big(d(u,v)\big)$$
$$\leq \frac{1}{4}\phi\big(d(x_1,x_2)+d(y_1,y_2)+d(z_1,z_2)+d(w_1,w_2)\big)$$
$$-\psi\left(\frac{d(x_1,x_2)+d(y_1,y_2)+d(z_1,z_2)+d(w_1,w_2)}{4}\right)$$

where $x_1, x_2, y_1, y_2, z_1, z_2, w_1, w_2 \in A$.

If $A = B$ in the above definition, the notion of proximally quadrupled weak (ψ,ϕ) contraction on A reduces to that of a quadrupled weak (ψ,ϕ) contraction.

3. MAIN RESULT

Let $(X, d, \preceq)$ be a partially ordered complete metric space endowed with the product space $A \times A \times A \times A$ satisfying the following conditions for $(x,y,z,w),(p,q,r,s) \in A \times A \times A \times A$,

$$(p,q,r,s) \preceq (x,y,z,w) \Leftrightarrow x \succeq p, y \preceq q, z \succeq r, w \preceq s.$$

Theorem 1. Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let A, B are non empty closed subsets of the metric space (X,d) such that $A_0 \neq \phi$. Let $F: A \times A \times A \times A \to B$ satisfy the following conditions:

1. F is continuous proximally quadrupled weak (ψ,ϕ) contraction on A having the proximal mixed monotone property on A such that $F(A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0$.
2. There exist elements (x_0, y_0, z_0, w_0) and (x_1, y_1, z_1, w_1) in $A_0 \times A_0 \times A_0 \times A_0$ such that

$$d\left(x_1, F\big(x_0, y_0, z_0, w_0\big)\right) = dist(A,B) \text{ with } x_0 \preceq x_1$$

$$d\left(y_1, F\big(y_0, z_0, w_0, x_0\big)\right) = dist(A,B) \text{ with } y_0 \succeq y_1,$$

$$d\left(z_1, F\big(z_0, w_0, x_0, y_0\big)\right) = dist(A,B) \text{ with } z_0 \preceq z_1 \text{ and}$$

$$d\big(w_1, F(w_0, x_0, y_0, z_0)\big) = dist(A,B) \text{ with } w_0 \succeq w_1,$$

Then there exists (x,y,z,w) in $A \times A \times A \times A$ such that $d\big(x, F(x,y,z,w)\big) = dist(A,B)$, $d\big(y, F(y,z,w,x)\big) = dist(A,B)$, $d\big(x, F(z,w,x,y)\big) = dist(A,B)$ and $d\big(w, F(w,x,y,z)\big) = dist(A,B)$.

Proof: By hypothesis, there exists elements $(x_0, y_0, z_{0,} w_0)$ and $(x_1, y_1, z_{1,} w_1)$ in $A_0 \times A_0 \times A_0 \times A_0$ such that

$$d\left(x_1, F\big(x_0, y_0, z_{0,} w_0\big)\right) = dist(A,B) \text{ with } x_0 \preceq x_1,$$

$$d\left(y_1, F\big(y_0, z_0, w_{0,} x_0\big)\right) = dist(A,B) \text{ with } y_0 \succeq y_1,$$

$$d\left(z_1, F\big(z_{0,} w_0, x_0, y_0\big)\right) = dist(A,B) \text{ with } z_0 \preceq z_1, \text{ and}$$

$$d\big(w_1, F(w_0, x_0, y_0, z_0)\big) = dist(A,B) \text{ with } w_0 \succeq w_1.$$

Since $(A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0$, then there exists element $(x_2, y_2, z_{2,} w_2) \in A_0 \times A_0 \times A_0 \times A_0$ such that

$$d\left(x_2, F\big(x_1, y_1, z_{1,} w_1\big)\right) = dist(A,B)$$

$$d\left(y_2, F\big(y_1, z_1, w_{1,} x_1\big)\right) = dist(A,B)$$

$$d\left(z_2, F\big(z_1, w_1, x_{1,} y_1\big)\right) = dist(A,B) \text{ and}$$

$$d\left(w_2, F\big(w_1, x_1, y_{1,} z_1\big)\right) = dist(A,B).$$

And also there exists element $(x_3, y_3, z_{3,} w_3) \in A_0 \times A_0 \times A_0 \times A_0$

$$d\left(x_3, F\big(x_2, y_2, z_{2,} w_2\big)\right) = dist(A,B)$$

$$d\left(y_3, F\big(y_2, z_2, w_{2,} x_2\big)\right) = dist(A,B)$$

$$d\left(z_3, F\big(z_2, w_2, x_{2,} y_2\big)\right) = dist(A,B)$$

$$d\left(w_3, F\big(w_2, x_2, y_{2,} z_2\big)\right) = dist(A,B).$$

Thus by lemma 1, 2, 3, 4 we have $x_1 \preceq x_2 \preceq x_3$; $y_1 \succeq y_2 \succeq y_3$; $z_1 \preceq z_2 \preceq z_3$ and $w_1 \succeq w_2 \succeq w_3$. Continuing in this way, we can construct sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ in A_0 such that

$$d\left(x_{n+1}, F\left(x_n, y_n, z_n, w_n\right)\right) = dist(A,B), \forall n \in N \text{ with } x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \ldots \tag{8}$$

$$d\left(y_{n+1}, F\left(y_n, z_n, w_n, x_n\right)\right) = dist(A,B), \forall n \in N \text{ with } y_0 \succeq y_1 \succeq \cdots \succeq y_n \succeq y_{n+1} \succeq \ldots \tag{9}$$

$$d\left(z_{n+1}, F\left(z_n, w_n, x_n, y_n\right)\right) = dist(A,B), \forall n \in N \text{ with } z_0 \preceq z_1 \preceq \cdots \preceq z_n \preceq z_{n+1} \preceq \ldots \tag{10}$$

$$d\left(w_{n+1}, F(w_n, x_n, y_n, z_n)\right) = dist(A,B), \forall n \in N \text{ with } w_0 \succeq w_1 \succeq \cdots \succeq w_n \succeq w_{n+1} \succeq \ldots \tag{11}$$

Then $d\left(x_n, F\left(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}\right)\right) = dist(A,B)$, $d\left(x_{n+1}, F\left(x_n, y_n, z_n, w_n\right)\right) = dist(A,B)$ and we also have $x_{n-1} \preceq x_n, y_{n-1} \succeq y_n, z_{n-1} \preceq z_n$ and $w_{n-1} \succeq w_n, \forall n \in N$. Since F is proximally quadrupled weak (ψ, ϕ) contraction on A, we have

$$\begin{aligned}
&\phi\left(d(x_n, x_{n+1})\right) \\
&\leq \frac{1}{4}\phi\left(d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n)\right) \\
&- \psi\left(\frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n)}{4}\right)
\end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned}
&\phi\left(d(y_n, y_{n+1})\right) \\
&\leq \frac{1}{4}\phi\left(d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n) + d(x_{n-1}, x_n)\right) \\
&- \psi\left(\frac{d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n) + d(x_{n-1}, x_n)}{4}\right)
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\phi\left(d(z_n, z_{n+1})\right) \\
&\leq \frac{1}{4}\phi\left(d(z_{n-1}, z_n) + d(w_{n-1}, w_n) + d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\right) \\
&- \psi\left(\frac{d(z_{n-1}, z_n) + d(w_{n-1}, w_n) + d(x_{n-1}, x_n) + d(y_{n-1}, y_n)}{4}\right)
\end{aligned} \tag{14}$$

and

$$\phi\big(d(w_n, w_{n+1})\big)$$
$$\leq \frac{1}{4}\phi\big(d(w_{n-1}, w_n) + d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n)\big)$$
$$-\psi\left(\frac{d(w_{n-1}, w_n) + d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n)}{4}\right) \tag{15}$$

From (12), (13), (14) and (15), we get

$$\phi\big(d(x_n, x_{n+1})\big) + \phi\big(d(y_n, y_{n+1})\big) + \phi\big(d(z_n, z_{n+1})\big) + \phi\big(d(w_n, w_{n+1})\big)$$
$$\leq \phi\big(d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n)\big)$$
$$-4\psi\left(\frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n)}{4}\right) \tag{16}$$

By property (iii) of ϕ, we have

$$\phi\big(d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(w_n, w_{n+1})\big)$$
$$\leq \phi\big(d(x_n, x_{n+1})\big) + \phi\big(d(y_n, y_{n+1})\big) + \phi\big(d(z_n, z_{n+1})\big) + \phi\big(d(w_n, w_{n+1})\big) \tag{17}$$

From (16) and (17), we get

$$\phi\big(d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(w_n, w_{n+1})\big)$$
$$\leq \phi\big(d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n)\big)$$
$$-4\psi\left(\frac{d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n)}{4}\right) \tag{18}$$

Since ϕ is non decreasing from (18), we get

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(w_n, w_{n+1})$$
$$\leq d(x_{n-1}, x_n) + d(y_{n-1}, y_n) + d(z_{n-1}, z_n) + d(w_{n-1}, w_n) \tag{19}$$

Let $\delta_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(w_n, w_{n+1})$, then the sequence $\{\delta_n\}$ is decreasing. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n\to\infty}\delta_n = \lim_{n\to\infty}[d(x_n,x_{n+1}) + d(y_n,y_{n+1}) + d(z_n,z_{n+1}) + d(w_n,w_{n+1})] = \delta \tag{20}$$

Now we have to show that $\delta = 0$. On the contrary, let $\delta > 0$. Taking limit as $n \to \infty$ on both sides of (18) and using the fact that ϕ is continuous, we have

$$\phi(\delta) \le \lim_{n\to\infty}\phi(\delta_{n-1}) - 4\lim_{n\to\infty}\psi\left(\frac{\delta_{n-1}}{4}\right)$$
$$= \phi(\delta) - 4\lim_{n\to\infty}\psi\left(\frac{\delta_{n-1}}{4}\right) < \phi(\delta),$$

Since $\lim_{t\to r}\psi(t) > 0, \forall\, r > 0$, which is a contradiction and hence we conclude that $\delta = 0$.

Thus,

$$\lim_{n\to\infty}\delta_n = \lim_{n\to\infty}[d(x_n,x_{n+1}) + d(y_n,y_{n+1}) + d(z_n,z_{n+1}) + d(w_n,w_{n+1})]$$
$$= 0 \tag{21}$$

Next we have to prove that $\{x_n\}$, $\{y_n\}$, $\{z_n\}\, and\, \{w_n\}$ are Cauchy sequences. On the contrary, let us assume that at least one of the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}\, and\, \{w_n\}$ is not a Cauchy sequence. This means that at least one of

$$\lim_{n,m\to\infty} d(x_n,x_m) = 0\,,\ \lim_{n,m\to\infty} d(y_n,y_m) = 0$$
$$\lim_{n,m\to\infty} d(z_n,z_m) = 0\,,\ \lim_{n,m\to\infty} d(w_n,w_m) = 0$$

is not true and consequently,

$$\lim_{n,m\to\infty}[d(x_n,x_m) + d(y_n,y_m) + d(z_n,z_m) + d(w_n,w_m)] = 0$$

is not true.

Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$, $\{x_{m_k}\}$ of $\{x_n\}$; $\{y_{n_k}\}$, $\{y_{m_k}\}$ of $\{y_n\}$; $\{z_{n_k}\}$, $\{z_{m_k}\}$ of $\{z_n\}$ and $\{w_{n_k}\}$, $\{w_{m_k}\}$ of $\{w_n\}$ such that n_k is the smallest index for which $n_k > m_k > k$,

$$d(x_{n_k},x_{m_k}) + d(y_{n_k},y_{m_k}) + d(z_{n_k},z_{m_k}) + d(w_{n_k},w_{m_k}) \ge \varepsilon \tag{22}$$

This means that

$$d(x_{n_k-1},x_{m_k}) + d(y_{n_k-1},y_{m_k}) + d(z_{n_k-1},z_{m_k}) + d(w_{n_k-1},w_{m_k}) < \varepsilon \tag{23}$$

Using triangle inequality from (19) and (20), we get

$$\begin{aligned}\varepsilon \leq\ & d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k}) \\ \leq\ & d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k}) + d(y_{n_k}, y_{n_k-1}) + d(y_{n_k-1}, y_{m_k}) + d(z_{n_k}, z_{n_k-1}) \\ & + d(z_{n_k-1}, z_{m_k}) + d(w_{n_k}, w_{n_k-1}) + d(w_{n_k-1}, w_{m_k}) \\ \leq\ & d(x_{n_k}, x_{n_k-1}) + d(y_{n_k}, y_{n_k-1}) + d(z_{n_k}, z_{n_k-1}) + d(w_{n_k}, w_{n_k-1}) + \varepsilon\end{aligned}$$

Taking limit as $k \to \infty$ and using (18), we obtain

$$\lim_{n,m\to\infty}[d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k})] = \varepsilon \tag{24}$$

By triangle inequality, we have

$$\begin{aligned}& d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k}) \\ & \leq d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k+1}) + d(x_{m_k+1}, x_{m_k}) + \\ & d(y_{n_k}, y_{n_k+1}) + d(y_{n_k+1}, y_{m_k+1}) + d(y_{m_k+1}, y_{m_k}) + \\ & d(z_{n_k}, z_{n_k+1}) + d(z_{n_k+1}, z_{m_k+1}) + d(z_{m_k+1}, z_{m_k}) + \\ & d(w_{n_k}, w_{n_k+1}) + d(w_{n_k+1}, w_{m_k+1}) + d(w_{m_k+1}, w_{m_k}) \\ & = [d(x_{n_k}, x_{n_k+1}) + d(y_{n_k}, y_{n_k+1}) + d(z_{n_k}, z_{n_k+1}) + d(w_{n_k}, w_{n_k+1})] \\ & + [d(x_{m_k+1}, x_{m_k}) + d(y_{m_k+1}, y_{m_k}) + d(z_{m_k+1}, z_{m_k}) + d(w_{m_k+1}, w_{m_k})] \\ & + [d(x_{n_k+1}, x_{m_k+1}) + d(y_{n_k+1}, y_{m_k+1}) + d(z_{n_k+1}, z_{m_k+1}) \\ & + d(w_{n_k+1}, w_{m_k+1})]\end{aligned} \tag{25}$$

By property of ϕ, we obtain

$$\begin{aligned}\phi(\gamma_k) \leq\ & \phi(\delta_{n_k}) + \phi(\delta_{m_k}) + \phi\left(d(x_{n_k+1}, x_{m_k+1})\right) + \phi\left(d(y_{n_k+1}, y_{m_k+1})\right) \\ & + \phi\left(d(z_{n_k+1}, z_{m_k+1})\right) + \phi\left(d(w_{n_k+1}, w_{m_k+1})\right)\end{aligned} \tag{26}$$

Where $\gamma_k = d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k})$

$$\delta_{n_k} = d(x_{n_k}, x_{n_k+1}) + d(y_{n_k}, y_{n_k+1}) + d(z_{n_k}, z_{n_k+1}) + d(w_{n_k}, w_{n_k+1})$$

$$\delta_{m_k} = d(x_{m_k+1}, x_{m_k}) + d(y_{m_k+1}, y_{m_k}) + d(z_{m_k+1}, z_{m_k}) + d(w_{m_k+1}, w_{m_k})$$

Since $x_{n_k} \succeq x_{m_k}$, $y_{n_k} \preceq y_{m_k}$, $z_{n_k} \succeq z_{m_k}$ and $w_{n_k} \preceq w_{m_k}$, using the fact that F is a proximally quadrupled weak (ψ, ϕ) contraction on A, we get

$$\begin{aligned}
&\phi\left(d(x_{n_k+1}, x_{m_k+1})\right) \\
&\quad \le \frac{1}{4}\phi\left(d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k})\right) \\
&\quad - \psi\left(\frac{d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k})}{4}\right) \\
&\quad = \frac{1}{4}\phi(\gamma_k) - \psi\left(\frac{\gamma_k}{4}\right) \qquad (27)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\phi\left(d(y_{n_k+1}, y_{m_k+1})\right) \\
&\quad \le \frac{1}{4}\phi\left(d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k}) + d(x_{n_k}, x_{m_k})\right) \\
&\quad - \psi\left(\frac{d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k})}{4}\right) \\
&\quad = \frac{1}{4}\phi(\gamma_k) - \psi\left(\frac{\gamma_k}{4}\right) \qquad (28)
\end{aligned}$$

$$\begin{aligned}
&\phi\left(d(z_{n_k+1}, z_{m_k+1})\right) \\
&\quad \le \frac{1}{4}\phi\left(d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k}) + d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k})\right) \\
&\quad - \psi\left(\frac{d(x_{n_k}, x_{m_k}) + d(y_{n_k}, y_{m_k}) + d(z_{n_k}, z_{m_k}) + d(w_{n_k}, w_{m_k})}{4}\right) \\
&\quad = \frac{1}{4}\phi(\gamma_k) - \psi\left(\frac{\gamma_k}{4}\right) \qquad (29)
\end{aligned}$$

and

$$\begin{aligned}
&\phi\left(d\left(w_{n_k+1}, w_{m_k+1}\right)\right) \\
&\qquad \leq \frac{1}{4}\phi\left(d\left(w_{n_k}, w_{m_k}\right) + d\left(x_{n_k}, x_{m_k}\right) + d\left(y_{n_k}, y_{m_k}\right) + d\left(z_{n_k}, z_{m_k}\right)\right) \\
&\qquad - \psi\left(\frac{d\left(x_{n_k}, x_{m_k}\right) + d\left(y_{n_k}, y_{m_k}\right) + d\left(z_{n_k}, z_{m_k}\right) + d\left(w_{n_k}, w_{m_k}\right)}{4}\right) \\
&\qquad = \frac{1}{4}\phi(\gamma_k) - \psi\left(\frac{\gamma_k}{4}\right) \qquad (30)
\end{aligned}$$

Using (27), (28), (29) and (30), we get

$$\phi(\gamma_k) \leq \phi\left(\delta_{n_k}\right) + \phi\left(\delta_{m_k}\right) + \phi(\gamma_k) - 4\psi\left(\frac{\gamma_k}{4}\right) \qquad (31)$$

Letting $k \to \infty$ and using (21), (24) and (31), we have

$$\phi(\varepsilon) \leq \phi(0) + \phi(0) + \phi(\varepsilon) - 4\psi\left(\frac{\varepsilon}{4}\right) < \phi(\varepsilon),$$

which is a contradiction and hence we can conclude that $\{x_n\}, \{y_n\}, \{z_n\}$ *and* $\{w_n\}$ are Cauchy sequences. Since A is a closed subset of a complete metric space X, these sequences have limits. Hence, there exists $x, y, z, w \in A$ such that $x_n \to x, y_n \to y, z_n \to z$ *and* $w_n \to w$. Then $(x_n, y_n, z_n, w_n) \to (x, y, z, w)$ in $A \times A \times A \times A$.

Since F is continuous, we have that $F(x_n, y_n, z_n, w_n) \to F(x, y, z, w)$, $F(y_n, z_n, w_n, x_n) \to F(y, z, w, x)$, $F(z_n, w_n, x_n, y_n) \to F(z, w, x, y)$ *and* $F(w_n, x_n, y_n, z_n) \to F(w, x, y, z)$.

But from (8), (9), (10) and (11), we know that the sequences $\left\{d\left(x_{n+1}, F(x_n, y_n, z_n, w_n)\right)\right\}$, $\left\{d\left(y_{n+1}, F(y_n, z_n, w_n, x_n)\right)\right\}$, $\left\{d\left(z_{n+1}, F(z_n, w_n, x_n, y_n)\right)\right\}$ and $\left\{d\left(w_{n+1}, F(w_n, x_n, y_n, z_n)\right)\right\}$ are constant sequences with the value $dist(A, B)$. Therefore, $d\left(x, F(x, y, z, w)\right) = dist(A, B)$, $d\left(y, F(y, z, w, x)\right) = dist(A, B)$, $d\left(z, F(z, w, x, y)\right) = dist(A, B)$ and $d\left(w, F(w, x, y, z,)\right) = dist(A, B)$. This completes our proof.

Corollary 1. Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let A be a non empty closed subset of the metric space (X,d). Let $F: A\times A\times A\times A\to A$ satisfy the following conditions:

1. F is continuous having proximal mixed monotone property and proximally quadrupled weak (ψ,ϕ) contraction on A.
2. There exists (x_0,y_0,z_0,w_0) and (x_1,y_1,z_1,w_1) in $A\times A\times A\times A$ such that $x_1 = F(x_0,y_0,z_0,w_0)$ with $x_0\preceq x_1$, $y_1 = F(y_0,z_0,w_0,x_0)$ with $y_0\succeq y_1$, $z_1 = F(z_0,w_0,x_0,y_0)$ with $z_0\preceq z_1$ and $w_1 = F(w_0,x_0,y_0,z_0)$ with $w_0\succeq w_1$.

Then there exists $(x,y,z,w)\in A\times A\times A\times A$ such that $d\big(x,F(x,y,z,w)\big)=0$,

$d\big(y,F(y,z,w,x)\big)=0$, $d\big(z,F(z,w,x,y)\big)=0$ and $d\big(w,F(w,x,y,z,)\big)=0$.

We also note that theorem 1 is still valid for F not necessarily continuous if A has the following property that

$\{x_n\}$ is a non decreasing sequence in A such that $x_n\to x$, then $x_n\preceq x$, (32)

$\{y_n\}$ is a non decreasing sequence in A such that $y_n\to y$, then $y_n\succeq y$, (33)

$\{z_n\}$ is a non decreasing sequence in A such that $z_n\to z$, then $z_n\preceq z$, (34)

and

$\{w_n\}$ is a non decreasing sequence in A such that $w_n\to w$, then $w_n\succeq w$, (35)

Theorem 2. *Assume the conditions* (32), (33), (34) *and* (35) *and* A_0 *is closed in X instead of continuity of F in Theorem* 1, *then the conclusion of Theorem* 1 *holds.*

Proof. Proceeding similar to Theorem 1, we claim that there exists sequences $\{x_n\},\{y_n\},\{z_n\}$ and $\{w_n\}$ in A satisfying the following conditions:

$$d\big(x_{n+1},F(x_n,y_n,z_n,w_n)\big)=dist(A,B) \text{ with } x_n\preceq x_{n+1}, \forall\, n\in N, \tag{36}$$

$$d\big(y_{n+1},F(y_n,z_n,w_n,x_n)\big)=dist(A,B) \text{ with } y_n\succeq y_{n+1}, \forall\, n\in N, \tag{37}$$

$$d\big(z_{n+1},F(z_n,w_n,x_n,y_n)\big)=dist(A,B) \text{ with } z_n\preceq z_{n+1}, \forall\, n\in N, \tag{38}$$

$$d\big(w_{n+1},F(w_n,x_n,y_n,z_n)\big)=dist(A,B) \text{ with } w_n\preceq w_{n+1}, \forall\, n\in N, \tag{39}$$

Moreover, $x_n \to x, y_n \to y, z_n \to z\ and\ w_n \to w$. From (32), (33), (34) and (35), we get $x_n \preceq x$, $y_n \succeq y$, $z_n \preceq z$ and $w_n \succeq w$ respectively. Note that the sequence $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{w_n\}$ are in A_0 and A_0 is closed. Therefore, $(x, y, z, w) \in A_0 \times A_0 \times A_0 \times A_0$. Since $F(A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0$, there exists $F(x, y, z, w)$, $F(y, z, w, x)$, $F(z, w, x, y)$ and $F(w, x, y, z)$ in B_0. Therefore, there exists $(x^*, y^*, z^*, w^*) \in A_0 \times A_0 \times A_0 \times A_0$ such that $d\big(x^*, F(x, y, z, w)\big) = dist(A, B), d\big(y^*, F(y, z, w, x)\big) = dist(A, B), d\big(z^*, F(z, w, x, y)\big) = dist(A, B)\ and\ d\big(w^*, F(w, x, y, z)\big) = dist(A, B)$.

Since $x_n \preceq x, y_n \succeq y, z_n \preceq z$ and $w_n \succeq w$, by using the fact that F is a proximally quadrupled weak (ψ, ϕ) contraction on A and using (36), (37), (38) and (39), we get

$$\begin{aligned}
&\phi\big(d(x_{n+1}, x^*)\big) \\
&\qquad \le \frac{1}{4}\phi\big(d(x_n, x) + d(y_n, y) + d(z_n, z) + d(w_n, w)\big) \\
&\qquad - \psi\left(\frac{d(x_n, x) + d(y_n, y) + d(z_n, z) + d(w_n, w)}{4}\right), \text{for all n}
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\phi\big(d(y_{n+1}, y^*)\big) \\
&\qquad \le \frac{1}{4}\phi\big(d(y_n, y) + d(z_n, z) + d(w_n, w) + d(x_n, x)\big) \\
&\qquad - \psi\left(\frac{d(y_n, y) + d(z_n, z) + d(w_n, w) + d(x_n, x)}{4}\right), \text{for all n}
\end{aligned}$$

$$\begin{aligned}
&\phi\big(d(z_{n+1}, z^*)\big) \\
&\qquad \le \frac{1}{4}\phi\big(d(z_n, z) + d(w_n, w) + d(x_n, x) + d(y_n, y)\big) \\
&\qquad - \psi\left(\frac{(z_n, z) + d(w_n, w) + d(x_n, x) + d(y_n, y)}{4}\right), \text{for all n}
\end{aligned}$$

and

$$\phi\big(d(w_n,w^*)\big)$$
$$\leq \frac{1}{4}\phi\big(d(w_n,w)+d(x_n,x)+d(y_n,y)+d(z_n,z)\big)$$
$$-\psi\left(\frac{d(w_n,w)+d(x_n,x)+d(y_n,y)+d(z_n,z)}{4}\right), \text{for all n}$$

Since $x_n \to x, y_n \to y, z_n \to z \; and \; w_n \to w$ by taking the limit on the above inequalities, we get $x = x^*$, $y = y^*$, $z = z^*$ and $w = w^*$. Hence, we get that $d\big(x, F(x,y,z,w)\big) = dist(A,B), d\big(y, F(y,z,w,x)\big) = dist(A,B), d\big(z, F(z,w,x,y)\big) = dist(A,B)$ and $d\big(w, F(w,x,y,z)\big) = dist(A,B)$. This complete the proof.

EXAMPLE 1. Let X={(0,0,0,1), (1,0,0,0), (-1,0,0,0), (0,0,0,-1)}∈ R^4 and consider the usual order

$$(x,y,z,w) \preceq (p,q,r,d) \Leftrightarrow x \preceq p, y \preceq q, z \preceq r, w \preceq s.$$

Thus, $(X,\preceq)$ is a partially ordered set. Besides, (X,d_4) is a complete metric space. Considering d_4 the Euclidean metric. Let $A = \{(0,0,0,1),(1,0,0,0)\}$ and $B = \{(0,0,0,-1),(-1,0,0,0)\}$ be closed subsets of X.

Then $dist(A,B) = \sqrt{2}$, $A = A_0$ and $B = B_0$. Let $F: A\times A\times A\times A \to B$ be defined as $F\Big((x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4),(z_1,z_2,z_3,z_4),(w_1,w,w_3,w_4)\Big)$=$(-x_4,-x_3,-x_2,-x_1)$.

Then, one can see that F is continuous such that $F(A_0\times A_0\times A_0\times A_0) \subseteq B_0$. Only comparable pairs of points in A are $x \preceq x$ for $x \in A$. Hence the proximal mixed monotone property and the proximally quadrupled weak (ψ,ϕ) contraction on A are obviously satisfied. However, F has many quadrupled best proximity points such as $(0,0,0,1),(0,0,0,1),(0,0,0,1),(0,0,0,1)$; $(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$; $(1,0,0,0),(1,0,0,0),(1,0,0,0),(1,0,0,0)$ etc. Hence not unique.

However, we can prove that the quadrupled best proximity points is in fact unique, provided that the product space $A\times A\times A\times A$ endowed with the partial ordered mentioned above has the following property:

Every pair of elements has either a lower bound or an upper bound.

This condition is equivalent to the following statement.

For every pair of $(x, y, z, w), (x^*, y^*, z^*, w^*) \in A \times A \times A \times A$, there exists $(a, b, c, d) \in A \times A \times A \times A$ which is comparable to (x, y, z, w) and (x^*, y^*, z^*, w^*).

Theorem 3. In addition to the hypothesis of Theorem 1 (resp. Theorem 2), suppose that for any two elements (x, y, z, w) and (x^*, y^*, z^*, w^*) in $A_0 \times A_0 \times A_0 \times A_0$ there exists $(z_1, z_2, z_3, z_4) \in A_0 \times A_0 \times A_0 \times A_0$ such that (z_1, z_2, z_3, z_4) is comparable to (x, y, z, w) and (x^*, y^*, z^*, w^*), then F has a unique quadrupled best proximity point.

Proof: From Theorem 1 (resp. Theorem 2), the set of quadrupled best proximity points of F is non empty. Suppose that there exists (x, y, z, w) and (x^*, y^*, z^*, w^*) in $A \times A \times A \times A$ which are quadrupled best proximity points. That is

$$d\big(x, F(x, y, z, w)\big) = dist(A, B)$$

$$d\big(y, F(y, z, w, x)\big) = dist(A, B)$$

$$d\big(z, F(z, w, x, y)\big) = dist(A, B)$$

$$d\big(w, F(w, x, y, z)\big) = dist(A, B)$$

and

$$d\big(x^*, F(x^*, y^*, z^*, w^*)\big) = dist(A, B)$$

$$d\big(y^*, F(y^*, z^*, w^*, x^*)\big) = dist(A, B)$$

$$d\big(z^*, F(z^*, w^*, x^*, y^*)\big) = dist(A, B)$$

$$d\big(w^*, F(w^*, x^*, y^*, z^*)\big) = dist(A, B)$$

We consider two cases:

Case I: Suppose (x, y, z, w) is comparable. Let (x, y, z, w) is comparable to (x^*, y^*, z^*, w^*) with respect to the ordering $A \times A \times A \times A$. Using the fact that F is a proximally quadrupled weak (ψ,ϕ) contraction on A to $d\big(x, F(x, y, z, w)\big) = dist(A, B)$

and $\left(x^*, F(x^*, y^*, z^*, w^*)\right) = dist(A, B)$, we get

$$\phi(d(x, x^*) \leq \frac{1}{4}\phi(d(x, x^*) + d(y, y^*) + d(z, z^*) + d(w, w^*) \\ - \psi\left(\frac{d(x, x^*) + d(y, y^*) + d(z, z^*) + d(w, w^*)}{4}\right) \tag{40}$$

Similarly, we get

$$\phi(d(y, y^*) \leq \frac{1}{4}\phi(d(y, y^*) + d(z, z^*) + d\left(w, w^*) + d(x, x^*)\right) \\ - \psi\left(\frac{d(y, y^*) + d(z, z^*) + d(w, w^*) + d(x, x^*)}{4}\right) \tag{41}$$

Also,

$$\phi(d(z, z^*) \leq \frac{1}{4}\phi(d(z, z^*) + d\left(w, w^*) + d(x, x^*) + d(y, y^*)\right) \\ - \psi\left(\frac{d(z, z^*) + d(w, w^*) + d(x, x^*) + d(y, y^*)}{4}\right) \tag{42}$$

And

$$\phi(d(w, w^*) \leq \frac{1}{4}\phi(d\left(w, w^*) + d(x, x^*) + d(y, y^*) + d(z, z^*)\right) \\ - \psi\left(\frac{d(w, w^*) + d(x, x^*) + d(y, y^*) + d(z, z^*)}{4}\right) \tag{43}$$

Adding (40), (41), (42) and (43), we get

$$\phi\left(d(x, x^*)\right) + \phi\left(d(y, y^*)\right) + \phi\left(d(z, z^*)\right) + \phi\left(d(w, w^*)\right) \\ \leq \phi\left(d(x, x^*) + d(y, y^*) + d(z, z^*) + d(w, w^*)\right) \\ - 4\psi\left(\frac{\phi(d(x, x^*) + d(y, y^*) + d(z, z^*) + d(w, w^*)}{4}\right) \tag{44}$$

By the property (iii) of ϕ, we obtain

$$\phi\left(d(x, x^*)\right) + \phi\left(d(y, y^*)\right) + \phi\left(d(z, z^*)\right) + \phi\left(d(w, w^*)\right) \\ \leq \phi\left(d(x, x^*) + d(y, y^*) + d(z, z^*) + d(w, w^*)\right) \tag{45}$$

From (44) and (45), we get

$$\phi\big(d(x,x^*)\big) + \phi\big(d(y,y^*)\big) + \phi\big(d(z,z^*)\big) + \phi\big(d(w,w^*)\big)$$
$$\le \phi\big(d(x,x^*), d(y,y^*), d(z,z^*), d(w,w^*)\big)$$
$$- 4\psi\left(\frac{\phi(d(x,x^*) + d(y,y^*) + d(z,z^*) + d(w,w^*)}{4}\right) \tag{46}$$

This implies that

$4\psi\left(\frac{d(x,x^*) + d(y,y^*) + d(z,z^*) + d(w,w^*)}{4}\right) \le 0.$ Using the property of ψ, we get

$d(x,x^*) + d(y,y^*) + d(z,z^*) + d(w,w^*) = 0.$ Hence $d(x,x^*) = d(y,y^*) = d(z,z^*) = d(w,w^*) = 0.$ So, $x = x^*, y = y^*, z = z^*$ and $w = w^*$.

Case II: Suppose (x,y,z,w) is not comparable. Let (x,y,z,w) be not comparable to (x^*,y^*,z^*,w^*), then there exists $(p_1,q_1,r_1,s_1) \in A_0 \times A_0 \times A_0 \times A_0$ which is comparable to (x,y,z,w) and (x^*,y^*,z^*,w^*) . Since $F(A_0 \times A_0 \times A_0 \times A_0) \subseteq B_0$, there exists $(p_2,q_2,r_2,s_2) \in A_0 \times A_0 \times A_0 \times A_0$ such that

$$d\big(p_2, F(p_1,q_1,r_1,s_1)\big) = dist(A,B)$$
$$d\big(q_2, F(q_1,r_1,s_1,p_1)\big) = dist(A,B)$$
$$d\big(r_2, F(r_1,s_1,p_1,q_1)\big) = dist(A,B)$$
$$d\big(s_2, F(s_1,p_1,q_1,r_1)\big) = dist(A,B)$$

Without loss of generality, assume that $(p_1,q_1,r_1,s_1) \preceq (x,y,z,w)$ (i.e $x \succeq p_1$, $y \preceq q_1$, $z \succeq r_1, w \preceq s_1$) . Note that $(p_1,q_1,r_1,s_1) \preceq (x,y,z,w)$ implies that $(y,z,w,x) \preceq (q_1,r_1,s_1,p_1)$. From Lemma 1, 2, 3 and 4, we get

$p_1 \preceq x, q_1 \succeq y, r_1 \preceq z$ and $s_1 \succeq w$

$d\big(p_2, F(p_1,q_1,r_1,s_1)\big) = dist(A,B), d\big(x, F(x,y,z,w)\big) = dist(A,B)$ implies that $p_2 \preceq x$.

$p_1 \preceq x$ and $q_1 \succeq y$

$d(q_2, F(q_1, r_1, s_1, p_1)) = dist(A, B), d(y, F(y, z, w, x)) = dist(A, B)$implies that $q_2 \succeq y$. $q_1 \succeq y$ and $r_1 \preceq z$

$d(r_2, F(r_1, s_1, p_1, q_1)) = dist(A, B), d(z, F(z, w, x, y)) = dist(A, B)$implies that $r_2 \preceq z$. $r_1 \preceq z$ and $s_1 \succeq w$

$d(s_2, F(s_1, p_1, q_1, r_1)) = dist(A, B), d(w, F(w, x, y, z)) = dist(A, B)$implies that $s_2 \succeq w$.

From the above four inequalities, we obtain $(p_2, q_2, r_2, s_2) \preceq (x, y, z, w)$. Continuing this process, we get sequences $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ and $\{s_n\}$ such that

$$d(p_{n+1}, F(p_n, q_n, r_n, s_n)) = dist(A, B)$$

$$d(q_{n+1}, F(q_n, r_n, s_n, p_n)) = dist(A, B)$$

$$d(r_{n+1}, F(r_n, s_n, p_n, q_n)) = dist(A, B)$$

$$d(s_{n+1}, F(s_n, p_n, q_n, r_n)) = dist(A, B)$$

with $(p_n, q_n, r_n, s_n) \preceq (x, y, z, w)$, $\forall n \in N$. By using the fact that F is a proximally quadrupled weak (ψ, ϕ) contraction on A, we get $p_n \preceq x$, $q_n \succeq y$, $r_n \preceq z$ and $s_n \succeq w$.

$$d(p_{n+1}, F(p_n, q_n, r_n, s_n)) = d(A, B), \qquad d(x, F(x, y, z, w)) = d(A, B) \text{ implies that}$$

$$\begin{aligned} &\phi(d(p_{n+1}, x) \\ &\leq \frac{1}{4}\phi\big(d(p_n, x) + d(q_n, y) + d(r_n, z) + d(s_n, w)\big) \\ &- \psi\left(\frac{d(p_n, x) + d(q_n, y) + d(r_n, z) + d(s_n, w)}{4}\right) \end{aligned} \tag{47}$$

Similarly, we can have

y $\preceq q_n$, $z \succeq r_n$, $w \preceq s_n$ and $x \succeq p_n$.

$$d(q_{n+1}, F(q_n, r_n, s_n, p_n)) = dist(A, B), d(y, F(y, z, w, x)) = dist(A, B) \text{ implies that}$$

$$\phi(d(q_{n+1},y) \leq \frac{1}{4}\phi\big(d(q_n,y)+d(r_n,z)+d(s_n,w)+d(p_n,x)\big)$$
$$-\psi\left(\frac{d(q_n,y)+d(r_n,z)+d(s_n,w)+d(p_n,x)}{4}\right) \quad (48)$$

$r_n \preceq z,\ s_n \succeq w,\ p_n \preceq x$ and $q_n \succeq y$.

$d\big(r_{n+1},F(r_n,s_n,p_n,q_n)\big)=dist(A,B), d\big(z,F(z,w,x,y)\big)=dist(A,B)$ implies that

$$\phi(d(z,r_{n+1}) \leq \frac{1}{4}\phi\big(d(z,r_n)+d(w,s_n)+d(x,p_n)+d(y,q_n)\big)$$
$$-\psi\left(\frac{\phi\big(d(z,r_n)+d(w,s_n)+d(x,p_n)+d(y,q_n)\big)}{4}\right) \quad (49)$$

and $w \preceq s_n,\ x \succeq p_n,\ y \preceq q_n$ and $z \succeq r_n$.

$d\big(s_{n+1},F(s_n,p_n,q_n,r_n)\big)=dist(A,B), d\big(w,F(w,x,y,z)\big)=dist(A,B)$ implies that

$$\phi(d(w,s_{n+1}) \leq \frac{1}{4}\phi\big(d(w,s_n)+d(x,p_n)+d(y,q_n)+d(z,r_n)\big)$$
$$-\psi\left(\frac{\phi\big(d(w,s_n)+d(x,p_n)+d(y,q_n)+d(z,r_n)\big)}{4}\right) \quad (50)$$

Adding (47), (48), (49) and (50), we obtain

$$\phi(d(p_{n+1},x)+\phi(d(q_{n+1},y)+\phi(d(r_{n+1},z)+\phi(d(s_{n+1},w)$$
$$\leq \phi\big(d(p_n,x)+d(q_n,y)+d(r_n,z)+d(s_n,w)\big)$$
$$-4\psi\left(\frac{d(p_n,x)+d(q_n,y)+d(r_n,z)+d(s_n,w)}{4}\right) \quad (51)$$

By the property (iii) of ϕ, we get

$$\phi\big(d(p_{n+1},x)+d(q_{n+1},y)+d(r_{n+1},z)+d(s_{n+1},w)\big)$$
$$\leq \phi\big(d(p_{n+1},x)\big)+\phi\big(d(q_{n+1},y)\big)+\phi\big(d(r_{n+1},z)\big)+\phi\big(d(s_{n+1},w)\big) \quad (52)$$

From (51) and (52), we obtain

$$\phi\big(d(p_{n+1},x)\big)+\phi\big(d(q_{n+1},y)\big)+\phi\big(d(r_{n+1},z)\big)+\phi\big(d(s_{n+1},w)\big)$$
$$\leq \phi\big(d(p_n,x)+d(q_n,y)+d(r_n,z)+d(s_n,w)\big)$$
$$-4\psi\left(\frac{d(p_n,x)+d(q_n,y)+d(r_n,z)+d(s_n,w)}{4}\right) \tag{53}$$

This implies that

$$\phi\big(d(p_{n+1},x)\big)+\phi\big(d(q_{n+1},y)\big)+\phi\big(d(r_{n+1},z)\big)+\phi\big(d(s_{n+1},w)\big)$$
$$\leq \phi\big(d(p_n,x)+d(q_n,y)+d(r_n,z)+d(s_n,w)\big) \tag{54}$$

Using the fact that ϕ is non decreasing, we get

$$d(p_{n+1},x)+d(q_{n+1},y)+d(r_{n+1},z)+d(s_{n+1},w)$$
$$\leq d(p_n,x)+d(q_n,y)+d(r_n,z)+d(s_n,w) \tag{55}$$

This means that the sequence $\{d(p_{n+1},x)+d(q_{n+1},y)+d(r_{n+1},z)+d(s_{n+1},w)\}$ is decreasing. Therefore, there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty}[d(p_n,x)+d(q_n,y)+d(r_n,z)+d(s_n,w)\,] = \alpha \tag{56}$$

We show that $\alpha = 0$. Suppose, to the contrary, that $\alpha > 0$. Taking the limit as $n \to \infty$ in (53), we have that

$$\phi(\alpha) \leq \phi(\alpha) - 4\lim_{n\to\infty}\psi\left(\frac{d(p_n,x)+d(q_n,y)+d(r_n,z)+d(s_n,w)}{4}\right) < \phi(\alpha).$$

This is a contradiction. Thus $\alpha = 0$, that is

$$\lim_{n\to\infty}[d(p_n,x)+d(q_n,y)+d(r_n,z)+d(s_n,w)\,] = 0 \tag{57}$$

So we have that

$p_n \to x, q_n \to y, r_n \to z$ and $s_n \to w$. Analogously, we can prove that $p_n \to x^*, q_n \to y^*, r_n \to z^*$ and $s_n \to w^*$. Therefore, $x = x^*, y = y^*, z = z^*$ and $w = w^*$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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