Available online at http://scik.org
J. Math. Comput. Sci. 9 (2019), No. 5, 533-550
https://doi.org/10.28919/jmcs/4158
ISSN: 1927-5307

INVARIANT SUBMANIFOLDS OF $N(\kappa)$-CONTACT METRIC MANIFOLDS

DIPANSHA KUMARI* AND H.G. NAGARAJA

Department of Mathematics, Bangalore University, Bengaluru – 560 056, INDIA

Abstract. In the present paper we study the nature of invariant submanifolds of $N(\kappa)$-contact metric manifolds and D-homothetically deformed $N(\kappa)$ contact metric manifolds. The conditions for totally geodesic submanifolds are derived.

Keywords: parallel; semi-parallel; pseudo-parallel; pseudo-2 parallel; Ricci-generalised pseudo-parallel; 2-Ricci-generalised pseudo-parallel.

2010 AMS Subject Classification: 53C25, 53C40, 53C50.

1. INTRODUCTION

A submanifold $(N,\tilde{g})$ of a Riemannian manifold (M,g) is totally geodesic if any geodesic on the submanifold N with its induced Riemannian metric $\tilde{g}$ is also a geodesic on the Riemannian manifold (M,g). For most of the Riemannian manifolds of dimension greater then 2, totally geodesic submanifolds do not exist. But the totally geodesic submanifolds occur if the manifold carries isometries. The study of invariant submanifolds was initiated by Bejancu and Papaghuic [3] and invariant submanifolds of almost contact manifolds was studied by Okumara in [9]. In 1969, Yano and Ishihara [11] have obtained conditions for an invariant submanifold of a normal

*Corresponding author
E-mail address: mohdfrd55@gmail.com
Received June 9, 2019

contact metric manifold to be totally geodesic in the case of codimension 2. Afterwards, In 1973, Kon [8] proved that invariant submanifold of normal contact metric manifold is totally geodesic if the second fundamental form of the immersion is covariantly constant. In general, an invariant submanifold of a Sasakian manifold is not totally geodesic. In this paper we are concerned the conditions under which invariant submanifolds of $N(\kappa)$- contact metric manifolds and D-homothetically deformed $N(\kappa)$- contact metric manifolds are totally geodesic.

2. Preliminaries

Let M be $(2m+1)$ dimensional almost contact metric manifold with the structure tensors (ϕ,ξ,η,g). where ϕ is a tensor field of type $(1,1)$, ξ a vector field, η a 1-form and g is a Riemannian metric on M [4]. Then

$$\begin{aligned} &\phi^2=-I+\eta\otimes\xi,\ \eta(\xi)=1,\ \phi\xi=0,\ \eta\cdot\phi=0, \\ &g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\ g(X,\xi)=\eta(X). \end{aligned} \tag{2.1}$$

for any $X,Y\in\Gamma(TM)$. Let Φ denote the 2-form in M and is given by
$\Phi(X,Y)=g(X,\phi Y)$. The κ-nullity distribution on a contact metric manifold M [10] for a real number κ is a distribution

$$N(\kappa):p\longrightarrow N_p(\kappa)=\{Z\in T_pM:R_M(X,Y)Z=\kappa[g(Y,Z)X-g(X,Z)Y]\}. \tag{2.2}$$

for any $X,Y\in T_pM$ where R_M denotes the Riemannian curvature tensor and T_pM denotes the tangent vector space of M at ant point $p\in M$.
If the characteristic vector field ξ of a contact metric manifold belongs to the κ-nullity distribution, then

$$R_M(X,Y)\xi=\kappa\{\eta(Y)X-\eta(X)Y\}. \tag{2.3}$$

A contact metric manifold with $\xi\in N(\kappa)$ is called a $N(\kappa)$-contact metric manifold. In an $N(\kappa)$-manifold the following relations hold:

$$(\nabla_X\phi)Y=g(X+hX,Y)\xi-\eta(Y)(X+hX), \tag{2.4}$$

$$h\xi=0, \tag{2.5}$$

$$h^2 = (\kappa - 1)\phi^2, \tag{2.6}$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.7}$$

$$\begin{aligned}(\nabla_Y h)X - (\nabla_X h)Y &= 2(\kappa - 1)g(Y, \phi X)\xi + (1-\kappa)(\eta(Y)\phi X - \eta(X)\phi Y) \\ &\quad + \eta(Y)\phi h X - \eta(X)\phi h Y,\end{aligned} \tag{2.8}$$

for all $X, Y \in \Gamma(TM)$, where h is a symmetric tensor.

Now let N be $(2n+1)$- dimensional immersed submanifold of M. Then the Gauss and Weingarten formulas are, respectively, given by

$$\nabla_X Y = \tilde{\nabla}_X Y + \sigma(X, Y) \tag{2.9}$$

and

$$\nabla_X V = -A_V X + \tilde{\nabla}^{\perp}_X V, \tag{2.10}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$, where σ denotes the second fundamental form, $\tilde{\nabla}^{\perp}$ the normal connection and A the shape operator. The second fundamental form and shape operator are related by

$$g(A_V X, Y) = g(\sigma(X, Y), V), \tag{2.11}$$

where g denotes the induced metric on N as well as the Riemannian metric g on M.
The covariant derivative of σ, is defined by

$$(\nabla_X \sigma)(Y, Z) = \tilde{\nabla}^{\perp}_X \sigma(Y, Z) - \sigma(\tilde{\nabla}_X Y, Z) - \sigma(Y, \tilde{\nabla}_X Z). \tag{2.12}$$

for any $X, Y, Z \in \Gamma(TN)$.
If $R_N(X, Y)Z$ denotes the Riemannian curvature tensor on the submanifold N of the manifold M then we have

$$\begin{aligned}R_M(X, Y)Z &= R_N(X, Y)Z + (\nabla_X \sigma)(Y, Z) - (\nabla_Y \sigma)(X, Z) + A_{\sigma(X,Z)}Y \\ &\quad - A_{\sigma(Y,Z)}X,\end{aligned} \tag{2.13}$$

for $X, Y, Z \in \Gamma(TN)$ [6].

The submanifold is totally geodesic if and only if $\sigma = 0$.
An immersion is said to be parallel and semi-parallel [2] if for all $X, Y \in \Gamma(TN)$ we get $\nabla \cdot \sigma = o$

and $R(X,Y)\cdot\sigma = 0$, respectively.
It is said to be pseudo-parallel [2] if for all $X,Y \in \Gamma(TN)$ we get

$$R(X,Y)\cdot\sigma = L_1 Q(g,\sigma), \tag{2.14}$$

where L_1 denotes a real function on N and $Q(E,T)$ [7] is defined by

$$Q(E,T)(X,Y,Z,W) = -T((X\wedge_E Y)Z,W) - T(Z,(X\wedge_E Y)W), \tag{2.15}$$

where $(X\wedge_E Y)Z$ is defined by

$$(X\wedge_E Y)Z = E(Y,Z)X - E(X,Z)Y. \tag{2.16}$$

Similarly, an immersion is said to be pseudo 2-parallel [2] if for all $X,Y \in \Gamma(TN)$,we get

$$R\cdot\nabla\sigma = L_2 Q(g,\nabla\sigma) \tag{2.17}$$

and Ricci generalized pseudo-parallel [2] if

$$R\cdot\sigma = L_2 Q(S,\sigma), \tag{2.18}$$

for all $X,Y \in \Gamma(TN)$.
The second fundamental form σ satisfying

$$(\nabla_X\sigma)(Y,Z) = A(X)\sigma(Y,Z), \tag{2.19}$$

where A is a non-zero one form, is said to be recurrent [7].

3. INVARIANT SUBMANIFOLD OF $N(\kappa)$-MANIFOLD

The invariant and anti-invariant submanifolds depend on the behaviour of almost contact metric structure ϕ. A submanifold N of an almost contact metric manifold is said to be invariant [3] if the structure vector field ξ is tangent to N at every point of N and ϕX is tangent to N for any vector field X tangent to N at every point of N, that is, if $X \in \Gamma(TN)$ then $\phi X \in \Gamma(TN)$ at every point of N.
Taking $Y = \xi$ in equation (2.9), we have

$$\nabla_X\xi = \tilde{\nabla}_X\xi + \sigma(X,\xi).$$

Using (2.7) in the above equation and equating the tangential and normal components we get

$$\tilde{\nabla}_X \xi = -\phi X - \phi hX \tag{3.1}$$

and

$$\sigma(X,\xi) = 0. \tag{3.2}$$

If R_N and S_N denote the Riemannian curvature tensor and Ricci tensor of the submanifold N. Using (3.2) in (2.13), we get

$$R_N(\xi,Y)\xi = \kappa\{\eta(Y)\xi - Y\}. \tag{3.3}$$

By the definition of Ricci tensor and using equation (2.6), we obtain

$$S_N(\xi,\xi) = 2\kappa n. \tag{3.4}$$

Now from (2.2) and (2.9) and a straightforward computation gives the following equations:

$$(\tilde{\nabla}_X \phi)Y = g(X+hX)\xi - \eta(Y)(X+hX) \tag{3.5}$$

and

$$\sigma(X,\phi Y) = \phi\sigma(X,Y) = \sigma(\phi X,Y), \tag{3.6}$$

Theorem 1. *Let N be an invariant submanifold of a $N(\kappa)$-manifold M. Then N is totally geodesic if and only if N is recurrent.*

Proof. Setting $X = \xi$ in (2.19) and using (2.12), we get

$$\tilde{\nabla}^{\perp}_Z \sigma(\xi,Y) - \sigma(\tilde{\nabla}_Z \xi,Y) - \sigma(\xi,\tilde{\nabla}_Z Y) = 0. \tag{3.7}$$

Using (3.1) and (3.2) in (3.7), we obtain

$$\sigma(\phi Z,Y) - \sigma(h\phi Z,Y) = 0. \tag{3.8}$$

Replacing Z by ϕZ in (3.8) and using (2.1), we get

$$-\sigma(Z,Y) + \sigma(hZ,Y) = 0. \tag{3.9}$$

Again replacing Z by hZ and using (2.6) and (2.1), we get

$$-\sigma(hZ,Y)-(\kappa-1)\sigma(Z,Y)=0. \tag{3.10}$$

Adding (3.9) and (3.10), we get $\kappa\sigma(Z,Y)=0$, which implies $\sigma(Z,Y)=0$,

for all $X,Y\in\Gamma(TN)$. Hence N is totally geodesic.

The converse is trivial. □

Corollary 3.1. *Let N be an invariant submanifold of a $N(\kappa)$-manifold M. Then N is totally geodesic if and only if N is parallel.*

Theorem 2. *Let N be an invariant submanifold of a $N(\kappa)$-manifold M. Then N is totally geodesic provided $L_2\neq\frac{1}{2n}$ if and only if N is Ricci-generalized pseudo-parallel.*

Proof. If the submanifold N is Ricci-generalized pseudo-parallel then from (2.18), we have

$$(R_M(X,Y)\cdot\sigma)(Z,W)=L_2Q(S_N,\sigma)(X,Y;Z,W). \tag{3.11}$$

Using (2.15) and (2.16), (3.11) can be written as

$$\begin{aligned}&R_N^{\perp}(X,Y)\sigma(Z,W)-\sigma(R_N(X,Y)Z,W)-\sigma(Z,R_N(X,Y)W)=\\&L_2[-S_N(Y,Z)\sigma(X,W)+S_N(X,Z)\sigma(Y,W)-S_N(Y,W)\sigma(Z,X)\\&+S_N(X,W)\sigma(Z,Y)].\end{aligned} \tag{3.12}$$

Taking $X=Z=\xi$ in (3.12) and using (3.2), we get

$$-\sigma(R_N(\xi,Y)\xi,W)=L_2S(\xi,\xi)\sigma(Y,W). \tag{3.13}$$

Using (3.3) and (3.4) in equation (3.13), we obtain

$$\kappa(1-2nL_2)\sigma(Y,W)=0.$$

Therefore we have $\sigma(Y,W)=0$, provided $L_2\neq\frac{1}{2n}$.

The converse is trivial. □

In the view of Theorem [3.2] one can easily prove the following theorem:

Theorem 3. *Let N be an invariant submanifold of a $N(\kappa)$-manifold M. Then N is totally geodesic provided $L_1\neq\kappa$ if and only if N is pseudo-parallel.*

We state the following Corollary.

Corollary 3.2. *Let N be an invariant submanifold of a $N(\kappa)$-manifold M. Then N is totally geodesic if and only if N is semi-parallel.*

Theorem 4. *Let N be an invariant submanifold of a $N(\kappa)$-manifold M. Then N is totally geodesic provided $L_1 \neq \kappa$ if and only if N is pseudo 2-parallel.*

Proof. If the submanifold N is pseudo 2-parallel then from (2.17), we have

$$(R_M(X,Y)\cdot\nabla\sigma)(Z,U,W) = L_1 Q(g,\nabla\sigma)(X,Y;Z,U,W). \tag{3.14}$$

This may be rewritten as

$$\begin{aligned}
&R_N^{\perp}(X,Y)(\nabla\sigma)(Z,U,W) - (\nabla\sigma)(R_N(X,Y)Z,U,W)\\
&-(\nabla\sigma)(Z,R_N(X,Y)U,W) - (\nabla\sigma)(Z,U,R_N(X,Y)W) =\\
&L_1[-(\nabla\sigma)((X\wedge_g Y)Z,U,W) - (\nabla\sigma)(Z,(X\wedge_g Y)U,W)\\
&-(\nabla\sigma)(Z,U,(X\wedge_g Y)W)],
\end{aligned} \tag{3.15}$$

where $(\nabla\sigma)(Z,U,W) = (\nabla_Z\sigma)(U,W)$. Now using (2.12) and (2.15) in (3.15), we have

$$\begin{aligned}
&R_N^{\perp}(X,Y)(\tilde{\nabla}_Z^{\perp}\sigma(U,W) - \sigma(\tilde{\nabla}_Z U,W) - \sigma(U,\tilde{\nabla}_Z W)) - \tilde{\nabla}^{\perp}_{R_N(X,Y)Z}\sigma(U,W)\\
&+\sigma(\tilde{\nabla}_{R_N(X,Y)Z}U,W) + \sigma(U,\tilde{\nabla}_{R_N(X,Y)Z}W) - \tilde{\nabla}_Z^{\perp}\sigma(R_N(X,Y)U,W)\\
&+\sigma(\tilde{\nabla}_Z R_N(X,Y)U,W) + \sigma(R_N(X,\tilde{Y})U,\nabla_Z W) - \tilde{\nabla}_Z^{\perp}\sigma(U,R_N(X,Y)W)\\
&+\sigma(\tilde{\nabla}_Z U,R_N(X,Y)W) + \sigma(U,\tilde{\nabla}_Z R_N(X,Y)W)\\
=&L_1[-\tilde{\nabla}^{\perp}_{(X\wedge_g Y)Z}\sigma(U,W) + \sigma(\tilde{\nabla}_{(X\wedge_g Y)Z}U,W) + \sigma(U,\tilde{\nabla}_{(X\wedge_g Y)Z}W)\\
&-\tilde{\nabla}_Z^{\perp}\sigma((X\wedge_g Y)U,W) + \sigma(\tilde{\nabla}_Z(X\wedge_g Y)U,W) + \sigma((X\wedge_g Y)U,\tilde{\nabla}_Z W)\\
&-\tilde{\nabla}_Z^{\perp}\sigma(U,(X\wedge_g Y)W) + \sigma(\tilde{\nabla}_Z U,(X\wedge_g Y)W) + \sigma(U,\tilde{\nabla}_Z(X\wedge_g Y)W)].
\end{aligned} \tag{3.16}$$

Setting $X = U = \xi$ in (3.16) and using (3.2), we get

$$\begin{aligned}
&-R_N^{\perp}(\xi,Y)\sigma(\tilde{\nabla}_Z\xi,W)+\sigma(\tilde{\nabla}_{R_N(\xi,Y)Z}\xi,W)-\tilde{\nabla}_Z^{\perp}\sigma(R_N(\xi,Y)\xi,W)\\
&+\sigma(\tilde{\nabla}_Z R_N(\xi,Y)\xi,W)+\sigma(R_N(\xi,Y)\xi,\tilde{\nabla}_Z W)+\sigma(\tilde{\nabla}_Z\xi,R_N(\xi,Y)\xi)\\
&=L_2[\sigma(\tilde{\nabla}_{(\xi\wedge_g Y)Z}\xi,W)-\tilde{\nabla}_Z^{\perp}\sigma((\xi\wedge_g Y)\xi,W)+\sigma(\tilde{\nabla}_Z(\xi\wedge_g Y)\xi,W)\\
&+\sigma((\xi\wedge_g Y)\xi,\tilde{\nabla}_Z W)+\sigma(\tilde{\nabla}_Z\xi,(\xi\wedge_g Y)W)].
\end{aligned} \tag{3.17}$$

Again plugging $W = \xi$ and using (3.1), (3.2) and (3.3), then the above equation (3.17) reduces to

$$(L_1-\kappa)\sigma(Y,\phi Z+\phi hZ)=0. \tag{3.18}$$

Replacing Z by ϕZ in (3.18)

$$(L_1-\kappa)\sigma(Y,-Z+hZ)=0. \tag{3.19}$$

Again replace Z by hZ in (3.19) to get

$$(L_1-\kappa)(\kappa-1)\sigma(Y,-hZ-Z)=0. \tag{3.20}$$

From (3.19) and (3.20), we get $\sigma(Y,Z)=0$, provided $L_1 \neq \kappa$ and
$\kappa \neq 1$. Therefore the submanifold is totally geodesic. The converse part is trivial. □

Theorem 5. *Let N be an invariant submanifold of a $N(\kappa)$-manifold M. Then N is totally geodesic $L_2 \neq \frac{1}{2n}$ if and only if N is 2-Ricci-generalized pseudo-parallel provided .*

Proof. If the submanifold N is 2-Ricci-generalized pseudo-parallel then we have

$$(R_M(X,Y)\cdot\nabla\sigma)(Z,U,W)=L_2 Q(S,\nabla\sigma)(X,Y;Z,U,W). \tag{3.21}$$

$$
\begin{aligned}
&R_N^{\perp}(X,Y)(\tilde{\nabla}_Z^{\perp}\sigma(U,W)-\sigma(\tilde{\nabla}_Z U,W)-\sigma(U,\tilde{\nabla}_Z W))-\tilde{\nabla}^{\perp}_{R_N(X,Y)Z}\sigma(U,W)\\
&+\sigma(\tilde{\nabla}_{R_N(X,Y)Z}U,W)+\sigma(U,\tilde{\nabla}_{R_N(X,Y)Z}W)-\tilde{\nabla}_Z^{\perp}\sigma(R_N(X,Y)U,W)\\
&+\sigma(\tilde{\nabla}_Z R_N(X,Y)U,W)+\sigma(R_N(X,\tilde{Y})U,\nabla_Z W)-\tilde{\nabla}_Z^{\perp}\sigma(U,R_N(X,Y)W)\\
&+\sigma(\tilde{\nabla}_Z U,R_N(X,Y)W)+\sigma(U,\tilde{\nabla}_Z R_N(X,Y)W)=\\
&L_1[-\tilde{\nabla}^{\perp}_{(X\wedge_S Y)Z}\sigma(U,W)+\sigma(\tilde{\nabla}_{(X\wedge_S Y)Z}U,W)+\sigma(U,\tilde{\nabla}_{(X\wedge_S Y)Z}W)\\
&-\tilde{\nabla}_Z^{\perp}\sigma((X\wedge_S Y)U,W)+\sigma(\tilde{\nabla}_Z(X\wedge_S Y)U,W)+\sigma((X\wedge_S Y)U,\tilde{\nabla}_Z W)\\
&-\tilde{\nabla}_Z^{\perp}\sigma(U,(X\wedge_S Y)W)+\sigma(\tilde{\nabla}_Z U,(X\wedge_S Y)W)+\sigma(U,\tilde{\nabla}_Z(X\wedge_S Y)W)].
\end{aligned}
\tag{3.22}
$$

Setting $X=U=\xi$ in (3.22) and using (3.2), we get

$$
\begin{aligned}
&-R_N^{\perp}(\xi,Y)\sigma(\tilde{\nabla}_Z\xi,W)+\sigma(\tilde{\nabla}_{R_N(\xi,Y)Z}\xi,W)-\tilde{\nabla}_Z^{\perp}\sigma(R_N(\xi,Y)\xi,W)\\
&+\sigma(\tilde{\nabla}_Z R_N(\xi,Y)\xi,W)+\sigma(R_N(\xi,Y)\xi,\tilde{\nabla}_Z W)+\sigma(\tilde{\nabla}_Z\xi,R_N(\xi,Y)\xi)\\
&=L_2[\sigma(\tilde{\nabla}_{(\xi\wedge_S Y)Z}\xi,W)-\tilde{\nabla}_Z^{\perp}\sigma((\xi\wedge_S Y)\xi,W)+\sigma(\tilde{\nabla}_Z(\xi\wedge_S Y)\xi,W)\\
&+\sigma((\xi\wedge_S Y)\xi,\tilde{\nabla}_Z W)+\sigma(\tilde{\nabla}_Z\xi,(\xi\wedge_S Y)W)].
\end{aligned}
\tag{3.23}
$$

Plugging $W=\xi$ and using (3.1), (3.2), (3.3) and (3.4) in (3.23), we get

$$(\kappa-2\kappa nL_2)\sigma(Y,\phi Z+\phi hZ)=0. \tag{3.24}$$

Replacing Z by ϕZ in (3.24), we obtain

$$(\kappa-2\kappa nL_2)\sigma(Y,-Z+hZ)=0. \tag{3.25}$$

Again replacing Z by hZ in (3.25), we get

$$(\kappa-2\kappa nL_2)(-\sigma(Y,hZ)-(\kappa-1)\sigma(Y,Z))=0 \tag{3.26}$$

From (3.25) and (3.26), we get $\sigma(Y,Z)=0$, provided $L_2\neq\frac{1}{2n}$.
Therefore the submanifold is totally geodesic. The converse part is trivial. □

From Theorem [3.1], Theorem [3.2], Theorem [3.3] , Theorem [3.4], Theorem [3.5], Corollary [3.1] and Corollary [3.2], we can state the following:

Theorem 6. *Let N be an invariant submanifold of a $N(\kappa)$-manifold M. Then the following statements are equivalent:*

1) *N is totally geodesic.*
2) *N is parallel.*
3) *N is semi-parallel.*
4) *N is recurrent.*
5) *N is pseudo-parallel (with $L_1 \neq \kappa$).*
6) *N is pseudo 2-parallel (with $L_1 \neq \kappa$).*
7) *N is Ricci-generalized pseudo-parallel (with $L_2 \neq \frac{1}{2n}$).*
8) *N is 2-Ricci-generalized pseudo-parallel (with $L_2 \neq \frac{1}{2n}$).*

4. Invariant submanifold of D-Homothetically deformed $N(\kappa)$-manifold

A D-homothetic deformation on an almost contact metric manifold $M(\phi,\xi,\eta,g)$ is defined by

$$\overline{\phi} = \phi,\ \overline{\xi} = \frac{1}{a}\xi,\ \overline{\eta} = a\eta,\ \overline{g} = ag + a(a-1)\eta\otimes\eta, \tag{4.1}$$

where a is a positive constant. It is clear that the D-homothetically deformed manifold $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is also an almost contact metric manifold and that

$$\overline{h} = \frac{1}{a}h. \tag{4.2}$$

Let $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ be the submanifold of D-homothetically deformed $N(\kappa)$-manifold $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$. Denote $\overline{\nabla}$ and $\overline{\tilde{\nabla}}$ by the Riemannian connections on $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ and $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ respectively and the Riemannian curvature tensors are denoted by $\overline{R}_M$ and $\overline{R}_N$. $\overline{\sigma}$ and $\overline{A}$ denote the second fundamental form and the shape operator respectively, the submanifold N of D-homothetically deformed $N(\kappa)$-manifold $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$.

We first view the relation between the covariant derivatives (resp. the Riemannian curvature tensors) on $M(\phi,\xi,\eta,g)$ and $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ in the following lemma and proposition.

Lemma 7. [5] *If $M(\phi,\xi,\eta,g)$ is a contact metric manifold with Riemannian connection ∇, the connection $\overline{\nabla}$ of the D-deformed manifold $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is given by*

$$\overline{\nabla}_X Y = \nabla_X Y + \frac{a-1}{a} g(hX,\phi Y)\xi - (a-1)\{\eta(X)\phi Y + \eta(Y)\phi X\}, \tag{4.3}$$

for any X,Y on M.

Proposition 8. *For any submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ of a D-deformed $N(\kappa)$ manifold the following relations hold:*

$$\overline{\tilde{\nabla}}_X \xi = -a\phi X - \phi hX \tag{4.4}$$

and

$$\overline{\sigma}(X,\xi) = 0. \tag{4.5}$$

Proof. From Gauss equation we may write

$$\overline{\nabla}_X Y = \overline{\tilde{\nabla}}_X Y + \overline{\sigma}(X,Y). \tag{4.6}$$

Putting $Y = \xi$ in (4.6) and using (4.3) and (2.7), we get

$$-a\phi X - \phi hX = \overline{\tilde{\nabla}}_X \xi + \overline{\sigma}(X,\xi). \tag{4.7}$$

On equating the tangential and normal parts we get the proposition. □

Proposition 9. [5] *Let $M(\phi,\xi,\eta,g)$ be a contact metric manifold with Riemannian curvature R_M. Then the Riemannian curvature $\overline{R}_M$ of D-deformed manifold is given by*

$$\begin{aligned}\overline{R}_M(X,Y)Z = {} & R_M(X,Y)Z + (a-1)\{g(Y,\phi Z)\phi X - g(X,\phi Z)\phi Y \\ & - 2g(X,\phi Y)\phi Z + \eta(X)(\nabla_Y\phi)Z - \eta(Y)(\nabla_X\phi)Z \\ & + \eta(Z)((\nabla_Y\phi)X - (\nabla_X\phi)Y)\} + \frac{a-1}{a}\{g((\nabla_Y\phi h)X \\ & - (\nabla_X\phi h)Y,Z)\xi + g(\phi hY,Z)\phi hX - g(\phi hX,Z)\phi hY\} \\ & + (a-1)^2\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\} \\ & + \frac{(a-1)^2}{a}\{\eta(Y)g(hX,Z)\xi - \eta(X)g(hY,Z)\xi\},\end{aligned} \tag{4.8}$$

for any X,Y,Z on M.

Now using (2.13), we can write

$$\overline{R}_M(X,Y)Z = \overline{R}_N(X,Y)Z + (\overline{\nabla}_X\overline{\sigma})(Y,Z) - (\overline{\nabla}_Y\overline{\sigma})(X,Z) + \overline{A}_{\overline{\sigma}(X,Z)}Y - \overline{A}_{\overline{\sigma}(Y,Z)}X. \tag{4.9}$$

Taking $Z = \overline{\xi}$ in (4.9), we have

$$\overline{R}_M(X,Y)\overline{\xi} = \overline{R}_N(X,Y)\overline{\xi} + (\overline{\nabla}_X\overline{\sigma})(Y,\overline{\xi}) - (\overline{\nabla}_Y\overline{\sigma})(X,\overline{\xi}) + \overline{A}_{\overline{\sigma}(X,\overline{\xi})}Y - \overline{A}_{\overline{\sigma}(Y,\overline{\xi})}X. \tag{4.10}$$

From the proposition[4.2], setting $Z = \overline{\xi}$ and using (2.6), we get

$$\begin{aligned}\overline{R}_M(X,Y)\overline{\xi} = \frac{1}{a}[&\kappa\{\eta(Y)X - \eta(X)Y\} + (a-1)\{\eta(X)(\nabla_Y\phi)\xi - \eta(Y)(\nabla_X\phi)\xi \\ &+ (\nabla_Y\phi)X - (\nabla_X\phi)Y\} + \frac{a-1}{a}\{g((\nabla_Y\phi h)X - (\nabla_X\phi h)Y,\xi)\xi\} \\ &+ (a-1)^2\{\eta(Y)X - \eta(X)Y\}].\end{aligned} \tag{4.11}$$

Taking into the account of (2.4) and (2.8), (4.11) reduces to

$$\overline{R}_M(X,Y)\overline{\xi} = \frac{(\kappa + a^2 - 1)}{a}\{\eta(Y)X - \eta(X)Y\} + \frac{2(a-1)}{a}\{\eta(Y)hX - \eta(X)hY\}. \tag{4.12}$$

Thus we can state the following:

Theorem 10. *Let $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ be a contact metric manifold obtained by D-homothetic deformation of an $N(\kappa)$-manifold $M(\phi,\xi,\eta,g)$ with Riemannian curvature tensor R_M. Then the Riemannian curvature tensor $\overline{R}_M$ of a D-deformed manifold is given by*

$$\overline{R}_M(X,Y)\xi = \overline{\kappa}_N\{\eta(Y)X - \eta(X)Y\} + \overline{\mu}_N\{\eta(Y)hX - \eta(X)hY\}, \tag{4.13}$$

where $\overline{\kappa}_N = \frac{(\kappa+a^2-1)}{a}$ and $\overline{\mu}_N = \frac{2(a-1)}{a}$.

Proposition 11. *In an invariant submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ of a D-deformed $N(\kappa)$ manifold the following relation holds:*

$$\overline{R}_N(\xi,X)\xi = \overline{\kappa}_N\{\eta(X)\xi - X\} - \overline{\mu}_N hX \tag{4.14}$$

and

$$\overline{S}_N(\xi,\xi) = 2n\overline{\kappa}_N. \tag{4.15}$$

Proof. Using (4.13) and (2.12) in (4.10) and from Proposition [4.1], we get (4.14). And (4.15) is straightforward. □

Now we prove main results of this paper.

Theorem 12. *An invariant submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ of D-homothetically deformed $N(\kappa)$-manifold $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is totally geodesic with $\kappa \neq (1-a^2)$ if and only if $\overline{\sigma}$ is recurrent .*

Proof. If the submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is recurrent then from (2.19), we have

$$\overline{\tilde{\nabla}}^{\perp}_X\overline{\sigma}(Y,Z)-\overline{\sigma}(\overline{\tilde{\nabla}}_XY,Z)-\overline{\sigma}(Y,\overline{\tilde{\nabla}}_XZ)=A(X)\overline{\sigma}(Y,Z). \tag{4.16}$$

Setting $Z=\overline{\xi}$ in (4.16) and making use of equations (4.4) and (4.5), we get

$$a\overline{\sigma}(\phi X,Y)+\overline{\sigma}(\phi hX,Y)=0. \tag{4.17}$$

Replacing X by ϕX in (4.17), we get

$$-a\overline{\sigma}(X,Y)+\overline{\sigma}(hX,Y)=0. \tag{4.18}$$

Again replacing X by hX in (4.18) and using (2.6), we get

$$-a\overline{\sigma}(hX,Y)-(\kappa-1)\overline{\sigma}(X,Y)=0. \tag{4.19}$$

From (4.18) and (4.19), we obtain $\overline{\sigma}(X,Y)=0$, provided $\kappa \neq (1-a^2)$. Therefore the submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is totally geodesic. The converse part is trivial. □

Corollary 4.1. *An invariant submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ of D-homothetically deformed manifold $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is totally geodesic with $\kappa \neq (1-a^2)$ if and only if $\overline{\sigma}$ is parallel .*

Theorem 13. *An invariant submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ of D-homothetically deformed $N(\kappa)$-manifold $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is totally geodesic provided $\overline{\kappa}_N^2(1-2nL_2)^2 \neq (1-\kappa)\overline{\mu}_N^2$ if and only if $\overline{\sigma}$ is Ricci-generalized pseudo-parallel.*

Proof. If the submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is Ricci-generalized pseudo-parallel then from (2.18) we have

$$(\overline{R}_M(X,Y)\cdot\overline{\sigma})(Z,W)=L_2Q(\overline{S}_N,\overline{\sigma})(X,Y;Z,W). \tag{4.20}$$

From (2.15) and (2.16), the above equation may be written as

$$\begin{aligned}
&\overline{R}_N^{\perp}(X,Y)\overline{\sigma}(Z,W)-\overline{\sigma}(\overline{R}_N(X,Y)Z,W)-\overline{\sigma}(Z,\overline{R}_N(X,Y)W)=\\
&L_2[-\overline{S}_N(Y,Z)\overline{\sigma}(X,W)+\overline{S}_N(X,Z)\overline{\sigma}(Y,W)-\overline{S}_N(Y,W)\overline{\sigma}(Z,X)\\
&+\overline{S}_N(X,W)\overline{\sigma}(Z,Y)].
\end{aligned} \tag{4.21}$$

Plugging $X=Z=\overline{\xi}$ in (4.21) and using (4.1), (4.5), (4.14) and (4.15), we obtain

$$\overline{\kappa}_N(1-2nL_2)\overline{\sigma}(Y,W)+\overline{\mu}_N\overline{\sigma}(hY,W)=0. \tag{4.22}$$

Replacing Y by $\overline{h}Y$ and using (4.1) and (2.6), we get

$$\overline{\kappa}_N(1-2nL_2)\overline{\sigma}(hY,W)-(\kappa-1)\overline{\mu}_N\overline{\sigma}(Y,W)=0. \tag{4.23}$$

From (4.22) and (4.23), we obtain
$\overline{\sigma}(y,W)=0$, provided $\overline{\kappa}_N^2(1-2nL_2)^2\neq(1-\kappa)\overline{\mu}_N$. Therefore the submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is totally geodesic. The converse part is trivial. □

Following the same steps of proof of Theorem[4.3], we have the following:

Theorem 14. *An invariant submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ of D-homothetically deformed manifold $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is totally geodesic provided $(\overline{\kappa}_N-L_1a^2)^2\neq(1-\kappa)\overline{\mu}_N^2$ if and only if $\overline{\sigma}$ is pseudo-parallel .*

Also we can state the following corollary:

Corollary 4.2. *An invariant submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ of D-homothetically deformed manifold $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is totally geodesic provided $\overline{\kappa}_N^2\neq(1-\kappa)\overline{\mu}_N^2$ if and only if $\overline{\sigma}$ is semi-parallel .*

Theorem 15. *An invariant submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ of D-homothetically deformed manifold $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is totally geodesic provided $(a\overline{\kappa}_N-L_1a^3+(\kappa-1)\overline{\mu}_N)\neq(\kappa-1)(-\overline{\kappa}_N+a\overline{\mu}_N+a^2L_1)$ if and only if $\overline{\sigma}$ is pseudo 2-parallel*

.

Proof. If the submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is pseudo 2-parallel then from (2.17), we have

$$(\overline{R}_M(X,Y)\cdot\overline{\nabla}\overline{\sigma})(Z,U,W) = L_1 Q(\overline{g},\overline{\nabla}\overline{\sigma})(X,Y;Z,U,W). \tag{4.24}$$

The equation (4.24) may be written as

$$\begin{aligned}
&\overline{R}_N^{\perp}(X,Y)(\overline{\nabla}\overline{\sigma})(Z,U,W) - (\overline{\nabla}\overline{\sigma})(\overline{R}_N(X,Y)Z,U,W)\\
&-(\overline{\nabla}\overline{\sigma})(Z,\overline{R}_N(X,Y)U,W) - (\overline{\nabla}\overline{\sigma})(Z,U,\overline{R}_N(X,Y)W) =\\
&L_1[-(\overline{\nabla}\overline{\sigma})((X\wedge_{\overline{g}}Y)Z,U,W) - (\overline{\nabla}\overline{\sigma})(Z,(X\wedge_{\overline{g}}Y)U,W)\\
&-(\overline{\nabla}\overline{\sigma})(Z,U,(X\wedge_{\overline{g}}Y)W)],
\end{aligned} \tag{4.25}$$

where $(\overline{\nabla}\overline{\sigma})(Z,U,W) = (\overline{\nabla}_Z\overline{\sigma})(U,W)$.

From (2.12) and (2.15), (4.25) becomes

$$\begin{aligned}
&\overline{R}_N^{\perp}(X,Y)(\overline{\tilde{\nabla}}_Z^{\perp}\overline{\sigma}(U,W) - \overline{\sigma}(\overline{\tilde{\nabla}}_Z U,W) - \overline{\sigma}(U,\overline{\tilde{\nabla}}_Z W)) - \overline{\tilde{\nabla}}^{\perp}_{\overline{R}_N(X,Y)Z}\overline{\sigma}(U,W)\\
&+\overline{\sigma}(\overline{\tilde{\nabla}}_{\overline{R}_N(X,Y)Z}U,W) + \overline{\sigma}(U,\overline{\tilde{\nabla}}_{\overline{R}_N(X,Y)Z}W) - \overline{\tilde{\nabla}}_Z^{\perp}\overline{\sigma}(\overline{R}_N(X,Y)U,W)\\
&+\overline{\sigma}(\overline{\tilde{\nabla}}_Z\overline{R}_N(X,Y)U,W) + \overline{\sigma}(\overline{R}_N(X,Y)U,\overline{\tilde{\nabla}}_Z W) - \overline{\tilde{\nabla}}_Z^{\perp}\overline{\sigma}(U,\overline{R}_N(X,Y)W)\\
&+\overline{\sigma}(\overline{\tilde{\nabla}}_Z U,\overline{R}_N(X,Y)W) + \overline{\sigma}(U,\overline{\tilde{\nabla}}_Z\overline{R}_N(X,Y)W) =\\
&L_1[-\overline{\tilde{\nabla}}^{\perp}_{(X\wedge_{\overline{g}}Y)Z}\overline{\sigma}(U,W) + \overline{\sigma}(\overline{\tilde{\nabla}}_{(X\wedge_{\overline{g}}Y)Z}U,W) + \overline{\sigma}(U,\overline{\tilde{\nabla}}_{(X\wedge_{\overline{g}}Y)Z}W)\\
&-\overline{\tilde{\nabla}}_Z^{\perp}\overline{\sigma}((X\wedge_{\overline{g}}Y)U,W) + \overline{\sigma}(\overline{\tilde{\nabla}}_Z(X\wedge_{\overline{g}}Y)U,W) + \overline{\sigma}((X\wedge_{\overline{g}}Y)U,\overline{\tilde{\nabla}}_Z W)\\
&-\overline{\tilde{\nabla}}_Z^{\perp}\overline{\sigma}(U,(X\wedge_{\overline{g}}Y)W) + \overline{\sigma}(\overline{\tilde{\nabla}}_Z U,(X\wedge_{\overline{g}}Y)W) + \overline{\sigma}(U,\overline{\tilde{\nabla}}_Z(X\wedge_{\overline{g}}Y)W)].
\end{aligned} \tag{4.26}$$

Setting $X = U = \overline{\xi}$ in equation (4.26) and using (4.5), we get

$$\begin{aligned}
&-\overline{R}_N^{\perp}(\xi,Y)\overline{\sigma}(\overline{\tilde{\nabla}}_Z\xi,W) + \overline{\sigma}(\overline{\tilde{\nabla}}_{\overline{R}_N(\xi,Y)Z}\xi,W) - \overline{\tilde{\nabla}}_Z^{\perp}\overline{\sigma}(\overline{R}_N(\xi,Y)\xi,W)\\
&+\overline{\sigma}(\overline{\tilde{\nabla}}_Z\overline{R}_N(\xi,Y)\xi,W) + \overline{\sigma}(\overline{R}_N(\xi,Y)\xi,\overline{\tilde{\nabla}}_Z W) + \overline{\sigma}(\overline{\tilde{\nabla}}_Z\xi,\overline{R}_N(\xi,Y)\xi)\\
&= L_2[\overline{\sigma}(\overline{\tilde{\nabla}}_{(\xi\wedge_{\overline{g}}Y)Z}\xi,W) - \overline{\tilde{\nabla}}_Z^{\perp}\overline{\sigma}((\xi\wedge_{\overline{g}}Y)\xi,W) + \overline{\sigma}(\overline{\tilde{\nabla}}_Z(\xi\wedge_{\overline{g}}Y)\xi,W)
\end{aligned} \tag{4.27}$$

$$+\overline{\sigma}((\xi\wedge_{\overline{g}}Y)\xi,\overline{\tilde{\nabla}}_Z W)+\overline{\sigma}(\overline{\tilde{\nabla}}_Z\xi,(\xi\wedge_{\overline{g}}Y)W)].$$

Again putting $W=\overline{\xi}$ in (4.27) and using (4.4), (4.5), (2.16) and (4.14), we obtain

$$(a\overline{\kappa}_N-L_1a^3+(\kappa-1)\overline{\mu}_N)\overline{\sigma}(Y,Z)+(-\overline{\kappa}_N+a\overline{\mu}_N+a^2L_1)\overline{\sigma}(Y,hZ)=0. \tag{4.28}$$

Replacing $Z=\overline{\xi}$ in (4.28), we have

$$-(a\overline{\kappa}_N-L_1a^3+(\kappa-1)\overline{\mu}_N)\overline{\sigma}(Y,hZ)+(\kappa-1)(-\overline{\kappa}_N+a\overline{\mu}_N+a^2L_1)\overline{\sigma}(Y,Z)=0. \tag{4.29}$$

From (4.28) and (4.29), we get $\overline{\sigma}=0$, provided $(a\overline{\kappa}_N-L_1a^3+(\kappa-1)\overline{\mu}_N)\neq(\kappa-1)(-\overline{\kappa}_N+a\overline{\mu}_N+a^2L_1)$. Therefore the submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is totally geodesic. The converse part is trivial. □

Theorem 16. *An invariant submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ of D-homothetically deformed manifold $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is totally geodesic provided $(-\overline{\kappa}_Na+2an\overline{\kappa}_NL_2-(\kappa-1)\overline{\mu}_N)^2\neq(1-\kappa)(-\overline{\kappa}_N+2na\overline{\kappa}_NL_2+a\overline{\mu}_N)^2$ if and only if $\overline{\sigma}$ is 2-Ricci-generalized pseudo-parallel .*

Proof. If the submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is 2-Ricci-generalized pseudo-parallel then we have

$$(\overline{R}_M(X,Y)\cdot\overline{\nabla}\overline{\sigma})(Z,U,W)=L_2Q(\overline{S}_N,\overline{\nabla}\overline{\sigma})(X,Y;Z,U,W). \tag{4.30}$$

The equation (4.30) may be written as

$$\begin{aligned}
&\overline{R}^{\perp}_N(X,Y)(\overline{\tilde{\nabla}}^{\perp}_Z\overline{\sigma}(U,W)-\overline{\sigma}(\overline{\tilde{\nabla}}_Z U,W)-\overline{\sigma}(U,\overline{\tilde{\nabla}}_Z W))-\overline{\tilde{\nabla}}^{\perp}_{\overline{R}_N(X,Y)Z}\overline{\sigma}(U,W)\\
&+\overline{\sigma}(\overline{\tilde{\nabla}}_{\overline{R}_N(X,Y)Z}U,W)+\overline{\sigma}(U,\overline{\tilde{\nabla}}_{R_N(X,Y)Z}W)-\overline{\tilde{\nabla}}^{\perp}_Z\overline{\sigma}(\overline{R}_N(X,Y)U,W)\\
&+\overline{\sigma}(\overline{\tilde{\nabla}}_Z\overline{R}_N(X,Y)U,W)+\overline{\sigma}(\overline{R}_N(X,Y)U,\overline{\tilde{\nabla}}_Z W)-\overline{\tilde{\nabla}}^{\perp}_Z\overline{\sigma}(U,\overline{R}_N(X,Y)W)\\
&+\overline{\sigma}(\overline{\tilde{\nabla}}_Z U,\overline{R}_N(X,Y)W)+\overline{\sigma}(U,\tilde{\nabla}_Z\overline{R}_N(X,Y)W)=\\
&L_2[-\overline{\tilde{\nabla}}^{\perp}_{(X\wedge_{\overline{S}}Y)Z}\overline{\sigma}(U,W)+\overline{\sigma}(\overline{\tilde{\nabla}}_{(X\wedge_{\overline{S}}Y)Z}U,W)+\overline{\sigma}(U,\overline{\tilde{\nabla}}_{(X\wedge_{\overline{S}}Y)Z}W)\\
&-\overline{\tilde{\nabla}}^{\perp}_Z\overline{\sigma}((X\wedge_{\overline{S}}Y)U,W)+\overline{\sigma}(\overline{\tilde{\nabla}}_Z(X\wedge_{\overline{S}}Y)U,W)+\overline{\sigma}((X\wedge_{\overline{S}}Y)U,\overline{\tilde{\nabla}}_Z W)\\
&-\overline{\tilde{\nabla}}^{\perp}_Z\overline{\sigma}(U,(X\wedge_{\overline{S}}Y)W)+\overline{\sigma}(\overline{\tilde{\nabla}}_Z U,(X\wedge_{\overline{S}}Y)W)+\overline{\sigma}(U,\overline{\tilde{\nabla}}_Z(X\wedge_{\overline{S}}Y)W)].
\end{aligned} \tag{4.31}$$

Setting $X = U = W = \overline{\xi}$ in (4.31) and using (4.4), (4.5), (2.16),(4.14) and (4.15), we get

$$(-a\overline{\kappa}_N + 2an\overline{\kappa}_N L_2 - (\kappa - 1)\overline{\mu}_N)\overline{\sigma}(Y,Z) - (-\overline{\kappa}_N + 2n\overline{\kappa}_N L_2 + a\overline{\mu}_N)\overline{\sigma}(Y,hZ) = 0. \tag{4.32}$$

Replacing Z by hZ in (4.32), we get

$$(-a\overline{\kappa}_N + 2an\overline{\kappa}_N L_2 - (\kappa - 1)\overline{\mu}_N)\overline{\sigma}(Y,hZ) + (-\overline{\kappa}_N + 2n\overline{\kappa}_N L_2 + a\overline{\mu}_N)(\kappa - 1)\overline{\sigma}(Y,Z) = 0. \tag{4.33}$$

From (4.32) and (4.33), we get $\overline{\sigma}(Y,Z) = 0$, provided
$(-\overline{\kappa}_N a + 2an\overline{\kappa}_N L_2 - (\kappa - 1)\overline{\mu}_N)^2 \neq (1-\kappa)(-\overline{\kappa}_N + 2na\overline{\kappa}_N L_2 + a\overline{\mu}_N)^2$.
Therefore the submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is totally geodesic. The converse part is trivial. □

We summarize the above results as follows:

Theorem 17. *For an invariant submanifold $N(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ of D-deformed manifold $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$, these statements are equivalent:*

1) *N is totally geodesic.*
2) *N is parallel (with $\kappa \neq (1-a^2)$).*
3) *N is semi-parallel (with $\overline{\kappa}_N^2 \neq (1-\kappa)\overline{\mu}_N^2$).*
4) *N is recurrent (with $\kappa \neq (1-a^2)$).*
5) *N is pseudo-parallel (with $(\overline{\kappa}_N - L_1 a^2)^2 \neq (1-\kappa)\overline{\mu}_N^2$).*
6) *N is pseudo 2-parallel (with $(a\overline{\kappa}_N - L_1 a^3 + (\kappa-1)\overline{\mu}_N) \neq (\kappa-1)(-\overline{\kappa}_N + a\overline{\mu}_N + a^2 L_1)$).*
7) *N is Ricci-generalized pseudo-parallel (with $\overline{\kappa}_N^2(1-2nL_2)^2 \neq (1-\kappa)\overline{\mu}_N^2$).*
8) *N is 2-Ricci-generalized pseudo-parallel (with $(-\overline{\kappa}_N a + 2an\overline{\kappa}_N L_2 - (\kappa-1)\overline{\mu}_N)^2 \neq (1-\kappa)(-\overline{\kappa}_N + 2na\overline{\kappa}_N L_2 + a\overline{\mu}_N)^2$).*

5. CONCLUSION

It has been shown in this paper that for an invariant submanifold of an $N(\kappa)$- contact metric manifold is totally geodesic condition is equivalent to parallel or pseudo-parallel or semi-parallelity of the submanifold. This holds for the D-homothetically deformed into $N(\kappa)$-manifolds also.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] B. S. Anitha, C. S. Bagewadi, Invariant submanifolds of Kenmotsu manifolds admitting quarter symmetric metric connection, Bull. Math. Anal. Appl. 4 (2012), no. 3, 99-114.

[2] C. S. Bagewadi, B. S. Anitha, Invariant submanifolds of trans-Sasakian manifolds, Reprint of Ukrain. Mat. Zh. 67 (2015), no. 10, 1309-1320.

[3] A. Bejancu, N. Papaghiuc, Semi-invariant submanifolds of a Sasakian manifold, An. St. Univ. Al. I. Cuza Iasi, 27 (1981), no. 1, 163-170.

[4] D. E. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 509, (1976).

[5] A. Carriazo, Martin-Molina, Veronica, Generalized (κ,μ)-space forms and D_a-homothetic deformations, Balkan J. Geom. Appl. 16 (2011), no. 1, 37-47.

[6] B. Y. Chen, Geometry of submanifolds, Pure and Applied Mathematics, Marcel Dekker Inc., New York, no.22, (1973).

[7] U. C. De, P. Majhi, On invariant submanifolds of Kenmotsu manifolds, J. Geom. 106 (2015), no. 1, 109-122.

[8] M. Kon, Invariant submanifolds of normal contact metric manifolds, Kodai Math. Sem. Rep., 25(1973), 330-336.

[9] M. Okumura, On contact metric immersion. Kodai Math. Sem. Rep. 20(1968), 389-409.

[10] T. Takahashi, S. Tanno, K-contact Riemannian manifolds isometrically immersed in a space of constant curvature, Tohoku Math. J. 23 (1971),no. 2, 535-539.

[11] K. Yano, S. Ishihara, Invariant submanifolds of an almost contact manifold. Kodai Mathematical Seminar Reports. Vol. 21. No. 3. Department of Mathematics, Tokyo Institute of Technology, 1969.