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QUADRATIC (s_1, s_2)-FUNCTIONAL INEQUALITY IN FUZZY NORMED SPACE

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Abstract. In this paper, we introduce and prove the Generalized Hyers-Ulam stability of Quadratic (s_1, s_2)-functional inequality in Fuzzy Normed space using the fixed point method.

Keywords: Generalized Hyers-Ulam(HU) stability; quadratic (s_1, s_2)-functional inequality; quadratic (s_1, s_2)-functional equation; fuzzy normed space.

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1. Introduction

Nearly two decades ago, Gl*á*nyi [8] proved that any h satisfies the Jordan-von Neumann functional equation

$$2h(x) + 2h(y) = h(xy) + h(xy^{-1})$$

if h satisfies the functional inequality

$$||2h(x) + 2h(y) - h(xy^{-1})|| \leq ||h(xy)||. \tag{1}$$

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Gl*á*nyi [9] and Fechner [5] proved the HU stability of the functional inequality (1). Park, Cho and Han [18] investigated and proved the HU Stability of the Cauchy additive functional inequality

$$||h(x)+h(y)+h(z)|| \leq ||h(x+y+z)||. \tag{2}$$

and the Cauchy-Jensen additive functional inequality

$$||h(x)+h(y)+2h(z)|| \leq ||2h\Big(\frac{x+y}{2}+z\Big)||. \tag{3}$$

The HU Stability is consequence of study of Ulam's [1] problem regarding stability of group homomorphism. A number of mathematicians namely Hyers [10], Aoki [2], Th.M.Rassias [19],Găvruta [7] studied HU Stability under various adaptations. Park [16],[17] introduced additive ρ-functional inequalities and proved their HU stability in Banach spaces and non-Archimedean Banach spaces. In this paper,we introduce and prove HU stability of quadratic (s_1,s_2)-functional inequality

$$F(F_1(x,y),t) \leq min\{F(s_1F_2(x,y),t),F(s_2F_3(x,y),t)\} \tag{4}$$

where

$F_1(x,y) = f(kx+y) - f(x+ky) - (k^2-1)[f(x)-f(y)]$

$F_2(x,y) = (k+1)^2 f(\frac{(kx+y)}{(k+1)}) - f(x+ky) - (k^2-1)[f(x)-f(y)]$

$F_3(x,y) = (k+1)^2 f(\frac{(kx+y)}{(k+1)}) - f(x+ky) - (k+1)^2(k^2-1)[f(\frac{x}{(k+1)}) - f(\frac{y}{(k+1)})]$

in Fuzzy Normed space, where k is a non zero positive integer; s_1 and s_2 are fixed non-zero real numbers with $\left(\frac{1}{s_1}+\frac{1}{s_2}\right) < 2$.

2. Preliminaries

The concept of fuzzy norm on a linear space was given by Katsaras [11] in 1984. Since then until now, the fuzzy norm has been defined in different ways by various mathematicians [3],[20],[6],[12].

2.1. Definition ([3],[15]). Let X be a real vector space. A function $F : X \times \mathbf{R} \to [0,1]$ is called a *fuzzy norm* on X if for all $a,b \in X$ and all $r,m,n \in \mathbf{R}$,

FN1: F(a, n) = 0 for $n \leq 0$;

FN2: a=0 iff F(a,n) = 1 for all $n > 0$;

FN3: F(ra, n) = $F(a, \frac{n}{|r|})$ if $r \neq 0$;

FN4: F(a+b,m+n) $\geq$ min{F(a,m),F(b,n)};

FN5: $lim_{n\to\infty}$F(a,n) =1,where F(a,.) is a non-decreasing function of **R**.

FN6: F(a,.) is continuous on **R**, for a $\neq 0$

The pair (X,F) is called a *fuzzy normed vector space*.

2.2. Definition ([3],[15]).

1. Let (X,F) be a fuzzy normed vector space. A sequence $\{a_n\}$ in X is said to be *convergent* if $\exists$ an $a \in X$ such that $\lim_{n\to\infty} F(a_n - a, r) = 1$ for all $r > 0$,where a is the limit of the sequence $\{a_n\}$, denoted by $F - \lim_{n\to\infty} a_n = a$.
2. Let (X,F) be a fuzzy normed vector space. A sequence $\{a_n\}$ in X is said to be *cauchy* if for each $\varepsilon > 0$ and each $r > 0$ there exists an $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$ and all $m > 0$,we have $F(a_{n+m} - a_n, r) > 1 - \varepsilon$.
3. The fuzzy norm is said to be *complete* if every cauchy sequence is convergent and the fuzzy normed vector space is called a *fuzzy Banach space*.
4. A mapping $f : X \to Y$ where X and Y are fuzzy normed vector spaces is continuous at a point $a_0 \in X$ if for each sequence $\{a_n\}$ converging to $a_0 \in X$, the sequence $\{f(a_n)\}$ converges to $f(a_0)$.If $f : X \to Y$ is continuous at each $a \in X$, then $f : X \to Y$ is said to be *continuous* on X.

2.3. Definition [13]. Let X be a set. A function $d : X \times X \to [0, \infty)$ is called a *generalized metric* on X if d satisfies the following conditions:

(1) $d(x,y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(2) $d(x,y) = d(y,x)$for all $x, y \in X$;

(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

2.4. Theorem [4]. Let (X,d) be a complete generalized metric space and $J : X \to X$ a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all non-negative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point $y^\star$ of J;

(3) $y^\star n$ is the unique fixed point of J in the set $Y=\{y\in X : d(J^{n_0}x,y)<\infty\}$;

(4) $d(y,y^\star)\leq(1/(1-L))d(y,Jy)$ for all $y\in Y$.

Throughout the paper, suppose that s_1 and s_2 are fixed nonzero real numbers with $\left(\frac{1}{s_1}+\frac{1}{s_2}\right)<2$ and k is a non zero positive integer. Also X and Y be real fuzzy normed space and fuzzy banach space respectively with norm F(.,t).

3. Quadratic (s_1,s_2)-Functional Inequality

3.1. Lemma. Let $f : X \to Y$ be a mapping with f(0)=0 and satisfies (4) for all $x,y\in X$ and all $t>0$.Then f is Quadratic.

Proof: Suppose that function f satisfies (4). By letting x=y in (4), we get

$$1\leq min\{F(s_1((k+1)^2f(x)-f((k+1)x)),t),(s_2((k+1)^2f(x)-f((k+1)x)),t)\}$$

$$\leq F((s_1+s_2)((k+1)^2f(x)-f((k+1)x)),2t)=F\left((k+1)^2f(x)-f((k+1)x),\frac{2t}{(s_1+s_2)}\right)$$

Therefore,

$$(k+1)^2f(x)=f((k+1)x) \tag{5}$$

Now from (4) and (5) we get

$$F(F_1(x,y),t)\leq min\{F(s_1F_1(x,y),t),F(s_2F_1(x,y),t)\}$$

$$=min\{F(F_1(x,y),\frac{t}{|s_1|}),F(F_1(x,y),\frac{t}{|s_2|})\}$$

$$\leq F\left(F_1(x,y),(\frac{1}{|s_1|}+\frac{1}{|s_2|})\frac{t}{2}\right)$$

i.e.

$$F(F_1(x,y),t)\geq F(F_1(x,y),\frac{t}{\zeta})$$

where $\zeta=\left\{\frac{1}{2}\left(\frac{1}{s_1}+\frac{1}{s_2}\right)\right\}$. Putting $\frac{t}{|\zeta|^{n-1}}$ instead of t, we get

$$F\left(F_1,\frac{t}{|\zeta|^{n-1}}\right)\geq F\left(F_1,\frac{t}{|\zeta|^{n}}\right)$$

Thus,for all $n \in \mathbf{Z}^{+}$ we have, $F(F_1,t)) \geq F\Big(F_1, \frac{t}{|\zeta|^n}\Big)$. Since $\zeta < 1$, therefore by taking limit $n \to \infty$ and using (FN5) , we get $F(F_1(x,y),t) = 1$ for all x,y $\in$ X , and hence $F_1(x,y) = 0$.So, $f : X \to Y$ is Quadratic.

3.2. Theorem. Let $\Psi : X^2 \to [0,\infty)$ be a function such that

$$\Psi(x,y) \leq \frac{L}{(k+1)^2}\Psi((k+1)x,(k+1)y)$$

for some $L < 1$ and for all x,y $\in$ X. Let $f : X \to Y$ be a mapping with f(0)=0 and satisfying $min\{F(F_1(x,y),t), \frac{t}{t+\Psi(x,y)}\} \leq$

$$min\{F(s_1F_2(x,y),t), F(s_2F_3(x,y),t)\} \tag{6}$$

where

$F_1(x,y) = f(kx+y) - f(x+ky) - (k^2-1)[f(x) - f(y)]$

$F_2(x,y) = (k+1)^2 f(\frac{(kx+y)}{(k+1)}) - f(x+ky) - (k^2-1)[f(x) - f(y)]$

$F_3(x,y) = (k+1)^2 f(\frac{(kx+y)}{(k+1)}) - f(x+ky) - (k+1)^2(k^2-1)[f(\frac{x}{(k+1)}) - f(\frac{y}{(k+1)})]$ for all $x,y \in X$ and all $t > 0$. Then $Q(x) = F - \lim_{n\to\infty}(k+1)^{2n} f\Big(\frac{x}{(k+1)^n}\Big)$ exists for all x $\in$ X and defines a Quadratic mapping $Q : X \to Y$ such that

$$F(f(x) - Q(x),t) \geq \frac{(2-2L)(k+1)t}{(2-2L)(k+1)t + \eta\Psi(x,x)} \tag{7}$$

for all $x \in X$, $t > 0$, where $\eta = \Big\{\frac{1}{|s_1|} + \frac{1}{|s_2|}\Big\}$.

Proof: Let $x = y$ in (6), we get

$$\frac{t}{t+\Psi(x,x)} \leq min\{F(s_1((k+1)^2 f(x) - f((k+1)x)),t), F(s_2((k+1)^2 f(x) - f((k+1)x)),t)\}$$

$$\leq min\Big\{F\Big((k+1)^2 f(x) - f((k+1)x), \frac{t}{|s_1|}\Big), F\Big((k+1)^2 f(x) - f((k+1)x), \frac{t}{|s_2|}\Big)\Big\}$$

$$\leq F\Big((k+1)^2 f(x) - f((k+1)x), \Big(\frac{1}{|s_1|} + \frac{1}{|s_2|}\Big)\frac{t}{2}\Big)$$

i.e.

$$F\Big(f(x) - (k+1)^2 f\Big(\frac{x}{(k+1)}\Big), \frac{\eta t}{2(k+1)}\Big) \geq \frac{t}{t+\Psi(x,x)} \tag{8}$$

Now let us consider the set

$$S = \{g : X \to Y\}$$

and a generalized metric on S,such that

$$d(g,h)=inf\Big(\varepsilon\in R^{+}:F(g(x)-h(x),\varepsilon t)\geq\frac{t}{t+\Psi(x,x)},\ for\ all\ x\in X,\ for\ all\ t>0\Big),$$

where $inf(\Psi)=+\infty$. Next, using lemma 2.1([14]) we can say that (S,d) is Complete.Now,let us consider a linear mapping $A:S\to S$ such that

$$Ag(x)=(k+1)^{2}g\Big(\frac{x}{(k+1)}\Big)$$

for all $x\in X$. Let $g,h\in S$ with $d(g,h)=\gamma$. Then

$$F(g(x)-h(x),\gamma t)\geq\frac{t}{t+\Psi(x,x)}$$

for all $x\in X,t>0$. Therefore,

$$F(Ag(x)-Ah(x),L\gamma t)=F\Big((k+1)^{2}g\Big(\frac{x}{(k+1)}\Big)-(k+1)^{2}h\Big(\frac{x}{(k+1)}\Big),L\gamma t\Big)$$

$$=F\Big(g\Big(\frac{x}{(k+1)}\Big)-h\Big(\frac{x}{(k+1)}\Big),\frac{L\gamma t}{(k+1)^{2}}\Big)\geq\frac{\frac{Lt}{(k+1)^{2}}}{\frac{Lt}{(k+1)^{2}}+\Psi(\frac{x}{(k+1)},\frac{x}{(k+1)})}$$

$$\geq\frac{\frac{Lt}{(k+1)^{2}}}{\frac{Lt}{(k+1)^{2}}+\frac{L}{(k+1)^{2}}\Psi(x,x)}=\frac{t}{t+\Psi(x,x)}$$

for all $x\in X,t>0$. Hence $d(Ag,Ah)=L\gamma$, i.e. $d(Ag,Ah)=Ld(g,h)$ for all $g,h\in S$. Also using (8), we can say that

$$d(f,Af)\leq\frac{\eta}{2(k+1)}.$$

Now, by Theorem (2.4), there exists a mapping $Q:X\to Y$ such that:

1. Q is a fixed point of A, i.e.,

$$Q(x)=(k+1)^{2}Q\Big(\frac{x}{(k+1)}\Big)\tag{9}$$

   for all $x\in X$. Since the mapping Q is a unique fixed point of A in the set

$$T=(g\in S:d(f,g)<\infty),$$

   thus Q is a unique mapping satisfying (9) such that there exists a $\varepsilon\in(0,\infty)$ satisfying

$$F(f(x)-Q(x),\varepsilon t)\geq\frac{t}{t+\Psi(x,x)}$$

   for all $x\in X$.

2. $d(A^n f, Q) \to 0$ as $n \to \infty$.This implies

$$Q(x) = F - \lim_{n\to\infty} (k+1)^{2n} f\Big(\frac{x}{(k+1)^n}\Big) \; for\ all\, x \in X.$$

3. $d(f,Q) \leq \frac{1}{1-L} d(f,Af)$, which implies $d(f,Q) \leq \frac{\eta}{2(k+1)-2(k+1)L}$.And thus inequality (7) is proved.Now by

$$min\Big\{F\Big((k+1)^{2n}F_1\Big(\frac{x}{(k+1)^n},\frac{y}{(k+1)^n}\Big),(k+1)^{2n}t\Big),\frac{t}{t+\Psi(\frac{x}{(k+1)^n},\frac{y}{(k+1)^n})}\Big\}$$
$$\leq min\Big\{F\Big((k+1)^{2n}s_1F_2\Big(\tfrac{x}{(k+1)^n},\tfrac{y}{(k+1)^n}\Big),(k+1)^{2n}t\Big),$$
$$F\Big((k+1)^{2n}s_2F_3\Big(\tfrac{x}{(k+1)^n},\tfrac{y}{(k+1)^n}\Big),(k+1)^{2n}t\Big)\Big\}$$

for all x, y $\in$ X, all $t > 0$ and all $n \in \mathbf{N}$. Now,by (6)

$$min\Big\{F\Big((k+1)^{2n}F_1\Big(\tfrac{x}{(k+1)^n},\tfrac{y}{(k+1)^n}\Big),t\Big),\tfrac{t/(k+1)^{2n}}{(t/(k+1)^{2n})+(L^n/(k+1)^{2n})\Psi(x,y)}\Big\}$$

(10) $$\leq min\Big\{F\Big((k+1)^{2n}s_1F_2\Big(\frac{x}{(k+1)^n},\frac{y}{(k+1)^n}\Big),t\Big),$$
$$F\Big((k+1)^{2n}s_2F_3\Big(\frac{x}{(k+1)^n},\frac{y}{(k+1)^n}\Big),t\Big)\Big\}$$

Since $\lim_{n\to\infty} \frac{t/(k+1)^{2n}}{(t/(k+1)^{2n})+(L^n/(k+1)^{2n})\Psi(x,y)} = 1$ for all x,y $\in$ X, all $t > 0$,therefore by lemma (3.1) the mapping $C : X \to Y$ is Quadratic.

3.3. Corollary. Let $\varsigma \geq 0$ and p be a real number with $p > 2$.Let X be a normed vector space with norm $||.||$ and (Y,N) be a fuzzy normed vector space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and

(11) $$min\Big\{F(F_1(x,y),t),\frac{t}{t+\varsigma(||x||^p+||y||^p)}\Big\} \leq min\{F(s_1F_2(x,y),t),F(s_2F_3(x,y),t)\Big\}$$

where $F_1(x,y), F_2(x,y)$ *and* $F_3(x,y)$ are as defined earlier for all $x,y \in X$ and all $t > 0$. Then $Q(x) = F - \lim_{n\to\infty}(k+1)^{2n} f\Big(\frac{x}{(k+1)^n}\Big)$ exists for all x $\in$ X and a Quadratic mapping $C : X \to Y$ such that

(12) $$F(f(x)-Q(x),t) \geq \frac{((k+1)^p-(k+1)^2)(k+1)t}{((k+1)^p-(k+1)^2)(k+1)t+\eta\varsigma||(k+1)x||^p}$$

for all $x \in X$, $t > 0$, where $\eta = \frac{1}{|s_1|} + \frac{1}{|s_2|}$.

Proof: The proof follows from above Theorem by taking $\Psi(x,y) = \varsigma(||x||^p + ||y||^p)$ for all $x,y \in X$ and $L = |k+1|^{2-p}$ and we get desired result.

3.4. Theorem. Let $\Psi : X^2 \to [0,\infty)$ be a function such that

$$\Psi(x,y) \leq (k+1)^2 L \Psi(\frac{x}{(k+1)}, \frac{y}{(k+1)})$$

for some $L < 1$ and for all x,y ∈ X. Let $f : X \to Y$ be a mapping with f(0)=0 and satisfying (6). Then $Q(x) = F - \lim_{n\to\infty} \frac{1}{(k+1)^{2n}} f((k+1)^n x)$ exists for all x ∈ X and defines a Quadratic mapping $C : X \to Y$ such that

$$F(f(x) - Q(x),t) \geq \frac{(2-2L)(k+1)^2 t}{(2-2L)(k+1)^2 t + \eta \Psi(x,x)} \tag{13}$$

for all $x \in X$, $t > 0$, where $\eta = \frac{1}{|s_1|} + \frac{1}{|s_2|}$.

Proof: It follows from (8) that, $F\left(f(x) - \frac{1}{(k+1)^2} f((k+1)x), \frac{\eta t}{2(k+1)^2}\right) \geq \frac{t}{t+\Psi(x,x)}$ for all $x \in X$ and all $t > 0$. Now consider linear mapping $A : S \to S$ such that

$$Ag(x) = \frac{1}{(k+1)^2} f((k+1)x)$$

for all $x \in X$,where (S,d) is the generalized metric space as defined in previous theorem. Then $d(f,Af) \leq \frac{\eta}{2(k+1)^2}$.Hence

$$d(f,C) \leq \frac{\eta}{2(k+1)^2 - 2(k+1)^2 L}$$

which proves inequality (13).Rest of the proof can be generated from (3.2).

3.5. Corollary. Let $\varsigma \geq 0$ and p be a real number with $0 < p < 2$.Let X be a normed vector space with norm $||.||$ and (Y,N) be a fuzzy normed vector space. Let $f : X \to Y$ be a mapping with $f(0) = 0$ and satisfying (11). Then $Q(x) = F - \lim_{n\to\infty} \frac{1}{(k+1)^{2n}} f((k+1)^n x)$ exists for all x ∈ X and a Quadratic mapping $C : X \to Y$ such that

$$F(f(x) - Q(x),t) \geq \frac{((k+1)^2 - (k+1)^p)t}{((k+1)^2 - (k+1)^p)t + \eta \varsigma ||x||^p} \tag{14}$$

for all $x \in X$, $t > 0$, where $\eta = \frac{1}{|s_1|} + \frac{1}{|s_2|}$.

Proof: The proof follows from above Theorem by taking $\Psi(x,y) = \varsigma(||x||^p + ||y||^p)$ for all $x,y \in X$ and $L = |k+1|^{p-2}$ and we get desired result.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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