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COVERING L −LOCALLY UNIFORM SPACES

JWNGSAR MOSHAHARY AND DIPAK KUMAR MITRA*

Department of Mathematical Sciences, Bodoland University, Kokrajhar 783370, India

Abstract: In this paper, we develop the notion of covering L −locally uniform spaces. Interior and closure operators were then introduced to show that every covering L −local uniformity induced a regular L −topology and vice-versa. Further, we have introduced the notion of weakly uniformly continuous functions in the class of covering L −local uniformities and studied some of its basic properties. We then established that the products of L −regular topologies are generated by the product covering L −locally uniform space. Towards the end of this paper, we have shown that every covering L −locally uniform spaces with countable base is pseudo-metrisable.

Keywords: L −topology; Covering L −locally uniform space; L −regularity; L −psuedo-metrisability.

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1. INTRODUCTION

To study about the uniform properties (such as completeness, uniform continuity and uniform convergence) in the setting of general topological spaces, uniform spaces were developed through entourage approach[23] and covering approach[21]. Efforts were made on reducing the conditions to develop weaker spaces, wherein results in uniform spaces could

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*Corresponding author

E-mail address: dkrmitra@gmail.com

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possibly be developed. As a result various generalisation of uniform spaces such as quasi-uniform, locally uniform spaces, locally quasi-uniform spaces, semi-uniform, semi-quasi-uniform spaces were developed leading to a broad spectrum of theory and applications in related fields.

One of the generalisation of uniform spaces namely locally uniform spaces were developed by James Williams [24] via localization of the triangle axiom through entourage approach. In [24], a topological space was shown to have a compatible local uniformity if and only if it is regular. Following this generalisation of uniform spaces many interesting and useful results on compactness, completeness and pseudo-metrizability were obtained in [24]. Further Vasudevan and Goel in [22] characterised James locally uniform spaces through covering approach.

Consequent to the development of the theory fuzzy topology, many spectacular and creative work about the theory of uniformities on various categories of fuzzy topological spaces have been accomplished by several authors including Hutton, Katsarsas, Lowen, Hu Cheng-Ming et. al. [13, 1, 15, 4] in the category $L-$TOP and $I-$TOP. Garcia et. al. [18] have introduced uniform spaces in a unifying framework of GL-monoid to include both the categories of Lowen Uniformity and Hutton type uniformities. The notion of covering fuzzy uniform spaces was introduced by Soetens et al. [20], Chandrika et al. [5, 2] and covering $L-$valued uniform space by García, at el. [14]. In a more recent development, in [7, 8, 6, 10, 12, 11, 9, 16, 17] Hazarika and Mitra have developed semi structure and localisaton of uniform and quasi-uniform spaces through entourage approach in the category $L-$TOP and $I-$TOP respectively. Subsequently, many interesting results on compactness, completeness were obtained. Problems regarding uniformly continuous and metrization are also considered. However, the localization of uniformity through covering approach has not been considered in the fuzzy settings.

In this paper, we developed the notion of covering $L-$locally uniform spaces. This generalised the notion of covering $L-$ uniform spaces in the sense of García, at el. [14]. Subsequently, several results on regular $L-$topological space, uniformly continuous and pseudo-metrization have been obtained. In our next article, we will study various important results related to compactness and uniform convergence.

Throughout this paper $(L, \leq, \wedge, \vee)$ denotes a fuzzy lattice with order reversing involution $'$; 0_L and 1_L are respectively inf and sup in L. X is an arbitrary (ordinary) set and L^X denotes the collection of all mappings $A: X \to L$. Any member of L^X is an $L-$fuzzy set. The $L-$fuzzy sets $x_\alpha: X \to L$ defined by $x_\alpha(y) = 0_L$ if $x \neq y$ and $x_\alpha(y) = \alpha$ if $x = y$ are the $L-$fuzzy points. The mappings $A: X \to L$ and $B: X \to L$ defined by $A(x) = 1_L$, $\forall x \in X$ and $B(x) = 0_L$, $\forall x \in X$ are denoted by $\underline{1}$ and $\underline{0}$ respectively. For any A, $B \in L^X$, the union and intersection of A and B are defined as $A \cup B(x) = A(x) \vee B(x)$ and $A \cap B(x) = A(x) \wedge B(x)$ respectively. Further, we say that $A \subseteq B$ if and only if $A(x) \leq B(x)$ and $x_\alpha \in A$ if and only if $\alpha < A(x)$, where x_α is an L-fuzzy point; complement A' of A is defined as $A'(x) = A(x)'$. An $L-$topology $\mathbb{F}$ on L^X is a subset of L^X closed under finite intersection and arbitrary union. In this case, the pair $(L^X, \mathbb{F})$ is known as L-topological space. The elements of $\mathbb{F}$ are called open sets and their complements are the closed sets. For any $A \in L^X$, the interior and closure of A in L-topological space $(L^X, \mathbb{F})$ are respectively denoted by A^o and $\bar{A}$. For basic definitions and results of product of $L-$topological spaces we refer to [3, 25]. Covering $L-$valued Uniformity referred to in this paper is in the sense of García, at el. [14]. For the problem of metrisation, we have consider the metric in the sense of Erceg[19].

2. PRELIMINARIES

This section includes basic definitions and results used in the subsequent sections.

Definition 2.1 [25] For any ordinary mapping $f: X \to Y$, the induced $L-$fuzzy mapping $f^{\to}: L^X \to L^Y$ and its L-fuzzy reverse mapping $f^{\leftarrow}: L^Y \to L^X$ respectively are defined as:

$$f^{\to}(A)(y) = \vee \{A(x) | x \in X,\ f(x) = y\},\ \ \forall\ A \in L^X,\ \forall\ y \in Y.$$

$$f^{\leftarrow}(B)(x) = B\big(f(x)\big),\ \ \forall\ B \in L^Y,\ \forall\ x \in X.$$

Symbol $f^{\to}$ and $f^{\leftarrow}$ always denote $f^{\to}$ to be the L-fuzzy mapping induced from an ordinary mapping f and $f^{\leftarrow}$ is the L-fuzzy reverse mapping of $f^{\to}$. Both the $L-$fuzzy mappings $f^{\rightarrow}$ and $f^{\leftarrow}$ are order preserving.

Definition 2.2 [25] Let X be a nonempty ordinary set and L be a fuzzy lattice. Then an operator $i: L^X \to L^X$ satisfying the following axioms is known as an interior operator on L^X

(IO1) $i(\underline{1}) = \underline{1}$.

(IO2) $i(A) \subseteq A, \forall A \in L^X$.

(IO3) $i(A \cap B) = i(A) \cap i(B), \forall A \in L^X$

(IO4) $i(i(A)) = i(A), \forall A \in L^X$.

Definition 2.3 [25] Let X be a nonempty ordinary set and L be a fuzzy lattice. Then an operator $c: L^X \to L^X$ satisfying the following axioms is known as a closure operator on L^X

(CO1) $c(\underline{0}) = \underline{0}$

(CO2) $A \subseteq c(A), \forall A \in L^X$.

(CO3) $c(A \cup B) = c(A) \cup c(B), \forall A, B \in L^X$.

(CO4) $c(c(A)) = c(A), \forall A \in L^X$.

Definition 2.4 [25] Let $(L^X, \mathbb{F})$ be an $L-$topological space. Then $(L^X, \mathbb{F})$ is said to be regular, if for every $G \in \mathbb{F}$ and $x_\alpha \leq G$, there is $A \in \mathbb{F}$ such that $x_\alpha \subseteq A \subseteq \overline{A} \subseteq G$.

Theorem 2.5 [25] Product of $L-$topological spaces is regular if and only if each of the factor space is regular.

Definition 2.6 [14] A collection $\mathcal{A}$ of L^X is called $L-$cover of L^X if $\bigcup \mathcal{A} = \underline{1}$. For any $\mathcal{A}, \mathcal{B} \subseteq L^X$ then $\mathcal{A}$ refines $\mathcal{B}$ if and only if for each $A \in \mathcal{A}$ there exits $B \in \mathcal{B}$ such that $A \subseteq B$. we write $\mathcal{A} \preccurlyeq \mathcal{B}$. The set of all $L-$covers of L^X, defined as $L - Cov(X)$, is a preordered set with respect to the relation '$\preccurlyeq$'.

Proposition 2.7 [14] For every $L-$covers $\mathcal{A}$ and $\mathcal{B}$of L^X, we have $\mathcal{A} \cap \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$ is also $L-$ cover of L^X .

Definition 2.8 [14] For each $A \in L^X$ and $\mathcal{A} \subseteq L^X$, the star of A with respect to $\mathcal{A}$ is defined as $st(A, \mathcal{A}) \coloneqq \{B \in \mathcal{A} : B \cap A \neq \underline{0}\}$. The collection $st(A) := \{st(A, \mathcal{A}) : A \in A\}$, is an

L −cover of L^X whenever $\mathcal{A}$ is cover.

Proposition 2.9 [14] Let $\mathcal{A},\mathcal{B} \subseteq L^X$ and $A, B \in L^X$. Then

1. If $\mathcal{A}$ is an L −cover of L^X , then $A \subseteq st(A,\mathcal{A})$ and, consequently, $\mathcal{A} \preccurlyeq st(\mathcal{A})$.
2. If $A \subseteq B$, then $st(A,\mathcal{A}) \subseteq st(B,\mathcal{A})$.
3. If $\mathcal{A} \preccurlyeq \mathcal{B}$, then $st(A,\mathcal{A}) \subseteq st(A,\mathcal{B})$.
4. $st(\bigcup \mathcal{B}, \mathcal{A}) = \bigcup_{B\in\mathcal{B}} st(B,\mathcal{A})$.
5. If $\mathcal{A}$ is an L −cover, then $st(st(A,\mathcal{A}),\mathcal{A}) \subseteq st(A, st(\mathcal{A}))$.
6. Let $f^{\rightarrow}: L^X \rightarrow L^Y$ be an L −fuzzy mapping and $\mathcal{B} \subseteq L^Y$. Also, let $f^{-1}(\mathcal{B}) = \{f^{\leftarrow}(B) : B \in \mathcal{B}\}$ and $C \in L^Y$. Then, $st(f^{\leftarrow}(C), f^{-1}(\mathcal{B})) \subseteq f^{\leftarrow}(st(C,\mathcal{B}))$

Remark 2.10 Let $\mathcal{A}$ and $\mathcal{B}$ be two L − covers of L^X such that $st(\mathcal{A}) \preccurlyeq \mathcal{B}$, then $st(A, st(\mathcal{A})) \subseteq st(A,\mathcal{B}), \forall\, A \in L^X$.

Definition 2.11 [14] A pair $(L^X, \mathfrak{U})$, consisting of L^X and a non-empty family $\mathfrak{U}$of L −covers of L^X, is said to be L −uniform space whenever the following conditions are satisfied.

(C1) $\mathcal{A} \preccurlyeq \mathcal{B}, \mathcal{A} \in \mathfrak{U} \Rightarrow \mathcal{A} \in \mathfrak{U}$.

(C2) For every $\mathcal{A},\mathcal{B} \in \mathfrak{U}, \mathcal{A} \cap \mathcal{B} \in \mathfrak{U}$.

(C3) For each $\mathcal{A} \in \mathfrak{U}$, there exits $\mathcal{B} \in \mathfrak{U}$ such that $st(\mathcal{B}) \preccurlyeq \mathcal{A}$.

3. COVERING L-LOCALLY UNIFORM SPACES

In this section, we introduce the notion of covering L −locally uniform spaces generalising the notion of covering L −uniform space in the sense of García et al.[14]. Interior operator and closure operators were then introduced for covering L −locally uniform spaces. Every regular L −topological spaces are characterised in terms of the developed notion.

Definition 3.1 A non-empty family $\mathcal{U}$ of L −covers of L^X is said to be a covering L −locally uniformity on L^X, if it satisfies the following axioms:

(LC1) $\mathcal{A} \preccurlyeq \mathcal{B}, \mathcal{A} \in \mathfrak{U} \Rightarrow \mathcal{A} \in \mathfrak{U}$. .

(LC2) For every $\mathcal{A},\mathcal{B} \in \mathfrak{U}, \mathcal{A} \cap \mathcal{B} \in \mathfrak{U}$.

(LC3) For each $\mathcal{A} \in \mathfrak{U}$ and $\forall\, x_\alpha \in L^X$, there exits $\mathcal{B} \in \mathfrak{U}$ such that

$st(x_\alpha, st(\mathcal{B})) \subseteq st(x_\alpha, \mathcal{A})$.

In that case we called the pair $(L^X, \mathfrak{U})$ as covering $L-$ locally uniform space. Let $\mathfrak{U}_1$ and $\mathfrak{U}_2$ be covering $L-$uniform spaces on L^X. If $\mathfrak{U}_1 \subset \mathfrak{U}_2$, then $\mathfrak{U}_2$ is called finer than $\mathfrak{U}_1$.

Definition 3.2 A non-empty family $\mathfrak{B}$ of $L-$covers of L^X is said to be a base for some covering $L-$ locally uniformity on L^X if it satisfies (LC2) and (LC3).

Clearly, by Remark 2.10, it follows that every covering $L-$ uniform space is covering $L-$locally uniform spaces. But the converse is not true as shown by the example cited below.

Example 3.3 Let $\mathrm{X} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}$ and $L = [0, 1]$.

Let $\mathcal{A} = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$, $\mathcal{B} = \{\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$.

Then $\mathcal{A} \cap \mathcal{B} = \{\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$.

Let $\mathfrak{B} = \{\mathcal{A}, \mathcal{B}, \mathcal{A} \cap \mathcal{B}\}$. Then clearly, $\mathcal{B}$ satisfies the axiom (LC2).

Now, $\mathrm{st}(\mathcal{A}) = \{a, b, c, d\}$, $\mathrm{st}(\mathcal{B}) = \{a, b, c, d\}$ and $\mathrm{st}(\mathcal{A} \cap \mathcal{B}) = \{a, b, c, d\}$.

Then, $\mathrm{st}(a, \mathrm{st}(\mathcal{A})) = \mathrm{st}(b, \mathrm{st}(\mathcal{A})) = \mathrm{st}(c, \mathrm{st}(\mathcal{A})) = \mathrm{st}(d, \mathrm{st}(\mathcal{A})) = \{a, b, c, d\}$

$\mathrm{st}(a, \mathrm{st}(\mathcal{B})) = \mathrm{st}(b, \mathrm{st}(\mathcal{B})) = \mathrm{st}(c, \mathrm{st}(\mathcal{B})) = \mathrm{st}(d, \mathrm{st}(\mathcal{B})) = \{a, b, c, d\}$

$\mathrm{st}(a, \mathrm{st}(\mathcal{A} \cap \mathcal{B})) = \mathrm{st}(b, \mathrm{st}(\mathcal{A} \cap \mathcal{B})) = \mathrm{st}(c, \mathrm{st}(\mathcal{A} \cap \mathcal{B})) = \mathrm{st}(d, \mathrm{st}(\mathcal{A} \cap \mathcal{B})) = \{a, b, c, d\}$.

Also $\mathrm{st}(a, \mathcal{A}) = \mathrm{st}(b, \mathcal{A}) = \mathrm{st}(c, \mathcal{A}) = \mathrm{st}(d, \mathcal{A}) = \{a, b, c, d\}$.

$\mathrm{st}(a, \mathcal{B}) = \mathrm{st}(b, \mathcal{B}) = \mathrm{st}(c, \mathcal{B}) = \mathrm{st}(d, \mathcal{B}) = \{a, b, c, d\}$.

$\mathrm{st}(a, \mathcal{A} \cap \mathcal{B}) = \mathrm{st}(b, \mathcal{A} \cap \mathcal{B}) = \mathrm{st}(c, \mathcal{A} \cap \mathcal{B}) = \mathrm{st}(d, \mathcal{A} \cap \mathcal{B}) = \{a, b, c, d\}$.

Thus $\mathfrak{B}$ satisfies the axioms (LC3) and consequently, $\mathfrak{B}$ is a base for some covering $L-$local uniformity on L^X. But for $\mathcal{A}$, there is no $\mathcal{B}$ such that $\mathrm{st}(\mathcal{B}) \preceq \mathcal{A}$. This implies $\mathcal{B}$ is not a base for covering $L-$uniformity on L^X.

Thus we may conclude that covering $L-$locally uniform spaces is a generalisation of covering $L-$uniform spaces in the sense of García et at.[14].

We now state the following Lemma in order to show that every covering $L-$local uniformity generates an $L-$topology.

Lemma 3.4 Let $(L^X, \mathfrak{U})$ be covering $L-$locally uniform space. Then the mapping, $\mathrm{int}\colon L^X \to L^X$ defined by $\mathrm{int}(\mathrm{A}) := \bigcup\{x_\alpha \in L^X : st(x_\alpha, \mathcal{A}) \subseteq A$ for some $\mathcal{A} \in \mathfrak{U}\}$, $\forall\, \mathrm{A} \in L^X$ is an interior operator on L^X .

Proof. Clearly, (IO1) $int(\underline{1}) = \underline{1}$ and (IO2) $\forall\, A \in L^X,\ int(A) \subseteq A$ satisfied trivially.

(IO3) Since L is completely distributive complete lattice, therefore by Proposition 2.1.5(v) in [25], L^X is also so. Hence, $int(A \cap B) = int(A) \cap int(B)$ follows immediately form (LC2).

(IO4) For any $A \in L^X$, let $x_\alpha \subseteq int(A)$, then there exits $\mathcal{A} \in \mathfrak{U}$ such that $\mathrm{st}(x_\alpha, \mathcal{A}) \subseteq \mathrm{A}$. Since $\mathfrak{U}$is a covering L-local uniformity, therefor for $x_\alpha \in L^X$ and $\mathcal{A} \in \mathfrak{U}$, there exits $\mathcal{B} \in \mathfrak{U}$ such that $st\big(x_\alpha, st(\mathcal{B})\big) \subseteq st(x_\alpha, \mathcal{A})$. Now by Proposition 2.9(6), we have $st(st(x_\alpha, \mathcal{B}), \mathcal{B}) \subseteq st(x_\alpha, st(\mathcal{B}))$. This implies $st(st(x_\alpha, \mathcal{B}), \mathcal{B}) \subseteq st(x_\alpha, \mathcal{A}) \subseteq \mathcal{A}$ .This further implies $st(x_\alpha, \mathcal{B}) \subseteq int(A)$. Thus $x_\alpha \subseteq int\big(int(A)\big)$ and consequently by (IO2),

$int(A) = int\big(int(A)\big)$.

Now by Theorem 2.2.21 in [25], we may conclude that

Theorem 3.5 Every covering $L-$locally uniformity $\mathfrak{U}$ on L^X , generates an $L-$topology on L^X.

In that case, we shall use the symbol $\mathbb{F}(\mathfrak{U})$ to denote the respective generated $L-$topology on L^X.Subsequently, $A^o = int(A),\ \forall A \in L^X$, where A^o is the interior of A in $\big(L^X, \mathbb{F}(\mathfrak{U})\big)$.

Theorem 3.6 Let $\mathfrak{U}_1$ and $\mathfrak{U}_2$ be two covering $L-$ local uniformities on L^X such that $\mathfrak{U}_2$ is finer than $\mathfrak{U}_1$. Then $\mathbb{F}(\mathfrak{U}_2)$ will finer than $\mathbb{F}(\mathfrak{U}_1)$.

Proof. Straightforward.

We now state the following important Lemmas:

Lemma 3.7 Let $(L^X, \mathfrak{U})$ be covering $L-$locally uniform space. Then the mapping, $cl: L^X \to L^X$ defined by $cl(A) = \cap\{\mathrm{st}(\mathrm{A}, \mathcal{A}) | \mathcal{A} \in \mathfrak{U}\}, \forall \mathrm{A} \in L^X$ is a closure operator on L^X.

Proof. Clearly, (CO1) $int(\underline{0}) = \underline{0}$ and (CO2) $\forall\, A \in L^X$, $A \subseteq cl(A)$ satisfied trivially.

(CO3) For any $\mathrm{A}, \mathrm{B} \in L^X$, we have $cl(A \cap B) = (\cap_{\mathcal{A} \in \mathfrak{U}} st(A \cup B, \mathcal{A})) = (\cap_{\mathcal{A} \in \mathfrak{U}} st(A, \mathcal{A}) \cup st(B, \mathcal{A}))$ [By Proposition 2.9 (5)]. Also since L is completely distributive complete lattice, therefore by Proposition 2.1.5 (v) in [25], L^X is also so. This implies

$cl(A \cap B) = (\cap_{\mathcal{A} \in \mathfrak{U}} st(A, \mathcal{A})) \cup (\cap_{\mathcal{A} \in \mathfrak{U}} st(B, \mathcal{A})) = cl(A) \cup cl(B)$.

(CO4) For any $A \in L^X$, we have $cl(cl(A)) = \bigcap_{\mathcal{A}\in\mathcal{U}} st(cl(A),\mathcal{A})$

$= \bigcap_{\mathcal{A}\in\mathfrak{U}} st(\bigcap_{\mathcal{A}\in\mathfrak{U}} st(A,\mathcal{A}),\mathcal{A}) = \bigcap_{\mathcal{A}\in\mathfrak{U}} st(st(A,\mathcal{A}),\mathcal{A})$.

$\subseteq \bigcap_{\mathcal{A}\in\mathfrak{U}} st\big(A, st(\mathcal{A})\big)$ (By Proposition 2.9 (6)).

$= \bigcap_{\mathcal{A}\in\mathfrak{U}} st(\bigcup_{x_\alpha\in A} x_\alpha, st(\mathcal{A}))$ (Since for any $A \in L^X, A = \bigcup_{x_\alpha\in A} x_\alpha$)

$= \bigcap_{\mathcal{A}\in\mathfrak{U}} \bigcup_{x_\alpha\in A} st\big(x_\alpha, st(\mathcal{A})\big)$ (By Proposition 2.9(5)).

$= \bigcup_{x_\alpha\in A} \bigcap_{\mathcal{A}\in\mathfrak{U}} st(x_\alpha, st(\mathcal{A})) \subseteq \bigcup_{x_\alpha\in A} \bigcap_{\mathcal{A}\in\mathfrak{U}} st(x_\alpha,\mathcal{A})$ (By LC3).

$= \bigcap_{\mathcal{A}\in\mathfrak{U}} \bigcup_{x_\alpha\in A} st(x_\alpha,\mathcal{A})$.

$= \bigcap_{\mathcal{A}\in\mathfrak{U}} st(\bigcup_{x_\alpha\in A} x_\alpha,\mathcal{A})$.

$= \bigcap_{\mathcal{A}\in\mathfrak{U}} st(A,\mathcal{A}) = cl(A)$.

Hence, by (CO2), we have $cl\big(cl(A)\big) = cl(A), \forall\, A \in L^X$.

Lemma 3.8 For every $L-$covers $\mathcal{A}$ and for each $A \in L^X$, we have $st(A,\mathcal{A}) = \bigcap\{B | st(B',\mathcal{A}) \subseteq A\}$.

Proof. It follows from the fact that for any $B \in L^X$, $B \subseteq st(A,\mathcal{A})$ if and only if $A \subseteq st(B,\mathcal{A})$ as $A\bigcap B \neq \underline{0}$ if and only if $B\bigcap A \neq \underline{0}$.

Lemma 3.9 Let $(L^X,\mathfrak{U})$ is covering $L-$locally uniform space, then $(cl(A))' = int(A')$

Proof. For any $A \in L^X$, we have

$int(A') = \bigcup\{x_\alpha \in L^X | st(x_\alpha,\mathcal{A}) \subseteq A'$ for some $\mathcal{A} \in \mathfrak{U}\}$.

$= \bigcup\{\bigcup\{x_\alpha \in L^X | st(x_\alpha,\mathcal{A}) \subseteq A'\}, A \in U\}$.

$= \bigcup\{st(A,\mathcal{A})' | \mathcal{A} \in \mathfrak{U}\}$ [By Lemma3.8].

Hence, $int(A')' = \bigcap\{st(A,\mathcal{A}) | \mathcal{A} \in \mathfrak{U}\} = cl(A)$.

Now by Lemma 3.7 and Lemma 3.9, we have the following:

Theorem 3.10 Let $(L^X,\mathfrak{U})$ be covering $L-$locally uniform space. Then for any $A \in L^X$, $\overline{A} = \bigcap\{st(A,\mathcal{A}) | \mathcal{A} \in \mathfrak{U}\}$, where $\overline{A}$ is the closure of A in $(L^X, \mathbb{F}(\mathfrak{U}))$.

Theorem 3.11 Let $(L^X,\mathcal{U})$ be covering $L-$locally uniform space. Then the topology

$(X, \mathbb{F}(\mathcal{U}))$ generated by covering L −locally uniform space is regular.

Proof. Let $(X, \mathfrak{U})$ be a covering L −locally uniform space. Now for any $x_\alpha \in L^X$ and $\mathcal{A} \in \mathfrak{U}$ there exits $\mathcal{B}$ such that $st(x_\alpha, st(\mathcal{B})) \subseteq st(x_\alpha, \mathcal{A})$. Again for $x_\alpha \in L^X$ and $\mathcal{B} \in \mathfrak{U}$ there exists $\mathcal{C} \in \mathfrak{U}$ such that $st(x_\alpha, st(\mathcal{C})) \subseteq st(x_\alpha, \mathcal{B})$. Then by (CO3), we have $cl(st(x_\alpha, st(\mathcal{C}))) \subseteq cl(st(x_\alpha, \mathcal{B}))$. But by definition of cl, we have $cl(st(x_\alpha, \mathcal{B})) \subseteq st(st(x_\alpha, \mathcal{B}), \mathcal{B})$. So, by Proposition 2.9 (6), we have $cl(st(x_\alpha, st(\mathcal{C}))) \subseteq st(st(x_\alpha, st(\mathcal{B}))$. This implies $cl(st(x_\alpha, st(\mathcal{C}))) \subseteq st(x_\alpha, \mathcal{A})$, as $st(x_\alpha, st(\mathcal{B})) \subseteq st(x_\alpha, \mathcal{A})$. Hence for each $x_\alpha \in L^X$ there exists a neighbourhood base at x_α consisting of closed sets and consequently the space is regular.

Theorem 3.12 Any regular L −topology generated by a covering L −locally uniform space.

Proof. Let $(L^X, \mathbb{F})$ be a regular L −topology and $\mathfrak{U}$ be the collection of all open covers in L^X, then it follows easily (LC1) and (LC2). The only thing left is for every $\mathcal{A} \in \mathfrak{A}$ there exits $\mathcal{B} \in \mathfrak{U}$ such that $st(x_\alpha, st(\mathcal{B})) \subseteq st(x_\alpha, \mathcal{A})$. Now by regularity of L^X, we have for $x_\alpha \in L^X$ and $\mathcal{A} \in \mathfrak{U}$ there exits an L −fuzzy open set G such that $x_\alpha \subseteq G \subseteq \overline{G} \subseteq st(x_\alpha, \mathcal{A})$. Again since G is open and $x_\alpha \subseteq G$, therefore there exists an open cover $\mathcal{B}$ such that, $st(x_\alpha, \mathcal{B}) \subseteq G$. But then $st(x_\alpha, st(\mathcal{B})) \subseteq st(G, \mathcal{B})$ (as $x_\alpha \subseteq G$). Thus $st(x_\alpha, st(\mathcal{B})) \subseteq st(G, \mathcal{B}) = G$ (as G is open) $\subseteq \overline{G} \subseteq st(x_\alpha, \mathcal{A})$. Also, by the construction of $\mathfrak{U}$, it follows from Lemma3.4 that the L −topologies $\mathbb{F}(\mathfrak{U})$ and $\mathbb{F}$ are identical. Hence the result.

By Theorem 2.5, we have the following Corollary.

Corollary 3.13 Let $\{(L^{X_t}, \mathbb{F}_t) | t \in \Lambda\}$ be a family of L-topological spaces. Then the product topology of L-topologies $\{\mathbb{F}_t | t \in \Lambda\}$ on L^X is generated by a covering L-local uniformity if and only if for each $t \in \Lambda, (L^{X_t}, \mathbb{F}_t)$ is generated by a covering L-local uniformity.

4. FUZZY UNIFORM CONTINUOUS FUNCTIONS

In this section we have established that every weakly uniform continuous function on covering L −locally uniform spaces are continuous with respect to the induced L −topologies. Towards the end of this section we have shown that the products of $L-$ regular topologies is generated by the product covering L −locally uniform spaces.

Definition 4.1 Let $(L^X, \mathfrak{U}_1)$ and $(L^Y, \mathfrak{U}_2)$ be two covering L −locally uniform spaces. Then a function $f^{\rightarrow}: (L^X, \mathfrak{U}_1) \rightarrow (L^Y, \mathfrak{U}_2)$, is called weakly uniformly continuous if and only if $f^{-1}(\mathcal{C}) \in \mathfrak{U}_1$, whenever $\mathcal{C} \in \mathfrak{U}_2$, where $f^{-1}(\mathcal{C}) = \{f^{\leftarrow}(C): C \in \mathcal{C}\}$.

Theorem 4.2 Every weakly uniform continuous function is continuous.

Proof. Let $f^{\rightarrow}: (L^X, \mathcal{U}_1) \rightarrow (L^Y, \mathfrak{U}_2)$ be a weakly uniformly continuous functions and $A \in L^Y$ be any member. Then by definition of int, we have $int(A) = \bigcup\{x_\alpha: st(x_\alpha, \mathcal{A}) \subseteq A, \text{for some } A \in \mathfrak{U}_2\}$. This implies

$$f^{\leftarrow}\left(int(A)\right) = \bigcup\{f^{\leftarrow}(x_\alpha): st(x_\alpha, A) \subseteq A \text{ for some } \mathcal{A} \in \mathfrak{U}_2\} \tag{1}$$

[Since by Theorem2.1.17(i)in[25], $f^{\leftarrow}$is arbitrary join preserving].

Since $f^{\leftarrow}$ is order preserving, therefore

$$st(x_\alpha, \mathcal{A}) \subseteq A \text{ implies } f^{\leftarrow}(st(x_\alpha, \mathcal{A})) \subseteq f^{\leftarrow}(A) \tag{2}$$

Then by Proposition2.9(6) and Line(2) we have

$$st(f^{\leftarrow}(x_\alpha, f^{-1}(\mathcal{A})) \subseteq f^{\leftarrow}(st(x_\alpha, \mathcal{A})) \subseteq f^{\leftarrow}(A).$$

Now from line (1), we have

$$f^{\leftarrow}\left(int(A)\right) \subseteq \bigcup\{f^{\leftarrow}(x_\alpha) |\, st(f^{\leftarrow}\left(x_\alpha, f^{-1}(\mathcal{A})\right) \subseteq f^{-1}(\mathcal{A}) \text{ for some } \mathcal{A} \in \mathfrak{U}_2\} \tag{3}$$

But since $f^{\rightarrow}$ is weakly uniformly continuous, therefore $\mathcal{A} \in \mathfrak{U}_2$ implies $f^{-1}(\mathcal{A}) \in \mathfrak{U}_1$.

So by Line (3), we have $f^{\leftarrow}(int(A)) \subseteq int(f^{\leftarrow}(int(A)))$. This implies $f^{\leftarrow}(int(A)) \in \mathbb{F}(\mathfrak{U}_1)$.

Hence $f^{\rightarrow}: (L^X, \mathbb{F}(\mathfrak{U}_1)) \rightarrow (L^Y, \mathbb{F}(\mathfrak{U}_2))$ is continuous.

Theorem 4.3 The composition of weakly uniformly continuous function is weakly uniformly continuous.

Proof. Let $f^{\rightarrow}: (L^X, \mathfrak{U}_1) \rightarrow (L^Y, \mathfrak{U}_2)$ and $g^{\rightarrow}: (L^Y, \mathfrak{U}_2) \rightarrow (L^Z, \mathfrak{U}_3)$ be two weakly uniformly continuous functions. Let $\mathcal{C} \in \mathcal{U}_3$ be any member. Then by Theorem 2.1.23(ii) in [25], we have $(g \circ f)^{\leftarrow}(\mathcal{C}) = f^{\leftarrow}(g^{\leftarrow}(\mathcal{C}))$. Since $g^{\rightarrow}$ is weakly uniformly continuous, therefore $\mathcal{C} \in \mathfrak{U}_3$ implies $g^{\leftarrow}(\mathcal{C}) \in \mathfrak{U}_2$. This further implies $f^{\leftarrow}(g^{\leftarrow}(\mathcal{C})) \in \mathfrak{U}_1$ as $f^{\rightarrow}$ is weakly uniformly continuous. Hence $(g \circ f)^{\leftarrow}(\mathcal{C}) \in \mathfrak{U}_1$ for every $\mathcal{C} \in \mathfrak{U}_3$. Hence the result.

Definition 4.4 Let $\{(L^{X_t}, \mathcal{U}_t) | t \in \Lambda\}$ be a family of covering L-locally uniform spaces, where

Λ is the index set. Let $X = \Pi_{t\in\Lambda} X_t$. The product covering L-local uniformity on L^X is defined as the coarsest covering L-local uniformity such that for every $t \in \Lambda$, projection $\pi_t^{\rightarrow}: L^X \rightarrow L^{X_t}$ is weakly uniformly continuous.

By Theorem 3.6, the following Theorem is now obvious.

Theorem 4.5 The L-topology generated by the product covering L-local uniformity is the product topology and conversely product of regular $L-$topologies is generated by product covering L-local uniformity.

5. PSEUDO METRIZABILITY IN COVERING L-LOCALLY UNIFORM SPACE

Theorem 5.1 If $(L^X, \mathfrak{U})$ is covering L-locally uniform space with countable base, then $(X, \mathbb{F}(\mathfrak{U}))$ has countable base.

Proof. Let $\mathfrak{U}^* = \{\mathcal{A}_n : n \in N\}$ be countable base for the covering $L-$locally uniform space.

For fixed n, let us define,

$$B_n = st(st(x_\alpha, \mathcal{A}_n), \mathcal{A}_m), \text{for some } m \in N, x_\alpha \in L^X.$$

By Lemma (3.4) it is clear that $int(B_n) = B_n$.

Let us denote the collection $\mathcal{B} = \{B_n : B_n = st(st(x_\alpha, \mathcal{A}_n), \mathcal{A}_m), m \in N\}$.

Let $x_\alpha \in B \subseteq L^X$ be any open set. Then $int(B) = B$, and since $\mathfrak{U}^*$ is base for $\mathfrak{U}^*$. By Covering $L-$locally uniform space, for $\mathcal{A}_j$ there exits $\mathcal{A}_\mathrm{k}$ such that $st(x_\alpha, st(\mathcal{A}_k)) \subseteq st(x_\alpha, \mathcal{A}_j)$.

Again by Proposition (2.9)(5) $st(st(x_\alpha, \mathcal{A}_k), \mathcal{A}_k) \subseteq st(x_\alpha, st(\mathcal{A}_k))$.

$$\Rightarrow st(st(x_\alpha, \mathcal{A}_k), \mathcal{A}_k) \subseteq st(x_\alpha, \mathcal{A}_j) \subseteq B.$$

$$\Rightarrow x_\alpha \in st(st(x_\alpha, \mathcal{A}_k), \mathcal{A}_k) \subseteq B.$$

$$\Rightarrow x_\alpha \in B_k \subseteq B.$$

On the other hand, each B_n is assign to some member of $\mathfrak{U}^*$, $\mathfrak{U}^*$ is countable implies $\mathcal{B}$ is countable.

Now by Theorem 6.4 in [19], we have the following result.

Theorem 5.2 Every covering $L-$locally uniform with countable base is pointwise pseudo-metrizable.

6. CONCLUSION

In this paper, we obtained covering *L*-locally uniform spaces by generalising covering *L*-uniform space in the sense García, at el. [14]. Interior and closure operators of covering *L*-local uniformity is topological in *L*- topology. We obatained one-one correspondance between regular *L*-topology and covering *L*-locally uniform spaces. Weakly uniformly continuous functions in the class of covering *L*-local uniformities and studied some of its basic properties. Further , we also obtained the products of *L*-regular topologies is generated by the product covering *L*-locally uniform space and every covering *L*-locally uniform spaces with countable base is pseudo-metrisable in the sense of Erceg [19].

Conflict of Interests

The authors declare that there is no conflict of interests.

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