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EDGE PRODUCT CORDIAL LABELING OF GRAPHS

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Abstract. The product cordial labeling is a variant of cordial labeling. We introduce a variant of product cordial labeling and name it as edge product cordial labeling. Unlike in product cordial labeling the roles of vertices and edges are interchanged. We investigate several results on this newly defined concept.

Keywords: Cordial labeling, Edge product cordial labeling, Edge product cordial graphs.

2000 AMS Subject Classification: 05C78

1. Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with order p and size q. For all standard terminology and notation we follow West [1]. We will give brief summary of definitions which are useful for the present investigations.

Definition 1.1 : A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called *a vertex labeling (or an edge labeling).*

For an extensive survey on graph labeling and bibliographic references we refer to Gallian [2].

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In 1987, Cahit [3] introduced cordial labeling as a weaker version of graceful labeling and harmonious labeling. In 2004, Sundaram et al. [4] have introduced product cordial labeling in which the absolute difference in cordial labeling is replaced by product of the vertex labels.

In this paper we introduce the edge analogue of product cordial labeling and investigate edge product cordial labeling for some standard graphs.

Definition 1.2 : For graph G, the edge labeling function is defined as $f : E(G) \rightarrow \{0, 1\}$ and induced vertex labeling function $f^* : V(G) \rightarrow \{0, 1\}$ is given as if $e_1, e_2, \ldots, e_n$ are the edges incident to vertex v then $f^*(v) = f(e_1)f(e_2)\ldots f(e_n)$.

Let us denote $v_f(i)$ is the number of vertices of G having label i under f^* and $e_f(i)$ is the number of edges of G having label i under f for $i = 1, 2$.

f is called *edge product cordial labeling* of graph G if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called *edge product cordial* if it admits edge product cordial labeling.

Definition 1.3 : A *unicyclic* graph is a connected graph with exactly one cycle.

Definition 1.4 : The *corona* $G_1 \odot G_2$ of two graph G_1 and G_2 is defined as a graph obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.

Definition 1.5 : The *crown* $(C_n \odot K_1)$ is obtained by joining a pendant edge to each vertex of C_n.

Definition 1.6 : The *armed crown* is a graph in which path P_2 is attached at each vertex of cycle C_n by an edge. It is denoted by AC_n where n is the number of vertices in cycle C_n.

Definition 1.7 : A *chord of cycle* C_n is an edge joining two non-adjacent vertices of cycle C_n.

Definition 1.8 : The *shell* S_n is the graph obtained by taking $n - 3$ concurrent chords in cycle C_n. The vertex at which all the chords are concurrent is called the apex vertex. The shell is also called *fan* f_{n-1}. Thus $S_n = f_{n-1} = P_{n-1} + K_1$

Definition 1.9 : The *wheel* W_n is defined to be the join $K_1 + C_n$. The vertex corresponding to K_1 is known as apex vertex, the vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges and edges joining apex and vertices of cycle are spoke edges. We continue to use this terminology related to wheel also for the graphs corresponding to definitions 1.10 to 1.15.

Definition 1.10 : The *helm* H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex.

Definition 1.11 : The *closed helm* CH_n is the graph obtained from a helm H_n by joining each pendant vertex to form a cycle.

Definition 1.12 : The *web* Wb_n is the graph obtained by joining the pendant vertices of a helm H_n to form a cycle and then adding a pendant edge to each vertex of outer cycle.

Definition 1.13 : The *flower* Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex of the helm.

Definition 1.14 : Let $e = uv$ be an edge of graph G and w is not a vertex of G. The edge e is *subdivided* when it is replaced by the edges $e' = uw$ and $e'' = wv$.

Definition 1.15 : The *gear graph* G_n is obtained from the wheel W_n by subdividing each of its rim edge.

2. Main results

Theorem 2.1 *The cycle C_n is edge product cordial graph for odd n and not edge product cordial graph for even n.*

Proof. Let $e_1, e_2, \ldots, e_n$ be edges of cycle C_n. To define $f : E(C_n) \rightarrow \{0, 1\}$ we consider following two cases.

Case 1: When n is odd.

$$f(e_i) = 0; \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$$
$$f(e_i) = 1; \quad \text{otherwise.}$$

In view of the above defined labeling patten we have

$$v_f(0) = v_f(1) + 1 = \frac{n+1}{2}$$
$$e_f(0) + 1 = e_f(1) = \frac{n+1}{2}$$

Thus in case 1 we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Case 2: When n is even.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to $\frac{n}{2}$ edges out of n edges. The edges with label 0 will give rise at least $\frac{n}{2} + 1$ vertices with label 0 and at most $\frac{n}{2} - 1$ vertices with label 1 out of total n vertices. Therefore $|v_f(0) - v_f(1)| \geq 2$. Thus the vertex condition for edge product cordial graph is violated. Hence C_n is not edge product cordial graph for even n.

Hence the cycle C_n is edge product cordial graph for odd n and not edge product cordial graph for even n.

Illustration 2.2 : C_5 and its edge product cordial labeling is shown in Fig. 1.

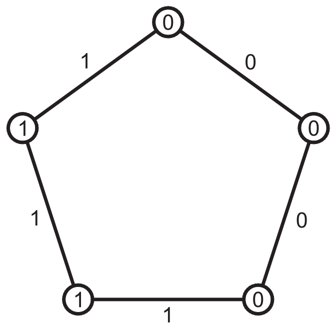

Fig. 1

Theorem 2.3 *The wheel W_n is not edge product cordial graph.*

Proof. Let $e_1, e_2, \ldots, e_{2n}$ be edges of wheel W_n. We consider following two cases.

Case 1: When n is odd.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to n edges out of $2n$ edges. The edges with label 0 will give rise at least

$\left\lceil \frac{n}{2} \right\rceil + 1$ vertices with label 0 and at most $\left\lfloor \frac{n}{2} \right\rfloor$ vertices with label 1 out of total $n+1$ vertices. Therefore $|v_f(0) - v_f(1)| \geq 2$. Thus the vertex condition for edge product cordial graph is violated. So W_n is not edge product cordial graph for odd n.

Case 2: When n is even.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to n edges out of $2n$ edges. The edges with label 0 will give rise at least $\frac{n}{2} + 2$ vertices with label 0 and at most $\frac{n}{2} - 1$ vertices with label 1 out of total $n+1$ vertices. Therefore $|v_f(0) - v_f(1)| \geq 3$. Thus the vertex condition for edge product cordial graph is violated. So W_n is not edge product cordial graph for even n.

Hence the wheel W_n is not edge product cordial graph.

Theorem 2.4 *All trees with order greater than 2 are edge product cordial.*

Proof. Tree of order 2 has only 1 edge. If we label that edge with 1 or 0 then both the incident vertices will get the same label and other label is not generated. Thus tree of order 2 is not edge product cordial graph.

For tree T of size $n (n \geq 2)$ let $e_1, e_2, \ldots, e_m$ are pendant edges. If $m \geq \left\lceil \frac{n}{2} \right\rceil$ then define the function $f : E(G) \to \{0, 1\}$ as $f(e_i) = 1$ for $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$.

Otherwise assign $f(e_i) = 1$ for $i = 1, 2, \ldots, m$. Consider $T_1 = T - \{e_1, e_2, e_3, \ldots, e_m\}$ and let $h_1, h_2, \ldots, h_k$ are pendant edges of T_1. If $m + k \geq \left\lceil \frac{n}{2} \right\rceil$ assign 1 to any l pendant edges of T_1 such that $m + k = \left\lceil \frac{n}{2} \right\rceil$. Also assign 0 to all the remaining edges. This gives edge product cordial labeling for T. If $m + k \leq \left\lceil \frac{n}{2} \right\rceil$ repeat the process until edge product cordial labeling is obtained.

Hence all trees with order greater than 2 are edge product cordial.

Illustration 2.5 : A tree and its edge product cordial labeling is shown in Fig. 2.

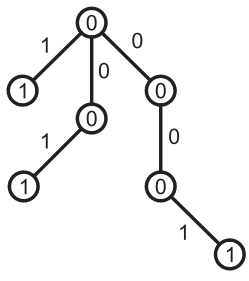

Fig. 2

Remark 2.6 For any unicyclic graph, $p = q$.

Corollary 2.7 *Any unicyclic graph of odd order is edge product cordial graph.*

Proof. Let the unicyclic graph is G and C_n be the cycle corresponding to it. If e is any cycle edge then $G - e$ will be a tree with edge product cordial labeling f.

Since $G - e$ is of even size, $e_f(0) = e_f(1)$ in $G - e$.

Let $f(e) = 1$ then function f is edge product cordial labeling for G.

Hence any unicyclic graph of odd order is edge product cordial graph.

Remark 2.8 Unicyclic graph of even order may or may not be edge product cordial graph as C_4 is not edge product cordial graph.

Theorem 2.9 *The crown $C_n \odot K_1$ is edge product cordial graph.*

Proof. Let $e_1, e_2, \ldots, e_n$ be the edges of cycle C_n and $e'_1, e'_2, \ldots, e'_n$ are pendant edges of crown $C_n \odot K_1$.

We define $f : E(C_n \odot K_1) \to \{0, 1\}$ as follows.

$$f(e_i) = 0; \quad \text{for all } i$$
$$f(e'_i) = 1; \quad \text{for all } i.$$

In view of the above defined labeling patten we have

$$v_f(0) = v_f(1) = n$$
$$e_f(0) = e_f(1) = n$$

Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence the crown $C_n \odot K_1$ is edge product cordial graph.

Illustration 2.10 : The crown $C_5 \odot K_1$ and its edge product cordial labeling is shown in Fig. 3.

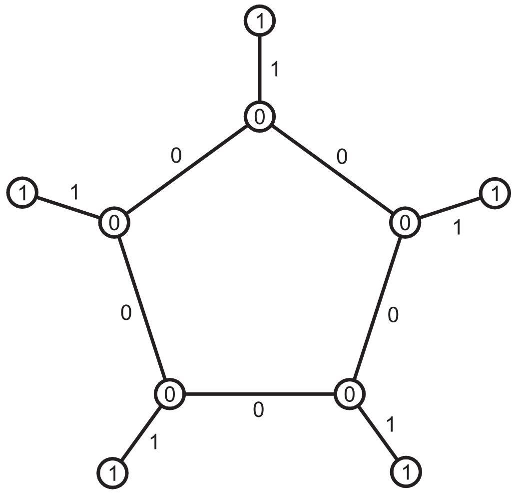

Fig. 3

Theorem 2.11 *The armed crown AC_n is edge product cordial graph.*

Proof. Let $e_1, e_2, \ldots, e_n$ be the edges of cycle C_n and u_i, w_i be the vertices and e_i^a be the edge of path P_i. To construct armed crown AC_n join vertex v_i of cycle C_n to vertex u_i with edge e_i'.

To define $f : E(AC_n) \to \{0, 1\}$ we consider following two cases.

<u>**Case 1:** When n is odd.</u>

In this case the graph is edge product cordial according to corollary 2.7 as for odd n the graph AC_n is a unicyclic graph of odd order.

<u>**Case 2:** When n is even.</u>

$$\begin{aligned} f(e_i) &= 0; \quad \text{for all } i \\ f(e_i^a) &= 1; \quad \text{for all } i \\ f(e_i') &= 0; \quad 1 \leq i \leq \frac{n}{2} \\ f(e_i') &= 1; \quad \text{otherwise.} \end{aligned}$$

In view of the above define labeling pattern we have

$$\begin{aligned} v_f(0) = v_f(1) &= \frac{3n}{2} \\ e_f(0) = e_f(1) &= \frac{3n}{2} \end{aligned}$$

Thus in each case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence the armed crown AC_n is edge product cordial graph.

Illustration 2.12 : The armed crown AC_5 and its edge product cordial labeling is shown in Fig. 4.

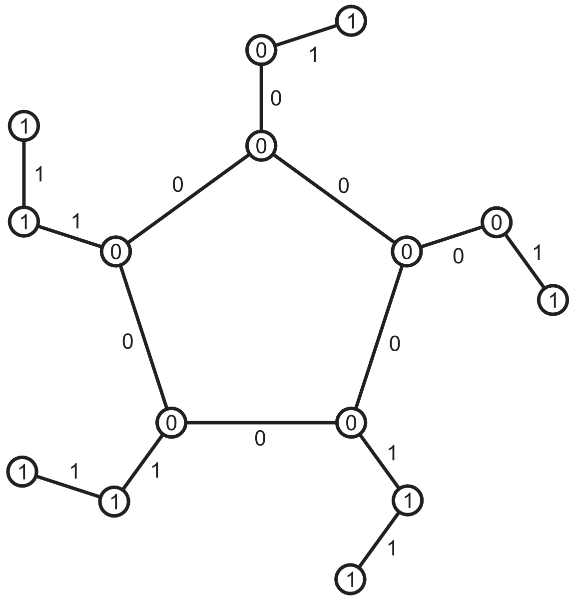

Fig. 4

Theorem 2.13 *The helm H_n is edge product cordial graph.*

Proof. Let $e_1, e_2, \ldots, e_n$ be the rim edges, $e'_1, e'_2, \ldots, e'_n$ are spoke edges and $e^a_1, e^a_2, \ldots, e^a_n$ are pendant edges of helm H_n.

To define $f : E(H_n) \to \{0,1\}$ we consider following two cases.

<u>**Case 1:** When n is odd.</u>

$$\begin{aligned} f(e_i) &= 0; \quad \text{for all } i \\ f(e^a_i) &= 1; \quad \text{for all } i \\ f(e'_i) &= 0; \quad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor \\ f(e'_i) &= 1; \quad \text{otherwise.} \end{aligned}$$

In view of the above define labeling pattern we have

$$v_f(0) - 1 = v_f(1) = n$$
$$e_f(0) = e_f(1) - 1 = \frac{3n-1}{2}$$

Case 2: When n is even.

$$\begin{aligned} f(e_i) &= 0; \quad \text{for all } i \\ f(e_i^a) &= 1; \quad \text{for all } i \\ f(e_i') &= 0; \quad 1 \leq i \leq \frac{n}{2} \\ f(e_i') &= 1; \quad \text{otherwise.} \end{aligned}$$

In view of the above define labeling pattern we have

$$v_f(0) - 1 = v_f(1) = n$$
$$e_f(0) = e_f(1) = \frac{3n}{2}$$

Thus in each case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence the helm H_n is edge product cordial graph.

Illustration 2.14 : H_5 and its edge product cordial labeling is shown in Fig. 5.

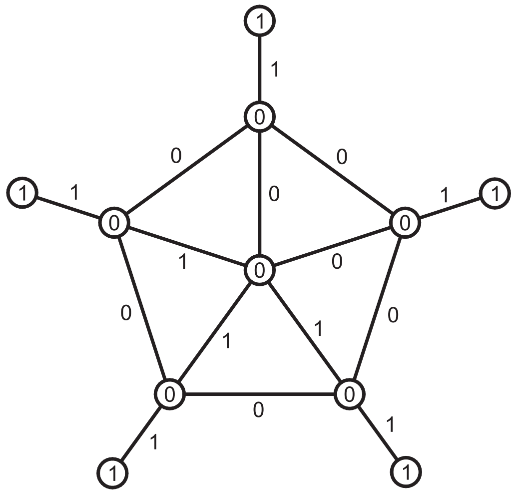

Fig. 5

Theorem 2.15 *The closed helm CH_n is edge product cordial graph.*

Proof. Let $e_1, e_2, \ldots, e_n$ be the edges of inner cycle, $e_1', e_2', \ldots, e_n'$ are spoke edges, $e_1^a, e_2^a, \ldots, e_n^a$ are the edges joining inner cycle and outer cycle and $e_1^b, e_2^b, \ldots, e_n^b$ are the edges of outer cycle of closed helm CH_n.

We define $f : E(CH_n) \to \{0, 1\}$ as follows.

$$f(e_i) = 0; \quad \text{for all } i$$
$$f(e'_i) = 0; \quad \text{for all } i$$
$$f(e^a_i) = 1; \quad \text{for all } i$$
$$f(e^b_i) = 1; \quad \text{for all } i.$$

In view of the above define labeling pattern we have

$$v_f(0) - 1 = v_f(1) = n$$
$$e_f(0) = e_f(1) = 2n$$

Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence the closed helm CH_n is edge product cordial graph.

Illustration 2.16 : The closed helm CH_5 and its edge product cordial labeling is shown in Fig. 6.

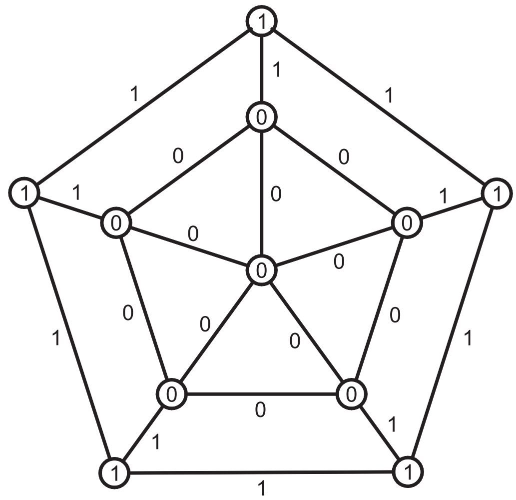

Fig. 6

Theorem 2.17 *The web Wb_n is edge product cordial graph.*

Proof. Let $e_1, e_2, \ldots, e_n$ be the edges of inner cycle, $e'_1, e'_2, \ldots, e'_n$ are spoke edges, $e^a_1, e^a_2, \ldots, e^a_n$ are the edges joining inner cycle and outer cycle, $e^b_1, e^b_2, \ldots, e^b_n$ are the edges of outer cycle and $e^c_1, e^c_2, \ldots, e^c_n$ are pendant edges of web Wb_n.

To define $f : E(Wb_n) \rightarrow \{0, 1\}$ we consider following two cases.

Case 1: When n is odd.

$$
\begin{aligned}
f(e_i) &= 0; \quad \text{for all } i\\
f(e_i') &= 0; \quad \text{for all } i\\
f(e_i^a) &= 1; \quad 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil\\
f(e_i^a) &= 0; \quad \text{otherwise}\\
f(e_i^b) &= 1; \quad \text{for all } i\\
f(e_i^c) &= 1; \quad \text{for all } i.
\end{aligned}
$$

In view of the above define labeling pattern we have

$$
\begin{aligned}
v_f(0) = v_f(1) &= \frac{3n+1}{2}\\
e_f(0) + 1 = e_f(1) &= \frac{5n+1}{2}
\end{aligned}
$$

Case 2: When n is even.

$$
\begin{aligned}
f(e_i) &= 0; \quad \text{for all } i\\
f(e_i') &= 0; \quad \text{for all } i\\
f(e_i^a) &= 1; \quad 1 \leq i \leq \frac{n}{2}\\
f(e_i^a) &= 0; \quad \text{otherwise}\\
f(e_i^b) &= 1; \quad \text{for all } i\\
f(e_i^c) &= 1; \quad \text{for all } i.
\end{aligned}
$$

In view of the above define labeling pattern we have

$$
\begin{aligned}
v_f(0) = v_f(1) + 1 &= \frac{3n+2}{2}\\
e_f(0) = e_f(1) &= \frac{5n}{2}
\end{aligned}
$$

Thus in each case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence the web Wb_n is edge product cordial graph.

Illustration 2.18 : Wb_5 and its edge product cordial labeling is shown in Fig. 7.

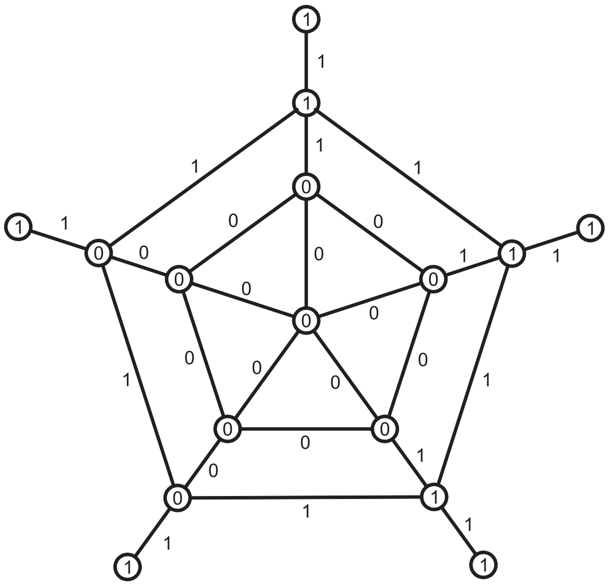

Fig. 7

Theorem 2.19 *The flower Fl_n is edge product cordial graph.*

Proof. Let $e_1, e_2, \ldots, e_n$ be the rim edges, $e'_1, e'_2, \ldots, e'_n$ are spoke edges and $e^a_1, e^a_2, \ldots, e^a_n$ are pendant edges of helm H_n. To obtain flower graph Fl_n join pendant vertices of H_n to apex of H_n by edges $e^b_1, e^b_2, \ldots, e^b_n$.

We define $f : E(Fl_n) \to \{0, 1\}$ as follows.

$$f(e_i) = 0; \quad \text{for all } i$$
$$f(e'_i) = 0; \quad \text{for all } i$$
$$f(e^a_i) = 1; \quad \text{for all } i$$
$$f(e^b_i) = 1; \quad \text{for all } i.$$

In view of the above define labeling pattern we have

$$v_f(0) - 1 = v_f(1) = n$$
$$e_f(0) = e_f(1) = 2n$$

Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Hence the flower Fl_n is edge product cordial graph.

Illustration 2.20 : The flower Fl_5 and its edge product cordial labeling is shown in Fig. 8.

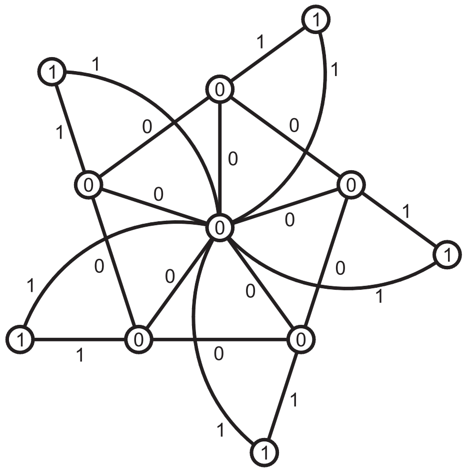

Fig. 8

Theorem 2.21 *The gear graph G_n is edge product cordial graph for odd n and not edge product cordial graph for even n.*

Proof. Let $e_1, e_2, \ldots, e_{2n}$ be the rim edges and $e'_1, e'_2, \ldots, e'_n$ are spoke edges of gear graph G_n. To define $f : E(G_n) \to \{0, 1\}$ we consider following two cases.

Case 1: When n is odd.

$$f(e_i) = 0; \quad 1 \leq i \leq n - 1$$
$$f(e_i) = 1; \quad \text{otherwise}$$
$$f(e'_i) = 0; \quad 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$$
$$f(e'_i) = 1; \quad \text{otherwise.}$$

In view of the above defined labeling patten we have

$$v_f(0) - 1 = v_f(1) = n$$
$$e_f(0) + 1 = e_f(1) = \frac{3n + 1}{2}$$

Thus in case 1 we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Case 2: When n is even.

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to $\frac{3n}{2}$ edges out of $3n$ edges. The edges with label 0 will give rise at least $n + 2$ vertices with label 0 and at most $n - 1$ vertices with label 1 out of total $2n + 1$ vertices. Therefore $|v_f(0) - v_f(1)| \geq 3$. Thus the vertex condition for edge product cordial graph is violated. Hence G_n is not edge product cordial graph for even n.

Hence the gear graph G_n is edge product cordial graph for odd n and not edge product cordial graph for even n.

Illustration 2.22 : The gear graph G_5 and its edge product cordial labeling is shown in Fig. 9.

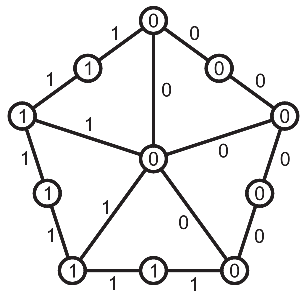

Fig. 9

Theorem 2.23 *The shell S_n is edge product cordial graph for odd n and not edge product cordial graph for even n.*

Proof. Let $e_1, e_2, \ldots, e_{n-1}$ be the edges incident to apex of shell S_n and $e'_1, e'_2, \ldots, e'_{n-2}$ are the other edges of shell S_n.

To define $f : E(S_n) \to \{0, 1\}$ we consider following two cases.

<u>**Case 1:** When n is odd.</u>

$$f(e_i) = 0; \quad 1 \leq i \leq \frac{n-1}{2}$$
$$f(e_i) = 1; \quad \text{otherwise}$$
$$f(e'_i) = 0; \quad 1 \leq i \leq \frac{n-1}{2} - 1$$
$$f(e'_i) = 1; \quad \text{otherwise.}$$

In view of the above defined labeling patten we have

$$v_f(0) - 1 = v_f(1) = \frac{n-1}{2}$$
$$e_f(0) + 1 = e_f(1) = n - 1$$

Thus in case 1 we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

<u>**Case 2:** When n is even.</u>

In order to satisfy the edge condition for edge product cordial graph it is essential to assign label 0 to $\frac{2n-4}{2}$ edges out of $2n-3$ edges. The edges with label 0 will give rise at least $\frac{n}{2} + 1$ vertices with label 0 and at most $\frac{n}{2} - 1$ vertices with label 1 out of total

n vertices. Therefore $|v_f(0) - v_f(1)| \geq 2$. Thus the vertex condition for edge product cordial graph is violated. Hence S_n is not edge product cordial graph for even n.

Hence the shell S_n is edge product cordial graph for odd n and not edge product cordial graph for even n.

Illustration 2.24 : The shell S_7 and its edge product cordial labeling is shown in Fig. 10.

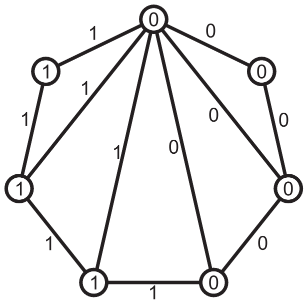

Fig. 10

3. Concluding Remarks

Labeling of discrete structure is a potential area of research. We have introduced the concept of edge product cordial labeling and derived several results on it. To investigate analogous results for different graphs as well as in the context of various graph labeling problems is an open area of research.

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