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CYCLIC GROUP ACTIONS ON ELLIPTIC SURFACES $E(2n)$

HONGXIA LI*

College of Arts and Sciences, Shanghai Maritime University, Shanghai 201306, China

Abstract. In this paper, we study the action of cyclic group $\mathbf{Z}_3$ on elliptic surfaces $E(2n), (n \geq 1)$. For convenience, we suppose the fixed point set of $\mathbf{Z}_3$ is composed of 2-spheres. We give a classification of this action. At the same time, we obtain the representation induced by $\mathbf{Z}_3$ on the second cohomology of $E(2n)$.

Keywords: cyclic group actions; fixed points; the second cohomology.

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1. Introduction

As to the classification of group actions on 4-manifolds, there are many results for pseudofree actions such as [9] and [10] and so on. A recent research is [3]. In that paper, they gave a complete description of the fixed-point set structure of a symplectic cyclic action of prime order on a minimal symplectic 4-manifold with $c_1^2 = 0$. In the case of topological manifolds, [4] showed that every closed, simply connected topological 4-manifold admits an action of cyclic group $\mathbf{Z}_p$ of any odd prime order. The action will be homologically trivial and pseudofree except in certain cases when $p = 3$.

*Corresponding author

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In this paper, we assume the fixed point set of $\mathbf{Z}_p$ is composed of 2-spheres S^2. By [4], this case occurs only when $p = 3$. Thus we only study the action of $\mathbf{Z}_3$ on elliptic surfaces. Meanwhile, we get the representation induced by $\mathbf{Z}_3$ on the second cohomology $H^2(E(2n))$.

We organize this paper as follows. In section 2, we give some preliminaries about the elliptic surfaces $E(2n)$ and group actions of $\mathbf{Z}_p$. In section 3, we prove the main results.

2. Preliminaries

In this paper, we always suppose $E(2n)(n \geq 1)$ be a minimal elliptic surface. $E(2n)$ is a simply connected 4-manifold which is defined as the $2n$-fold fiber sum of copies of $E(1)$, where $E(1) = \mathbf{C}P^2 \sharp \overline{\mathbf{C}P^2}$ being equipped with an elliptic fibration.

Note that $\text{sign}(E(2n)) = -16n$ and $\chi(E(2n)) = 24n$. Thus $E(2) = E(1)\sharp_{\mathbb{T}^2} E(1)$ is the $K3$-surface. To see this just note that the Euler characteristic are additive under taking fiber connected sums over a torus. Hence $\text{sign}(E(2)) = -16$ and $\chi(E(2)) = 24$ which characterizes $K3$ surface [6].

Let $\mathbf{Z}_p$ be a cyclic group action on 4-manifold X with odd prime order p. $g : X \to X$ is a generator of $\mathbf{Z}_p$. Let $F = \text{Fix}(g)$ denote the fixed point set of $\mathbf{Z}_p$. By Local Smith theory, F consists of isolated points and surfaces. Besides Edmonds (Proposition 2.4 in [5]) proved that all fixed surfaces are 2-spheres if and only if the representation g_* on $H^2(X)$ is a permutation representation. In general, we have the well known Lefschetz fixed point formula

$$\chi(F) = \Lambda(g) = 2 + \text{Trace}[g_* : H^2(X) \to H^2(X)].$$

For g and $\hat{g}$ defined as above, we have

$$\text{Spin}(\hat{g}, X) = \text{ind}_{\hat{g}} D = \text{tr}(\hat{g}|_{kerD}) - \text{tr}(\hat{g}|_{cokerD}),$$

where $\text{Spin}(\hat{g}, X)$ denote the spin number of X under the action of g, D denotes the Dirac operator. Especially, when $X = E(2n)$ is an elliptic surface, we have the following Lefschetz formula about the spin number.

Lemma 2.1 *Let $X = E(2n)(n \geq 1)$ be a minimal elliptic surface. $g : X \to X$ is a cyclic group action on X with order p(where p is odd prime). $\hat{g}$ is the lift of g which preserves trivial Spin^c structure. Assume the fixed point set X^g is composed of isolated points P_j and connected 2-manifolds F_k, then we have the following formula for Spin number*

$$\begin{aligned} \mathrm{Spin}(\hat{g}, X) = & -\frac{1}{4}\sum_{P_j} \epsilon(P_j, \hat{g}) csc(\alpha_j/2) csc(\beta_j/2) \\ & + \frac{1}{4}\sum_{F_k} \epsilon(F_k, \hat{g}) cos(\theta_k/2) csc^2(\theta_k/2)[F_k]\cdot[F_k], \end{aligned} \tag{1}$$

where α_j(resp β_j) denotes $2\pi l_{\alpha_j}/p$ (resp $2\pi l_{\beta_j}/p$) $(0 < \alpha_j,\ \beta_j < \pi)$, $\theta_k = 2\pi l_{\theta_j}/p$ $(0 < \theta_k < \pi)$, $\epsilon(P_j, \hat{g})$ and $\epsilon(F_k, \hat{g})$ are ± 1, the sign depends on the action of g on the Spin bundle.

As to the proof of this lemma we can refer to [1, 2, 7, 8].

For closed, simply connected, topological 4-manifold X, [5] give a direct sum decomposition of $H^2(X) = T \oplus C \oplus R$, where T is t summands of trivial type, C is c summands of cyclotomic type and R is r summands of regular type. And the following relation exists.

$$\begin{aligned} t + c(p-1) + rp = \beta_2(X), \\ \beta_1(F) = c, \\ \beta_0(F) + \beta_2(F) = t + 2, \end{aligned}$$

where β_i is the i-th Betti number.

3. Main results

At first, we study the action of $G = \mathbf{Z}_3$ on $X = K3$ surface.

Suppose the fixed point set is composed of disjoint union of m copies of 2-sphere S^2. Then by [5], the representation of G on $H^2(K_3)$ is permutation. For convenience, we denote by F = Fix(g) the disjoint union of m copies of 2-sphere S^2, and denote by $N = [S^2] \cdot [S^2]$ the selfintersection number of S^2. Then from fomula (1), we have

$$\mathrm{Spin}(\hat{g}, X) = \mathrm{Spin}(\hat{g^2}, X) = -\frac{1}{6}N.$$

Theorem 3.1 *Suppose there exists a smooth cyclic group action G of order 3 on $K3$ surface and the fixed point set of G is mS^2. Then the only possible case is $m = 12$ and the representation of G on $H^2(K_3) = 22\mathbf{Z}$.*

Proof. On the one hand, the ordinary Lefschetz formula should hold: $L(g, X) = \chi(F) = 2 + \mathrm{tr}(g|_{H^2(X)}) = 2m$. Since $\mathrm{tr}(g|_{H^2(X)}) \le b_2 = 22$, we have $m \le 12$.

Note that

$$\chi(X/G) = \frac{1}{3}(24 + 2\chi(F)) = 8 + \frac{4}{3}m,$$
$$\mathrm{sign}(X/G) = \frac{1}{3}(-16 + 2\mathrm{sign}(F)) = \frac{1}{3}(-16 + 2\mathrm{sign}(F)).$$

From formula (1), we have

$$-\frac{1}{8}\mathrm{sign}(F) = -\frac{mN}{6}.$$

Thus

$$\mathrm{sign}(X/G) = \frac{8}{9}(-6 + mN).$$

Since $\mathrm{sign}(X/G) \in \mathbf{Z}$, we have $mN \equiv 6 \mod 9$. Suppose $mN = 9k + 6,\ k \in \mathbf{Z}$, then

$$b_+^G = \frac{1}{2}(\chi(X/G) + \mathrm{sign}(X/G) - 2) = 3 + 4k + \frac{2m}{3}.$$

Hence $m \equiv 0 \mod 3$, that is $m = 3, 6, 9, 12$. Since $b_+^G \le b_+ = 3$, we have the following results,

Case 1. When $b_+^G = 1$, we have $m = 3, k = -1, N = -1$.

Case 2. When $b_+^G = 3$, we have $m = 12, k = -2, N = -1$.

On the other hand, by the G-index theorem,

$$\mathrm{ind}_{\hat{g}} D_X = k_0 + \zeta k_1 + \zeta^2 k_2 = -\frac{mN}{6},$$
$$\mathrm{ind}_{\hat{g}^2} D_X = k_0 + \zeta^2 k_1 + \zeta k_2 = -\frac{mN}{6},$$
$$\mathrm{ind}_1 D_X = k_0 + k_1 + k_2 = 2.$$

The solution of these equations can be the following two cases,

Case 1. $k_0 = 1, k_1 = k_2 = \frac{1}{2}$.

Case 2. $k_0 = 2, k_1 = k_2 = 0$.

Since $k_i \in \mathbf{Z}$, case 1 can not exist.

From [5], $H^2(K_3) = T \oplus C \oplus R$. Meanwhile, we have the following relations.

$$t + c(p-1) + rp = \beta_2(K3),$$

$$\beta_1(F) = c,$$

$$\beta_0(F) + \beta_2(F) = t + 2.$$

Since $\beta_2(K3) = 22$, $\beta_0(F) = \beta_2(F) = m$, $\beta_1(F) = 0$, we have $r = c = 0$ and $t = 22$ for case 2. Thus the representation of G on $H^2(K_3) = 22\mathbf{Z}$ and theorem 3.1 is proved.

Next, we study the $G = \mathbf{Z}_3$ action on elliptic surface $X = E(4)$. We obtain the following result.

Theorem 3.2 *Suppose there exists a smooth cyclic group action G of order 3 on the elliptic surface $E(4)$ and the fixed point set of G is the disjoint union of mS^2. Then the action belongs to one of the three types in Table 1, where $R(G)$ in table 1 denotes the representation induced by G on $H^2(E(4))$.*

TABLE 1. The classification of actions

Type	m	b_+^G	b_-^G	b_2^G	Sign(X/G)	N	$R(G)$
A_1	3	1	17	18	-16	-2	$4\mathbf{Z} \oplus 14\mathbf{Z}(G)$
A_2	6	3	19	22	-16	-1	$10\mathbf{Z} \oplus 12\mathbf{Z}(G)$
A_3	24	7	39	46	-32	-1	$46\mathbf{Z}$

Proof. On the one hand, we have $m \leq 24$ by the Lefschetz formula.

Note that $\text{sign}(F) = \frac{4mN}{3}$, thus

$$\text{sign}(X/G) = \frac{1}{3}(-32 + 2\text{sign}(F)) = \frac{8}{9}(-12 + mN).$$

Since $\mathrm{sign}(X/G) \in \mathbf{Z}$, we have $mN \equiv 12 \mod 9$. Thus $mN = 9k+12,\ k \in \mathbf{Z}$. Meanwhile,

$$\chi(X/G) = \frac{1}{3}(24 + 2\chi(F)) = 8 + \frac{4}{3}m.$$

Then we have

$$b_+^G = \frac{1}{2}(\chi(X/G) + \mathrm{sign}(X/G) - 2) = 7 + 4k + \frac{2m}{3}.$$

Hence $m \equiv 0 \mod 3$. That is $m = 3, 6, 9, 12, 15, 18, 21, 24$. Since b_+^G is 1 or 3 or 5 or 7, we obtain the following results,

Case 1. When $b_+^G = 1$, we have $m = 3, k = -2, N = -2$.

Case 2. When $b_+^G = 3$, we have $m = 6, k = -2, N = -1$.

Case 3. When $b_+^G = 5$, we have $m = 3, k = -1, N = 1$.

Case 4. When $b_+^G = 5$, we have $m = 15, k = -3, N = -1$.

Case 5. When $b_+^G = 7$, we have $m = 24, k = -4, N = -1$.

On the other hand, by the G-index theorem, we have

$$\mathrm{ind}_{\hat{g}} D_X = k_0 + \zeta k_1 + \zeta^2 k_2 = -\frac{mN}{6},$$
$$\mathrm{ind}_{\hat{g}^2} D_X = k_0 + \zeta^2 k_1 + \zeta k_2 = -\frac{mN}{6},$$
$$\mathrm{ind}_1 D_X = k_0 + k_1 + k_2 = 4.$$

For case 3 and 4, there are some k_j which do not belong to $\mathbf{Z}$. Thus case 3 and 4 can not exist.

Besides, from the direct sum decomposition $H^2(E(4)) = T \oplus C \oplus R$, we have

$$t + 2c + 3r = \beta_2(E(4)),$$
$$\beta_1(F) = c,$$
$$\beta_0(F) + \beta_2(F) = t + 2,$$

which together with the facts

$$\beta_2(E(4)) = 46, \beta_0(F) = \beta_2(F) = m,\ \beta_1(F) = 0$$

gives the following results.

For case 1, $r = 14,\ t = 4$. Thus the representation of G on $H^2(E(4))$ is $4\mathbf{Z} \oplus 14\mathbf{Z}(G)$.

For case 2, $r = 12,\ t = 10$. Thus the representation of G on $H^2(E(4))$ is $10\mathbf{Z} \oplus 12\mathbf{Z}(G)$.

For case 5, $r = 0,\ t = 46$. Thus the representation of G on $H^2(E(4))$ is $46\mathbf{Z}$. This completes the proof of theorem 3.2.

As to the $G = \mathbf{Z}_3$ action on $E(2n)(n > 2)$, readers can study as above.

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