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A NEW MONOTONICALLY STABLE DISCRETE MODEL FOR THE SOLUTION OF DIFFERENTIAL EQUATIONS EMANATING FROM THE EVAPORATING RAINDROP

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Abstract. Most of the models of the dynamics of evaporating raindrop have been created using Ordinary Differential Equations. Some models assumed that evaporation is affected by air resistance that is negligible and proportional to the velocity of transition others assumed that the air resistance is significant and proportional to the square of the velocity. Because of the significant differences in the basic assumption used by various modelers, the solution of the resulting equations produced differed values There are yet no work to confirm these dependencies. In this work we obtain a class of hybrid finite difference schemes that can be used to obtain a reliable approximate solution to some of these differential equation models. The schemes were found to possess the same monotonic properties as the analytic solution.

Keywords: monotonically stable discrete model; differential equations emanating from the evaporating raindrop; hybrid finite difference schemes.

2010 AMS Subject Classification: 34C60, 34K60.

1.0 INTRODUCTION

An equation of state is a representation of a dynamic system as a function of state variables such as pressure, temperature, volume and number of particles which are used to measure the status of such system under a given set of physical conditions that may have effect on the measured values.

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Evaporation is a type of vaporization of a liquid that occurs from the surface of a liquid into a gaseous phase that is not saturated with the evaporating substance. Evaporation happens when atoms or molecules escape from the liquid and turn into a vapor. Factors that may affect the way and the rate at which evaporation occurs include: Saturation of the air, humidity, air pressure and temperature of the liquid. The shape of a raindrop depends upon its size and the amount of air resistance present during it drops. Some are round, others appear flattened at the bottom, and then there are those that resemble jellybeans. For the purpose of this study, all the raindrops will be considered as spheres. (see [3, 7]).

Most existing mathematical models of this dynamical system were developed based on the principle of Newton's law of motion. Each of these models considered two major variations. Either that the raindrops are small particle bodies that is affected by air resistance significantly or that the raindrops are big particles that floats through the air vacuums with insignificant air resistance (see [6, 8]). Generally when a body in motion is treated as a particle rotational effects are unimportant (or negligible), and also when the size of the object is negligible relative to the scale of the system, only translational motion is relevant.

2.0 MATERIALS AND METHODS

The following ordinary differential equations have been considered while deriving a hybrid finite difference scheme that can simulation the dynamics of the motion of an evaporating raindrop.

2.1 **Model I** [6]

These authors came-up with the Ordinary Differential Equation given below

$$\frac{dv}{dt} + \frac{3\left(\frac{k}{\rho}\right)}{t\left(\frac{k}{\rho}\right)+r_0} v = g \quad (1)$$

ρ is the density of rainwater

r_0 is the initial radius of the raindrop

$r = \frac{k}{\rho}t + r_0$ is the radius of the raindrop at time t, k is a constant

g is acceleration due to gravity

2.2 **Model II** [6]

If we consider the small object theory we can arrive at the following model for the evaporating raindrop:

$$F = m\frac{dv}{dt} = mg - kv \qquad (5)$$

Which gives a solution of the form

$$v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) \qquad (6)$$

We will use a combination of interpolation and Nonstandard methods

3.0 DERIVATION OF THE SIMULATION MODEL

To obtain a simulation model which can accommodate the assumption on air resistance and the decay thru evaporation. We assume raindrop water has a density of 0.99823 grams/cubic centimeter and a temperature of $4^{o}C$ degrees Celsius. The water is evaporating with a speed proportional to the surface area. We are suggesting a new simulation model with the following components.

$$y = a_0 + a_1 x - \frac{a_0 \beta g}{(\alpha x + \beta)^3} + e^{-\alpha x} \qquad (7)$$

a_0 is a known and measurable constant, α and β are simulation parameters. This interpolation function will be used to derive a finite difference scheme for solution of first order Ordinary Differential Equation. To get the hybrid scheme, we will apply nonstandard method to the step size by replacing it with a normalized function (Obayomi&Oke2003). For the scheme:

Differentiating (7)

$$y' = a_1 + 3a_0 \alpha \beta g (\alpha x + \beta)^{-4} + (-\alpha) e^{-\alpha x} \qquad (8)$$

$$y' = a_0 (1 + 3(\alpha x + \beta)^{-4} \alpha + (-\alpha) e^{-\alpha x} \qquad (9)$$

$$y'' = -12 a_0 \alpha^2 \beta g (\alpha x + \beta)^{-5} + \alpha^2 e^{-\alpha x} \qquad (10)$$

$$y''' = 60 a_0 \alpha^3 (\alpha x + \beta)^{-6} - \alpha^3 e^{-\alpha x} \qquad (11)$$

$$a_0 = \frac{y'' - \alpha^2 e^{-\alpha x}}{(-12\, \alpha^2 \beta g (\alpha x + \beta)^{-5})} \qquad (12)$$

or

$$a_0 = \frac{y''' + \alpha^3 e^{-\alpha x}}{(60 \alpha^3 (\alpha x + \beta)^{-6})} \qquad (13)$$

$$a_1 = y' + \alpha e^{-\alpha x} - \frac{(\alpha x + \beta) \alpha^2 e^{-\alpha x}}{4\alpha} + \frac{(\alpha x + \beta) y''}{4\alpha} \qquad (14)$$

3.1 Generation of the Finite Difference Model

$$y = a_0 + a_1 x - \frac{a_0 \beta g}{(\alpha x + \beta)^3} + a_0 e^{-\alpha x}$$

$$y_n = a_0 + a_1 x_n - \frac{a_0 \beta g}{(\alpha x_n + \beta)^3} + a_0 e^{-\alpha x_n} \tag{15}$$

$$y_{n+1} = a_0 + a_1 x_{n+1} - \frac{a_0 \beta g}{(\alpha x_{n+1} + \beta)^3} + a_0 e^{-\alpha x_{n+1}} \tag{16}$$

$$y_{n+1} - y_n =$$

$$= a_1 \left[(x_{n+1} - x_n) + a_0 \beta g \{ (\alpha x_{n+1} + \beta)^{-3} - (\alpha x_{n+1} + \beta)^{-3} \} + a_0 (e^{-\alpha x_{n+1}} - e^{-\alpha x_n}) \right] \tag{17}$$

$$\left.\begin{aligned} &\text{Let } x_n = a + nh \qquad x_{n+1} = a + (n+1)h \\ &\qquad x_{n+1} - x_n = h \end{aligned}\right\} \tag{18}$$

Putting (17) in (18), we have

$$y_{n+1} - y_n = a_1 h + a_0 \beta g \{ (\alpha[a + (n+1)h] + \beta)^{-3} - (\alpha[a + nh] + \beta)^{-3} \} +$$
$$(e^{-\alpha(a+nh)} \{ e^{-\alpha h} - 1 \}) \tag{19}$$

$$M = \{ (\alpha[a + (n+1)h] + \beta)^{-3} - (\alpha[a + nh] + \beta)^{-3} \}$$

$$N = (e^{-\alpha(a+nh)} \{ e^{-\alpha h} - 1 \})$$

$$y_{n+1} = y_n + a_1 h + a_0 \beta g M + N \tag{20}$$

$$a_0 = \frac{y'' - \alpha^2 e^{-\alpha x}}{(-12\, \alpha^2 \beta g (\alpha x + \beta)^{-5})}$$

$$a_1 = y' + \alpha e^{-\alpha x} - \frac{(\alpha x + \beta) \alpha^2 e^{-\alpha x}}{4\alpha} + \frac{(\alpha x + \beta) y''}{4\alpha}$$

$$y_{n+1} = y_n + hy' + h\alpha e^{-\alpha x} \left(1 - \frac{\alpha(\alpha x + \beta)}{4\alpha}\right) + \frac{h(\alpha x + \beta) y''}{4\alpha} + \frac{(\alpha^2 e^{-\alpha x} - y'')\beta g}{(12\, \alpha^2 \beta g (\alpha x + \beta)^{-5})} M + N \tag{21}$$

$$y_{n+1} =$$

$$y_n + hy' + \frac{3h\alpha \beta g (\alpha x + \beta)^{-4} - M \beta g)}{(12\, \alpha^2 \beta g (\alpha x + \beta)^{-5})} y'' + h\alpha e^{-\alpha x} \left(1 - \frac{\alpha(\alpha x + \beta)}{4\alpha}\right) + \frac{(\alpha^2 e^{-\alpha x}) M \beta g}{(12\, \alpha^2 \beta g (\alpha x + \beta)^{-5})} + N$$

$$y_{n+1} = y_n + hy' + T y'' + U + V + N \tag{22}$$

$$T = \frac{3h\alpha \beta g (\alpha x + \beta)^{-4} - M \beta g)}{(12\, \alpha^2 \beta g (\alpha x + \beta)^{-5})},\ U = h\alpha e^{-\alpha x} \left(1 - \frac{\alpha(\alpha x + \beta)}{4\alpha}\right),\ V = \frac{(\alpha^2 e^{-\alpha x}) M \beta g}{(12\, \alpha^2 \beta g (\alpha x + \beta)^{-5})}.$$

4.0 QUALITATIVE PROPERTIES OF THE NEW SCHEME

Definition [1]

Any algorithm for solving a differential equation in which the approximation y_{n+1} to the solution at x_{n+1} can be calculated iff x_n, y_n *and* h are known is called a one step method. It is a common practice to write the functional dependence y_{n+1} on the quantities $x_n, y_n and\ h$ in the form $y_{n+1}= y_n+\phi(x_n, y_n, h)$, where $\phi(x_n, y_n, h)\ is\ the$ incremental function.

Definition [2]

A numerical scheme with an incremental $\phi(x_n, y_n, h)$ is said to be consistent with the initial value problem $y' = f(x, y)\ , y(x_0) = y_0$ if the incremental function is identically zero at t_0 when $h = 0$.

Theorem [1]

Let the incremental function of the scheme defined in the one step scheme above be continuous and jointly as a function of its arguments in the region defined by

$x \in [a, b] and\ y \in (-\infty\ , \infty), 0 \leq h \leq h_0$ where $h_0 > 0$ and let there exists a constant L such that $\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) \leq L|y_n - y_n^*|$ for all (x_n, y_n, h) and (x_n, y_n^*, h)

in the region just defined then the relation $(x_n, y_n, 0) = (x_n, y_n^*)$ is a necessary condition for the convergence of the new scheme.

Theorem [2]

Let $y_n = y(x_n)$ and $p_n = p(x_n)$ denote two different numerical solution of the differential equation with the initial condition specified a

$y_0= y(x_0) = \xi$ and $p_0= p(x_0) = \xi^*$ respectively such that $|\xi-\xi^*|<\varepsilon$ $\varepsilon>0$.

If the two numerical estimates are generated by the integration scheme, we have

$y_{n+1}= y_n+h\phi(x_n, y_n, h)$

$p_{n+1}= p_n+h\phi(x_n, p_n, h)$

The condition that $|y_{n+1} - p_{n+1}| \leq$ K $|\xi-\xi^*|$ is the necessary and sufficient condition for the stability and convergence of the schemes.

4.1 Proof of Convergence

$$\mathrm{y}_{n+1} = \mathrm{y}_n + \mathrm{h}y' + T\, \mathrm{y}'' + U + V + \mathrm{N}$$

$$y_{n+1} = y_n + \mathrm{h}f_n + Tf_n' + U + V + \mathrm{N} \tag{23}$$

The incremental function can be written as

$$\phi(x_n, y_n, h) = \mathrm{h}f_n + Tf_n' + U + V + \mathrm{N}$$

$$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) = \mathrm{h}[f(x_n, y_n, h) - f(x_n, y_n^*, h)] + T[f'(x_n, y_n, h) - f'(x_n, y_n^*, h)]$$

$$= \mathrm{h}[f(x_n, y_n) - f(x_n, y_n^*)] + T[f'(x_n, y_n) - f'(x_n, y_n^*)] \tag{24}$$

$$= \mathrm{h}[\frac{\partial f(x_n,\bar{y})}{\partial y_n}(y_n - y_n^*)] + T[\frac{\partial f'(x_n,\bar{y})}{\partial y_n}(y_n - y_n^*)] \tag{25}$$

L1 = $\text{SUP}_{(x_n,y_n)\in D}\ \frac{\partial f(x_n,\bar{y})}{\partial y_n}$ and

L2 = $\text{SUP}_{(x_n,y_n)\in D}\ \frac{\partial f'(x_n,\bar{y})}{\partial y_n}$

then $\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h)$ = h$[L1(y_n - y_n^*)]$ + $T[L2(y_n - y_n^*)]$ (26)

Let L = |h.L1+T.L2|

$\phi(x_n, y_n, h) - \phi(x_n, y_n^*, h) \leq L|y_n - y_n^*|$, which is the condition for convergence.

4.2 Consistency of the Schemes

$y_{n+1} = y_n + \text{M}f_n + Nf_n' + \text{Q}f_n''$ (27)

Then

$y_{n+1} = y_n + h\ \phi(x_n, y_n, h)$

When $h = 0$, M$= 0, N = 0, Q = 0$

$\Rightarrow y_{n+1} = y_n$ and the incremental function is identically zero when $h = 0$

$\Rightarrow \phi(x_n, y_n, 0) \equiv 0$, which proves consistency.

4.3 Stability of the Schemes

Consider the equation

$y_{n+1} = y_n + \{h\}f_n(x_n, y_n) + \{T\}f_n'(x_n, y_n)$ (28)

Let $p_{n+1} = p_n + \{h\}f_n(x_n, P_n) + \{T\}f_n'(x_n, P_n)$

$y_{n+1} - p_{n+1} = y_n - p_n + h[f_n(x_n, y_n) - f_n(x_n, P_n)] + \{T\}[f_n'(x_n, y_n) - f_n'(x_n, P_n)]$

$= y_n - p_n +$ h$[\frac{\partial f(x_n,p_n)}{\partial p_n}(y_n - p_n)] + T[\frac{\partial f'(x_n,\ p_n)}{\partial p_n}(y_n - p_n)]$

L1 = $\text{SUP}_{(x_n,y_n)\in D}\ \frac{\partial f(x_n,p_n)}{\partial p_n}$ and

L2 = $\text{SUP}_{(x_n,y_n)\in D}\ \frac{\partial f'(x_n,p_n)}{\partial p_n}$

$y_{n+1} - p_{n+1} = y_n - p_n + \text{h}.L1(y_n - p_n) + T.L2(y_n - p_n)$

$|y_{n+1} - p_{n+1}| = |y_n - p_n| + [h.L1 + T.L2]\ |(y_n - p_n)|$

Let L = $|1 + [h.L1 + T.L2]|$

$|y_{n+1} - p_{n+1}| \leq \text{L}\ |y_n - p_n|$ (29)

Let $y_0 = y(x_0) = \xi$ and $p_0 = p(x_0) = \xi^*$ then

$|y_{n+1} - p_{n+1}| \leq \text{K}\ |\xi - \xi^*|$ (30)

$\Rightarrow y_{n+1} = y_n$ and the incremental function is identically zero when $h = 0$

$\Rightarrow \phi(x_n, y_n, 0) \equiv 0$

5.0 APPLICATION AND NUMERICAL EXPERIMENT

5.1 Application of the finite difference schemes to Model I.

We now derive a scheme for the first model

$$\frac{dv}{dt}+\frac{3(\frac{k}{\rho})}{t(\frac{k}{\rho})+r_0}v=g \tag{31}$$

$$v(t)=\frac{g}{4}t+\frac{\rho g r_0}{4k}-\frac{\rho g r_0}{4k}\left\{\frac{\rho g r_0}{kt+\rho r_0}\right\}^3$$

$v_0=0\,,k=-0.01,\rho=0.99823\ \beta\,,\ \propto$ are simulation parameters

$$y'=\frac{dv}{dt}=g-\frac{3\propto y}{\propto x+\beta} \tag{32}$$

$$y''+3\propto y'(\alpha x+\beta)^{-1}-3\propto^2 y(\alpha x+\beta)^{-2}=0$$

$$y''=\frac{3\propto^2 y}{(\alpha x+\beta)^2}-\frac{3\propto y'}{(\alpha x+\beta)} \tag{33}$$

$$y'''+3\propto y''(\alpha x+\beta)^{-1}-6\propto^2 y'(\alpha x+\beta)^{-2}+6\propto^3 y(\alpha x+\beta)^{-3}=0$$

$$y'''=\frac{d^3v}{dt^3}=\frac{6\propto^2 y'}{(\alpha x+\beta)^2}-\frac{6\propto^3 y}{(\alpha x+\beta)^3}-\frac{3\propto y''}{(\alpha x+\beta)} \tag{34}$$

The new scheme Newh will be obtained by substituting the derivatives above and applying it to

$$y_{n+1}=y_n+hy'+\frac{h\,3\alpha\beta g(\alpha x+\beta)^{-4}-\mathrm{M}\beta g)}{(12\,\alpha^2\beta g(\alpha x+\beta)^{-5})}y''+h\alpha e^{-\alpha x}\left(1-\frac{\alpha(\alpha x+\beta)}{4\alpha}\right)+\frac{(\alpha^2 e^{-\alpha x})M\beta g}{(12\,\alpha^2\beta g(\alpha x+\beta)^{-5})}+\mathrm{N}$$

$$y_{n+1}=y_n+hy'+T\,y''+U+V+\mathrm{N} \tag{35}$$

$$T=\frac{3h\alpha\beta g(\alpha x+\beta)^{-4}-\mathrm{M}\beta g)}{(12\,\alpha^2\beta g(\alpha x+\beta)^{-5})},\ U=h\alpha e^{-\alpha x}\left(1-\frac{\alpha(\alpha x+\beta)}{4\alpha}\right),\ V=\frac{(\alpha^2 e^{-\alpha x})M\beta g}{(12\,\alpha^2\beta g(\alpha x+\beta)^{-5})}$$

$$\mathrm{M}=\{(\alpha[a+(n+1)h]+\beta)^{-3}-(\alpha[a+nh]+\beta)^{-3}\}$$

$$\mathrm{N}=(e^{-\alpha(a+nh)}\{e^{-\alpha h}-1\})$$

The hybrid scheme NEWSINr is obtained by replacing h with $\psi=\sin(rh)$ and the hybrid scheme NEWEXP by replacing h with $\psi=\frac{(e^{\lambda h}-1)}{\lambda}$, the choice of this denominator is informed by the works of Anguluv, Lubuma [5] and Obayomi, Oke [9].

5.2 Application of the finite difference scheme to Model II

$$F=mg-kv \tag{36}$$

$$m\frac{dv}{dt}=mg-kv$$

$$v(t)=\frac{mg}{k}\left(1-e^{-\frac{k}{m}t}\right) \tag{37}$$

$v' = \frac{dv}{dt} = g - \frac{k}{m}v$

$y' = g - \frac{k}{m}y$

$$y'' = -\frac{k}{m}\left\{g - \frac{k}{m}y\right\} \quad (38)$$

$y''' = \left(\frac{k}{m}\right)^2 y'$

$$y''' = \left(\frac{k}{m}\right)^2 \left\{g - \frac{k}{m}y\right\} \quad (39)$$

The new scheme Newh will be obtained by substituting the derivatives above and applying it to

$$y_{n+1} = y_n + hy' + T\,y'' + U + V + N \quad (40)$$

$T = \frac{3h\alpha\beta g(\alpha x+\beta)^{-4} - M\beta g)}{(12\,\alpha^2\beta g(\alpha x+\beta)^{-5})}$, $U = h\alpha e^{-\alpha x}\,(1 - \frac{\alpha(\alpha x+\beta)}{4\alpha})$, $V = \frac{(\alpha^2 e^{-\alpha x})M\beta g}{(12\,\alpha^2\beta g(\alpha x+\beta)^{-5})}$

$M = \{(\alpha[a + (n+1)h] + \beta)^{-3} - (\alpha[a + nh] + \beta)^{-3}\}$

$N = (e^{-\alpha(a+nh)}\{e^{-\alpha h} - 1\})$

The hybrid scheme NEWSINr is obtained by replacing h with $\psi = \sin(rh)$ and the hybrid scheme NEWEXP by replacing h with $\psi = \frac{(e^{\lambda h} - 1)}{\lambda}$ $r, \lambda \in \mathbb{R}$

5.3 Experimentation and Result

The following are the 3D graphs obtained from the schemes when applied to the two models. We have used same parameters, step size, denominator functions and simulation parameters ∝ and β to test the two models.

5.3.1 Graph for the schemes of Model I

Solution curves for $\frac{dv}{dt} + \frac{3(\frac{k}{\rho})}{t(\frac{k}{\rho}) + r_0}v = g,\ y(0) = 0$ are presented below

Graph of Schemes with parameters: h=0.01, ∝=0.05, β=600, r=0.995, λ =-0.995

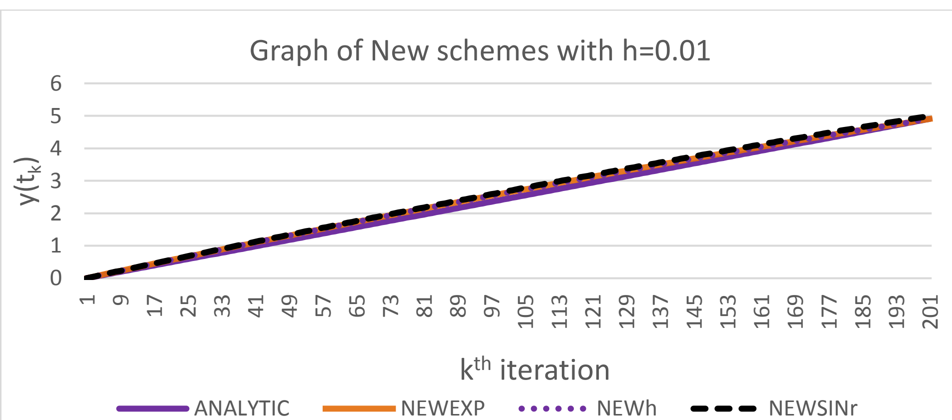

Fig 1: solution curves for the schemes of Model I

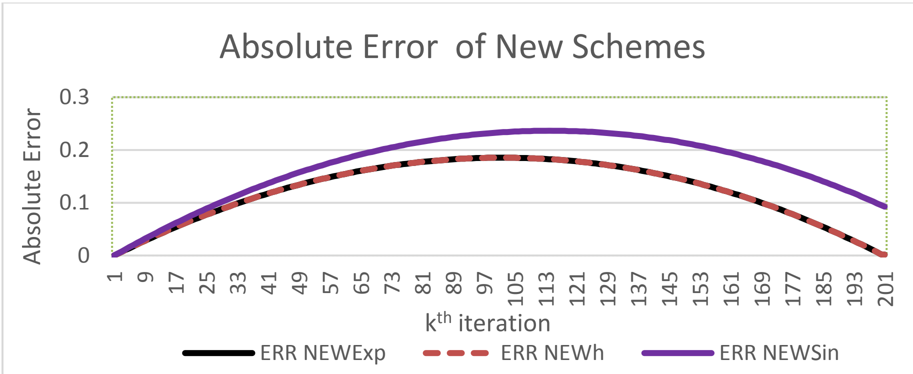

Fig 2: Graph of absolute error for the schemes in Fig1

4.3.2 The Graph for the schemes of Model II

Solution curves for $\frac{dv}{dt} = g - \frac{k}{m}v,\ y(0) = 0$ are presented below

Graph of Schemes with parameters h=0.01, ∝=0.05, β=600, r=0.995, λ =-0.995

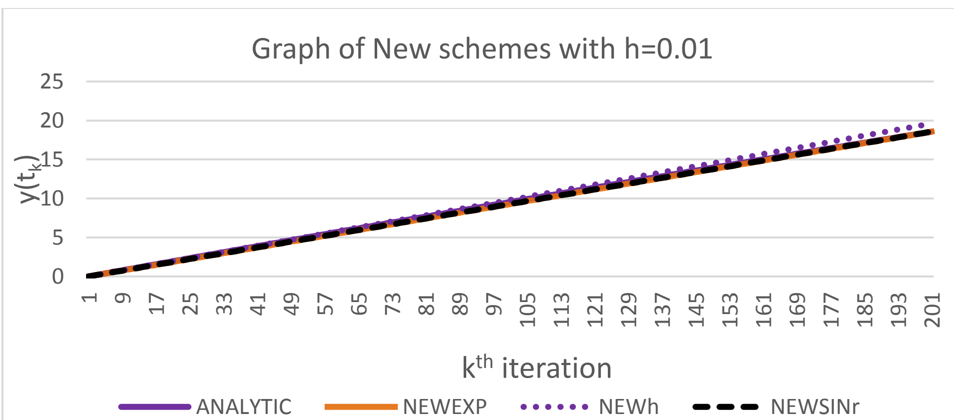

Fig 3: solution curves for the schemes of model II

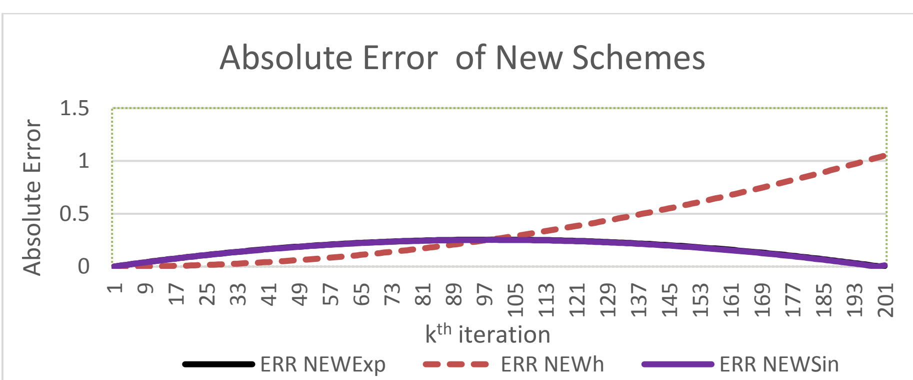

Fig 4: Graph of absolute Error for schemes in Fig. 3

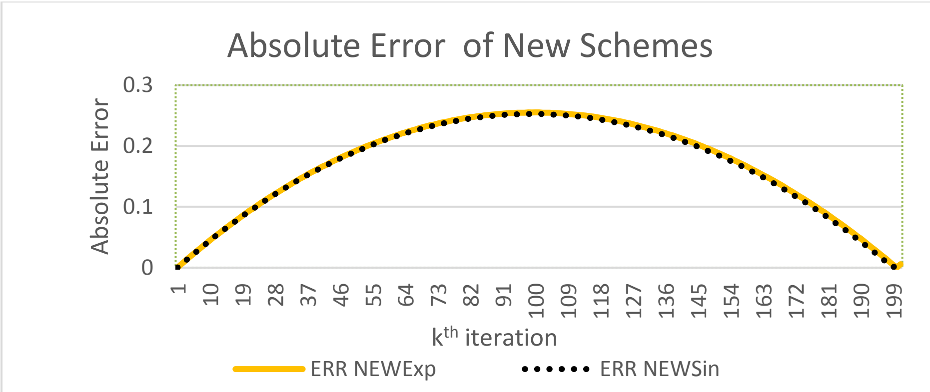

Fig 5: Graph of absolute Error for schemes in Fig. 3

5.0 DISCUSSION AND CONCLUSION

The derived simulation models have been tested with the control parameters ∝ and β. We also applied the Non-Standard method by modifying denominator function which also provide have parameters λ and r that can be chosen to obtain iteratively assigned step size as denominator. The discrete model worked for only those models that assumed negligible air resistance, but it failed for the other models of this dynamical phenomena. We therefore present the application of the schemes to two initial value problems which are two different models of the evaporating raindrop. The solution curves of the schemes follow the analytical solutions of the respective equations monotonically as shown in Figs. 1and 3. The numerical properties of the schemes like linear stability, convergence and consistency has been proved analytically. During the course of simulating the equations we varied this control parameters to obtain family of curves that are very close to the analytic solution and also have the same dynamics as the original equation. The scheme NEWh is the Standard scheme because the denominator function remains the step size throughout the iteration processes, but this scheme possesses the highest absolute error of deviation from the analytic solution, this confirms the good qualities of Non-standard modeling. The choice of appropriate values for variables λ and r can be determined using the conditions set by Angueluv, Lubuma [5] and extended by Obayomi, Oke [9]. The graph of Absolute error (see Figs 2, 4, 5) has demonstrated this quality. The same values of these parameters were used to execute the iterations. The schemes of Model II produced absolute errors of less than 0.03. The result of the schemes is consistent with literature. We can conclude that the discrete model can be used to simulate the dynamics of evaporating raindrop as proposed.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] Henrici P. Discrete Variable Methods in ODE. John Willey & Sons, New York. 1962.

[2] Fatunla S. O. Numerical Methods for Initial Values Problems on Ordinary Differential Equations. Academic Press, New York. 1988.

[3] Humphreys, W. J. Physics of the Air, 279, New York: Dover, 1964.

[4] Mickens R. E. Non-standard Finite Difference Models of Differential Equations. World Scientific, Singapore. 115, 144-162, 1994.

[5] Anguelov, R, and Lubuma J.M.S. Nonstandard finite difference method by nonlocal approximation. Math. Comput. Simul. 6(2003), 465-475.

[6] D.G. Zill and R.M. Cullen Differential Equations with boundary value problems (sixth Edition) Brooks /Cole Thompson Learning Academic Resource Center. 2005, pp. 90 -101.

[7] Riel, Herbert. Introduction to the Atmosphere. 3rd ed. New York: McGraw Hill, 1978: 107

[8] T. P. Dreyer. Modelling with Ordinary Differential Equations. CRC Press, New York, 1993.

[9] Obayomi A.A. and Oke M.O. Development of new Nonstandard denominator function for Finite Difference schemes. J. Nigerian Assoc. Math. Phys. 33(2016), 50-60.