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SOME CHARACTERIZATIONS OF NEUTROSOPHIC G-SUBMODULES

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Abstract. In this paper, we have studied some new features of the algebraic structure G-module in neutrosophic domain and derived the algebraic properties of neutrosophic G-submodules. The quotient of neutrosophic G-submodules and restriction of neutrosophic G-submodules to the neutrosophic G-submodules are defined and discussed.

Keywords: neutrosophic set; G-Module; neutrosophic G-submodule; quotient of neutrosophic G-submodule; restriction of neutrosophic G-submodule.

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1. Introduction

In 1965, Lotfi A. Zadeh [10] introduced the concept of vagueness in mathematical modelling. A number of generalisations of the fundamental concept of set theory have come up. As a generalization of fuzzy set theory, intuitionistic fuzzy set theory was proposed by Attanassov [1] in 1986 in which each element is associated with a degree of membership and non membership values. Again as a generalization of fuzzy set and intuitionistic fuzzy set, neutrosophic set was defined with three different types of membership values by Smarandache in 1995. In the

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real world, the practical problems are related with incomplete, indeterminate and inconsistent informations. Neutrosophic set is a powerful tool and most appropriate frame work for dealing with incomplete, indeterminate and inconsistent information.

The algebraic structure of set theory dealing with uncertainty has been studied by some authors. In 1971, Azriel Rosenfield presented a seminal paper on fuzzy subgroup and W.J. Liu developed the concept of fuzzy normal subgroup and fuzzy sub ring. Combining neutrosophic set theory with abstract algebra is an emerging trend in the area of mathematical research. Neutrosophic algebraic structures and its properties give us a strong mathematical background to explain applied mathematical concepts in engineering, data mining and economics. Here we deal with G modules in the domain of of neutrosophic set. Neutrosophy is a new branch of philosophy and logic which studies the origin and features of Neutralities in nature. Each proposition in neutrosophic logic is approximated to have the percentage of truth (T), the percentage of indeterminacy (I) and the percentage of falsity (F).

2. Preliminaries

Definition 2.1. [3] Let $(G,*)$ be a group. A vector space M over the field K is called a *G-module* if for every $g \in G$ and $m \in M$; $\exists$ a product (called the action of G on M), $g \cdot m \in M$ which satisfies the following axioms;

(1) $1_G \cdot m = m; \quad \forall m \in M \quad (1_G$ being the identity element of $G)$

(2) $(g*h)\cdot m = g\cdot(h\cdot m); \quad \forall m \in M$ and $g,h \in G$

(3) $g\cdot(k_1m_1+k_2m_2) = k_1(g\cdot m_1)+k_2(g\cdot m_2); \forall k_1,k_2 \in K; m_1,m_2 \in M\ ; g \in G.$

Example 2.1. [5] Let $G=\{1,-1,i,-i\}$ and $M=\mathbb{C}^n;\ (n\geq 1)$. Then M is a vector space over $\mathbb{C}$ and under the usual addition and multiplication of complex numbers we can show that M is a G-module.

Definition 2.2. [4] Let M be a G-module. A vector subspace N of M is a *G-submodule* if N is also a G-module under the same action of G.

Definition 2.3. [4] *Let M and M^* be G modules. A mapping $f : M \to M^*$ is called a G module homomorphism if it satisfies the following conditions*

(1) $f(am_1+bm_2)=af(m_1)+bf(m_2)$

(2) $f(gm)=gf(m)\ \forall\ a,b\in F,m_1,m_2\in M,g\in G$

Definition 2.4. [8] *Let M and M^* be G modules and let $f:M\to M^*$ be a G module homomorphism. Then*

$$Ker\ f=\{m\in M:f(m)=0^*\}$$

is a G -submodule of M and

$$Im\ f=\{f(m):m\in M\}$$

is a G -submodule of M^*.

Definition 2.5. [8] *Let M be a G module and N be a G-submodule of M. Then the quotient module $M/N=\{x+N:x\in M\}$ under the action $g(x+N)=gx+N$ is also a G-module.*

Definition 2.6. [9] *A Neutrosophic set A on the universal set X is defined as*

$$A=\{(x,t_A(x),i_A(x),f_A(x)):x\in X\}$$

where $t_A,i_A,f_A:X\to(^-0,1^+)$. The three components t_A,i_A and f_A represent membership value (Percentage of truth), indeterminacy (Percentage of indeterminacy) and non membership value (Percentage of falsity) respectively. These components are functions of non standard unit interval $(^-0,1^+)$.

Remark 2.1. [9] *If $t_A,i_A,f_A:X\to[0,1]$, then A is known as single valued neutrosophic set (SVNS).*

Definition 2.7. [6, 9] *Let A and B be two Neutrosophic sets on X. Then A is contained in B, denoted as $A\subseteq B$ if and only if $A(x)\leqslant B(x)\ \forall\ x\in X$, this means that*

$$t_A(x)\leq t_B(x),i_A(x)\leq i_B(x),f_A(x)\geq f_B(x)$$

Definition 2.8. [9] *The complement of $A=\{(x,t_A(x),i_A(x),f_A(x)):x\in X\}$ is denoted by A^C and defined as $A^C=\{(x,f_A(x),1-i_A(x),t_A(x)):x\in X\}$ and $(A^C)^C=A$*

Definition 2.9. [2, 9] *Let A and B be two Neutrosophic sets of X*

(1) *The union C of A and B is denoted by $C = A \cup B$ and defined as*

$$C(x) = A(x) \vee B(x)$$

where $C(x) = \{(x, t_C(x), i_C(x), f_C(x)) : x \in X\}$ where $t_C(x) = t_A(x) \vee t_B(x)$, $i_C(x) = i_A(x) \vee i_B(x)$ and $f_C(x) = f_A(x) \wedge f_B(x)$

(2) *The intersection C of A and B is denoted by $C = A \cap B$ and is defined as*

$$C(x) = A(x) \wedge B(x)$$

where $C(x) = \{(x, t_C(x), i_C(x), f_C(x)) : x \in X\}$ where $t_C(x) = t_A(x) \wedge t_B(x)$, $i_C(x) = i_A(x) \wedge i_B(x)$ and $f_C(x) = f_A(x) \vee f_B(x)$.

Definition 2.10. [2] *Let A,B be neutrosophic sets on module M. Then their sum $A+B$ is a neutrosophic set on M, defined as follows*

$$t_{A+B}(x) = \vee\{t_A(y) \wedge t_B(z) | x = y+z, y, z \in M\}$$

$$i_{A+B}(x) = \vee\{i_A(y) \wedge i_B(z) | x = y+z, y, z \in M\}$$

$$f_{A+B}(x) = \wedge\{f_A(y) \vee f_B(z) | x = y+z, y, z \in M\}$$

Definition 2.11. [7] *For any neutrosophic subset $A = \{(x, t_A(x), i_A(x), f_A(x)) : x \in X\}$, the support A^* of the neutrosophic set A can be defined as*

$$A^* = \{x \in X : \ t_A(x) > 0, i_A(x) > 0, f_A(x) < 1\}.$$

3. NEUTROSOPHIC G- SUBMODULE

Definition 3.1. *Let G be a group and M be a G module over a field F. A neutrosphic G-submodule is a neutrosophic set $A = \{(x, t_A(x), i_A(x), f_A(x)) : x \in M\}$ in M such that the following conditions are satisfied.*

(1) $t_A(ax+by) \geq t_A(x) \wedge t_A(y)$
$i_A(ax+by) \geq i_A(x) \wedge i_A(y)$
$f_A(ax+by) \leq f_A(x) \vee f_A(y)$, $\forall\, x, y \in M, a, b \in F$

(2) $t_A(gm) \geq t_A(m)$

$i_A(gm) \geq i_A(m)$

$f_A(gm) \leq f_A(m)\ \forall\ g \in G, m \in M$

Remark 3.1. Let A be a neutrosophic G-submodule of a G-module M, then it is obvious that

$$t_A(0) \geq t_A(m),\ i_A(0) \geq i_A(m) \text{ and } f_A(0) \leq f_A(m)\ \forall\ m \in M.$$

Proposition 3.1. *Let M be a G -module over F and A be neutrosophic G-submodule of M.Then the support A^* is a G-su module of M.*

Proof. Let $x, y \in A^*$, $a, b \in K, g \in G$ then

$$t_A(x) > 0, i_A(x) > 0, f_A(x) < 1$$

$$t_A(y) > 0, i_A(y) > 0, f_A(y) < 1$$

Then by definition of neutrosophic G-submodule

$$t_A(ax+by) \geq\ t_A(x) \wedge t_A(y) \geq 0.$$

$$i_A(ax+by) \geq i_A(x) \wedge i_A(y) \geq 0.$$

$$f_A(ax+by) \leq f_A(x) \vee f_A(y) \leq 1.$$

Therefore $ax+by \in A^*$

Now consider $t_A(gx) \geq t_A(x) \geq 0$. Similarly $i_A(gx) \geq i_A(x) \geq 0$ and $f_A(gx) \leq f_A(x) \leq 1$. Therefore $gx \in A^*$. Hence A^* is a G-sub module of M.

□

Definition 3.2. *Let A_i, $i \in J$ be an arbitrary non empty family of neutrosophic set of G-module M, then*

(1) $\bigcap_{i \in J} A_i = \{x, t_{\bigcap_{i\in J} A_i}(x), i_{\bigcap_{i\in J} A_i}(x), f_{\bigcap_{i\in J} A_i}(x) : x \in M\}$ *where*

$$t_{\bigcap_{i\in J} A_i}(x) = \bigwedge_{i\in J} t_{A_i}(x)$$

$$i_{\bigcap_{i\in J} A_i}(x) = \bigwedge_{i\in J} i_{A_i}(x)$$

$$f_{\bigcap_{i\in J} A_i}(x) = \bigvee_{i\in J} f_{A_i}(x)$$

(2) $\bigcup_{i\in J} A_i = \{x, t_{\bigcup_{i\in J} A_i}(x), i_{\bigcup_{i\in J} A_i}(x), f_{\bigcup_{i\in J} A_i}(x) : x \in M\}$ *where*

$$t_{\bigcup_{i\in J} A_i}(x) = \bigvee_{i\in J} t_{A_i}(x)$$

$$i_{\bigcup_{i\in J} A_i}(x) = \bigvee_{i\in J} i_{A_i}(x)$$

$$f_{\bigcup_{i\in J} A_i}(x) = \bigwedge_{i\in J} f_{A_i}(x)$$

Theorem 3.1. *Let M be a G module over F and A, B be two neutrosophic G- submodules of M. Then $A \cap B$ is also a neutrosophic G-submodule of M.*

Proof. Let $x, y \in M$ and $a, b \in F$, 'then

$$\begin{aligned} t_{A\cap B}(ax+by) &= t_A(ax+by) \wedge t_B(ax+by) \\ &\geq \{t_A(x) \wedge t_A(y)\} \wedge \{t_B(x) \wedge t_B(y)\} \\ &= \{t_A(x) \wedge t_B(x)\} \wedge \{t_A(y) \wedge t_B(y)\} \\ &= t_{A\cap B}(x) \wedge t_{A\cap B}(y) \end{aligned}$$

Similarly

$$i_{A\cap B}(ax+by) \geq i_{A\cap B}(x) \wedge i_{A\cap B}(y)$$

$$f_{A\cap B}(ax+by) \leq f_{A\cap B}(x) \vee f_{A\cap B}(y)$$

For $g \in G$,

$$\begin{aligned} t_{A\cap B}(gx) &= t_A(gx) \wedge t_B(gx) \\ &\geq t_A(x) \wedge t_B(x) \\ &= t_{A\cap B}(x) \end{aligned}$$

Similarly

$$i_{A\cap B}(gx) \geq i_{A\cap B}(x)$$

$$f_{A\cap B}(gx) \leq f_{A\cap B}(x)$$

Hence $A \cap B$ is neutrosophic G module on M. □

Corollary. Let M be a G module over F and $\{A_i : i = 1, 2, ..., n\}$ be a family of neutrosophic G-module of M. Then $M = \cap_{i=1}^{n} Ai$ is also neutrosophic G-submodule of M.

Theorem 3.2. *Let M_1 and M_2be G -modules over F and A,B be neutrosophic G-submodules of M_1 and M_2 respectively.Then $A\times B$ is also neutrosophic G-submodule of $M_1\times M_2$.*

Proof. Let $x=(x_1,y_1), y=(x_2,y_2)\in M_1\times M_2$ where $x_1,x_2\in M_1,\ y_1,y_2\in M_2$ and $a,b\in F$, then

$$\begin{aligned}
t_{A\times B}(ax+by) &= t_{A\times B}\{a(x_1,y_1)+b(x_2,y_2)\}\\
&= t_{A\times B}\{(ax_1+bx_2, ay_1+by_2)\}\\
&= t_A(ax_1+bx_2)\wedge t_B(ay_1+by_2)\\
&\geq \{t_A(x_1)\wedge t_A(x_2)\}\wedge\{t_B(y_1)\wedge t_B(y_2)\}\\
&= \{t_A(x_1)\wedge t_B(y_1)\}\wedge\{t_A(x_2)\wedge t_B(y_2)\}\\
&= t_{A\times B}((x_1,y_1)\wedge t_{A\times B}((x_2,y_2)\\
&= t_{A\times B}(x)\wedge t_{A\times B}(y)
\end{aligned}$$

Similarly

$$i_{A\times B}(ax+by)\geq i_{A\times B}(x)\wedge i_{A\times B}(y)$$

$$f_{A\times B}(ax+by)\leq f_{A\times B}(x)\vee f_{A\times B}(y)$$

For $g\in G$ and $x=(x_1,y_1)\in M_1\times M_2$, then

$$\begin{aligned}
t_{A\times B}(gx) &= t_{A\times B}\{g(x_1,y_1)\}\\
&= t_{A\times B}\{gx_1, gy_1\}\\
&= t_A(gx_1)\wedge t_B(gy_1)\\
&\geq t_A(x_1)\wedge t_B(y_1)\\
&= t_{A\times B}(x_1,y_1)\\
&= t_{A\times B}(x)
\end{aligned}$$

Similarly

$$i_{A\times B}(gx)\geq i_{A\times B}(x)$$

$$f_{A\times B}(gx)\leq f_{A\times B}(x)$$

Hence $A\times B$ is also a neutrosophic G modules on $M_1\times M_2$. □

Theorem 3.3. *Let M be a G-module over F. Let A and B be two neutrosophic G-submodules of M. Then $A+B$ is also a neutrosophic G-submodule of M.*

Proof. Let $x,y \in M$ and assume $\wedge\{t_{A+B}(x), t_{A+B}(y)\} = e$. Let $h > 0$, then

$$e-h < t_{A+B}(x) = \vee\{t_A(a) \wedge t_B(b) : x = a+b,\ a,b \in M\}$$

$$e-h < t_{A+B}(y) = \vee\{t_A(c) \wedge t_B(d) : y = c+d,\ c,d \in M\}$$

$\Rightarrow\ \exists\, a,b \in M$ such that $x = a+b$

$$e-h < \{t_A(a) \wedge t_B(b) : x = a+b,\ a,b \in M\}$$

$$e-h < \{t_A(c) \wedge t_B(d) : y = c+d,\ c,d \in M\}$$

$\Rightarrow$

$$e-h < t_A(a),\ e-h < t_B(b),\ e-h < t_A(c),\ e-h < t_B(d)$$

$\Rightarrow$

$$e-h < t_A(a) \wedge t_A(c) \leq t_A(a+c)$$

$$e-h < t_B(b) \wedge t_B(d) \leq t_B(b+d)$$

We get $x+y = (a+b)+(c+d) = (a+c)+(b+d)$ such that

$$e-h < t_A(a+c) \wedge t_B(b+d)\ \textit{such that}\ x+y = a+c+b+d$$

$\Rightarrow$

$$e-h < \vee\{t_A(a+c) \wedge t_B(b+d)\} = t_{A+B}(x+y)$$

Since h is arbitrary, it follows that

$$t_{A+B}(x+y) \geq e = t_{A+B}(x) \wedge t_{A+B}(y)$$

Similarly, we can prove that

$$i_{A+B}(x+y) \geq i_{A+B}(x) \wedge i_{A+B}(y)$$

$$f_{A+B}(x+y) \leq f_{A+B}(x) \vee f_{A+B}(y)$$

Then, let $b = \vee(t_{A+B}(x), t_{A+B}(y)) = t_{A+B}(x)\ (say)$ and let $h > 0$

$$b-h < t_{A+B}(x) = \vee\{t_A(a) \wedge t_B(b) : x = a+b,\ a,b \in M\}$$

$\Rightarrow\ a,b \in M$ *such that* $x = a+b$

$$b-h < \{t_A(a) \wedge t_B(b) : x = a+b,\ a,b \in M\}$$

$\Rightarrow$

$$b-h < t_A(a) \leq t_A(ka),\ b-h < t_B(b) \leq t_B(kb), \forall\ k \in F$$

Now, $kx = k(a+b) = ka+kb$

$\Rightarrow$

$$b-h < \{t_A(ka) \wedge t_B(kb)\}$$

$\Rightarrow$

$$b-h < \vee\{t_A(ka) \wedge t_B(kb) : kx = k(a+b)\} = t_{A+B}(kx)$$

Since h is arbitrary

$\Rightarrow$

$$t_{A+B}(kx) \geq b = t_{A+B}(x)$$

Similarly, we can prove that

$$i_{A+B}(kx) \geq i_{A+B}(x),\ f_{A+B}(kx) \leq f_{A+B}(x)$$

Now, let $g \in G,\ x,a,b \in M$

$$t_A(ga) \geq t_A(a),\ t_B(gb) \geq t_B(b)\ \Rightarrow t_A(ga) \wedge t_B(gb) \geq t_A(a) \wedge t_B(b)$$

Also, $gx = g(a+b) = ga+gb$

$$\begin{aligned} t_{A+B}(x) &= \vee\{t_A(a) \wedge t_B(b) : x = a+b\} \\ &\leq \vee\{t_A(ga) \wedge t_B(gb) : gx = ga+gb\} \\ &\leq t_{A+B}(gx) \end{aligned}$$

i.e. $t_{A+B}(gx) \geq t_A(x)$. Similarly we can prove that

$$i_{A+B}(gx) \geq i_A(x),\ f_{A+B}(gx) \leq f_A(x)$$

$\therefore A+B$ is neutrosophic G- submodule on M. $\square$

Definition 3.3. *Let A be a neutrosophic G-submodule of M and N be a G-submodule of M. Then the restriction of A to N is denoted by $A|_N$ and it is a neutrosophic set of N defined as follows $A|_N(x) = (x, t_{A|_N}(x), i_{A|_N}(x), f_{A|_N}(x))$ where*

$$t_{A|_N}(x) = t_A(x)$$

$$i_{A|_N}(x) = i_A(x)$$

$$f_{A|_N}(x) = f_A(x)$$

$\forall x \in N$.

Proposition 3.2. *Let A be a neutrosophic G-submodule of a G-module of M over F. Let N be a G-submodule of M. Then $A|_N$ is a neutrosophic G-submodule of N.*

Proof. Let $a, b \in F$ and $x, y \in N$, then $ax + by \in N$

Consider

$$\begin{aligned} t_{A|_N}(ax+by) &= t_A(ax+by) \\ &\geq t_A(x) \wedge t_A(y) \\ &= t_{A|_N}(x) \wedge t_{A|_N}(y) \end{aligned}$$

Similarly,we can prove that

$$i_{A|_N}(ax+by) \geq i_{A|_N}(x) \wedge t_{A|_N}(y), f_{A|_N}(ax+by) \leq f_{A|_N}(x) \vee f_{A|_N}(y)$$

.

For $g \in G$ and $x \in N$, we have

$$t_{A|_N}(gx) = t_A(gx) \geq t_A(x)$$

similarly we can prove that $i_{A|_N}(gx) \geq i_A(x),\ f_{A|_N}(gx) \leq f_A(x)$

Hence $A|_N$ is neutrosophic G-submodule of N □

Definition 3.4. *Let M be a G-module over F and N be a G-submodule of M. Then the neutrosophic set A_N of M/Ndefined as*

$$A_N(x+N) = (x+N, t_{A_N}(x+N), i_{A_N}(x+N), f_{A_N}(x+N)),$$

where

$$t_{A_N}(x+N) = \vee t_A(x+n) : n \in N$$

$$i_{A_N}(x+N) = \vee i_A(x+n) : n \in N$$

$$f_{A_N}(x+N) = \wedge f_A(x+n) : n \in N, \forall x \in M$$

Proposition 3.3. *Let M be a G-module over F. Let N be a G- submodule of M. Then A_N is a neutrosophic quotient G-submodule of M with respect to N or factor neutrosophic G-module of A on M relative to G-submodule N.*

Proof. Consider $x+N,\ y+N \in M/N,\ g \in G, a, b \in K$. By definition

$$\begin{aligned}
t_{A_N}\{a(x+N)+b(y+N)\} &= t_{A_N}\{(ax+by)+N\} \\
&= \vee\{t_A[(ax+by)+n] : n \in N\} \\
&\geq \vee\{t_A[(ax+by)+an_1+bn_2] : n = an_1+bn_2,\ n_1, n_2 \in M\} \\
&= \vee\{t_A[a(x+n_1)+b(y+n_2)] : n_1, n_2 \in N\} \\
&\geq \vee\{t_A(x+n_1) \wedge t_A(y+n_2) : n_1, n_2 \in N\} \\
&\geq [\vee\{t_A(x+n_1) : n_1 \in N\}] \wedge [\vee\{t_A(y+n_2) : n_2 \in N\}] \\
&= t_{A_N}(x+N) \wedge t_{A_N}(y+N)
\end{aligned}$$

Similarly

$$i_{A_N}\{a(x+N)+b(y+N)\} \geq i_{A_N}(x+N) \wedge i_{A_N}(y+N)$$

$$f_{A_N}\{a(x+N)+b(y+N)\} \leq f_{A_N}(x+N) \vee f_{A_N}(y+N)$$

Consider $\forall\ x \in M, g \in G$

$$\begin{aligned}
t_{A_N}\{g(x+N)\} &= t_{A_N}\{gx+N\} \\
&= \vee\{t_A(g(x+n)) : n \in N\} \\
&\geq \vee\{t_A(x+n) : n \in N\} \\
&= t_{A_N}(x+N)
\end{aligned}$$

Similarly

$$i_{A_N}\{g(x+N)\} \geq i_{A_N}(x+N)$$

$$f_{A_N}\{g(x+N)\} \leq f_{A_N}(x+N)$$

Hence A_N is a neutrosophic G -submodule of M/N.

□

4. Conclusion

Neutrosophic G- submodule is one of the generalizations of an algebraic structure, G-module. This paper has developed a combination of an algebraic structure G-module with neutrosophic set. The algebraic property of neutrosophic G-submodules and its sum, product, intersection, Cartesian product , quotient and restriction of neutrosophic G-submodules are defined. Also we can extend the study of neutrosophic G-submodules in homomorphism and isomorphism of neutrosophic G-submodules.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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