Available online at http://scik.org

J. Math. Comput. Sci. 10 (2020), No. 1, 51-67

https://doi.org/10.28919/jmcs/4287

ISSN: 1927-5307

STABILITY ANALYSIS OF A DETERMINISTIC VACCINATION MODEL WITH NON-MONOTONIC INCIDENCE RATE

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Abstract: In this study, a deterministic epidemic model with vaccination and non-monotonic incidence rate is considered. This model also included the effect of temporary immunity. The model shows a disease free and an endemic equilibrium. Threshold R_0 (also known as basic reproduction number) is obtained, which gives the complete dynamics of the disease. If this threshold is less than unity, the disease-free equilibrium exists and infection disappears. If it is greater than unity, the endemic equilibrium exists and infection persists. The local and global stability of disease-free and endemic equilibrium are established. Global stability of positive equilibrium is proved by using a geometric approach given by Li and Muldowney. Numerical simulations are also given to support theoretical findings.

Keywords: endemic; non-monotonic incidence; basic reproduction number; stability; equilibrium.

2010 AMS Subject Classification: 92D30, 92B05, 34D23.

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Received September 2, 2019

1. INTRODUCTION

Epidemic models are significant tools in analyzing the spread and control of infectious diseases. Quarantine, treatment and vaccination are most commonly used methods to control spread of infectious diseases. Of these, vaccination is a proven prophylactic approach and is used in healthy individuals to prevent occurrence of infectious diseases [1, 2]. Many infectious diseases like measles, mumps, rubella, hepatitis B, influenza are reduced to a great extent by the use of vaccination. Several clinical results [3] have shown that the vaccination does not give permanent resistance to the disease. Once effect of a vaccine wanes from the body, susceptibility towards the disease increases again. Therefore, in order to prevent the infection and eradication of the disease, the vaccination in population must reach its optimal level. A mathematical study [4] on a model for childhood diseases with non-permanent immunity has shown that the disease will persist in the population if the vaccination coverage level is below a definite value. An SEIV epidemic model [5] with a nonlinear incidence and a waning preventive vaccination has formulated and prove that there is always a backward bifurcation for increasing the rate at which infected individuals are treated. An epidemic model [6] has included partial temporary immunity and saturated incidence to obtain the critical vaccination coverage necessary for eradication of the disease.

In epidemic models, incidence is the rate at which susceptible persons become infectious. Various incidence rates have been investigated by researchers. Bilinear, nonlinear, standard, saturated, specific nonlinear, general incidence rates are used in epidemic models and provided a detailed analysis of the proposed models [7-12].

Motivated from the work of [6], we are proposing an epidemic model with vaccination and non-monotonic incidence rate $\beta SI/(1+aI^2)$ [12]. This incidence rate increases with small I and decreases with large I. It also measures the psychological effect from the behavioral change of susceptible community when infective population becomes larger. This is important because the number of adequate contacts between infective and susceptible population decreases at high stage of infection due to the quarantine or isolation of infectives or the protective actions taken by the susceptibles [12].

2. MATHEMATICAL MODEL

The model is formulated as under:

$$\left.\begin{aligned}
\frac{dS}{dt} &= (1-p)A - \frac{\beta SI}{1+aI^2} - \mu S + \theta V + (1-q)\gamma I \\
\frac{dE}{dt} &= \frac{\beta SI}{1+aI^2} - \mu E - \sigma E \\
\frac{dI}{dt} &= \sigma E - \mu I - \gamma I \\
\frac{dV}{dt} &= pA - \theta V - \mu V + q\gamma I
\end{aligned}\right\} \qquad (1)$$

where S , E , I and V denotes susceptible, exposed, infectious and vaccinated persons respectively and initial conditions are $S(0), E(0), I(0), V(0) > 0$. The total population at time t is given by $N = S + E + I + V$.

All the parameters used in model (1) are positive. The constant A is the recruitment rate of susceptible individuals, μ is the natural death rate in the population, β is the average number of contacts of a person per unit time, θ is the rate at which vaccine wanes (that is, it becomes gradually weaker), γ is the rate at which infected individuals are recovered, σ is the rate at which exposed population becomes infectious, p is the fraction of A who are vaccinated ($0 \le p \le 1$), q is the fraction of recovered people who get disease acquired immunity ($0 \le q \le 1$) and a is the parameter which measures the psychological effect of disease on the population when the infective population becomes larger.

3. EQUILIBRIUM POINTS AND BASIC REPRODUCTION NUMBER

In this section, we determine all equilibrium states of model and basic reproduction number. Since $N = S + E + I + V$, we have $N'(t) = A - \mu N$. As $t \to \infty$, N approaches to the carrying capacity A/μ. It follows that the solutions of the system (1) remains bounded in the biologically meaningful and positively invariant region, defined by

$$\Re = \left\{(S,E,I,V) \in R_4^+ : S,E,I,V \ge 0, S+E+I+V \le A/\mu\right\}.$$

The disease-free equilibrium of the system (1) is $P_0 = (S_0, 0, 0, V_0)$ and is given by

$$S_0 = \frac{A}{\mu}\left\{\frac{(1-p)\mu+\theta}{\mu+\theta}\right\},\ V_0 = \frac{Ap}{\mu+\theta} \tag{2}$$

The basic reproduction number (R_0) is defined as the average number of secondary infections when one infective individual is entered into fully susceptible population [13]. It is very useful parameter which determines whether an infection will spread through the population or not. We obtained R_0 by next generation matrix method described in [13].

$$R_0 = \frac{\beta\sigma S_0}{(\mu+\gamma)(\mu+\sigma)} = \frac{A\beta\sigma[(1-p)\mu+\theta]}{\mu(\mu+\gamma)(\mu+\sigma)(\mu+\theta)} \tag{3}$$

Next, solving (1) for positive equilibrium, we get $S = \frac{S_0}{R_0}(1+aI^2)$, $V = \frac{pA+q\gamma I}{\mu+\theta}$, $E = \left(\frac{\mu+\gamma}{\sigma}\right)I$ and I is given by

$$a_1 I^2 + a_2 I + a_3 = 0 \tag{4}$$

where, $a_1 = a\mu(\mu+\theta)(\mu+\gamma)(\mu+\sigma)$

$$a_2 = \beta[(\mu+\theta)(\mu+\gamma)(\mu+\sigma) - \sigma\gamma\{(1-q)\mu+\theta\}]$$

$$a_3 = \mu(\mu+\theta)(\mu+\gamma)(\mu+\sigma)(1-R_0)$$

Here $a_1, a_2 > 0$. Thus from equation (4), it is clear that

(i) If R_0 is less than or equal to one, then there is no positive equilibrium.

(ii) If R_0 is greater than one, then there exist a unique positive (endemic) equilibrium $P_* = (S_*, E_*, I_*, V_*)$ and is given by

$$S_* = \frac{S_0}{R_0}(1+aI_*^2),\ V_* = \frac{pA+q\gamma I_*}{\mu+\theta},\ E_* = \left(\frac{\mu+\gamma}{\sigma}\right)I_* \tag{5}$$

and $$I_* = \frac{-\beta[(\mu+\theta)(\mu+\gamma)(\mu+\sigma) - \sigma\gamma\{(1-q)\mu+\theta\}] + \sqrt{D}}{2a\mu(\mu+\theta)(\mu+\gamma)(\mu+\sigma)} \tag{6}$$

where, $D=\beta^2[(\mu+\theta)(\mu+\gamma)(\mu+\sigma)-\sigma\gamma\{(1-q)\mu+\theta\}]^2-4a\{\mu(\mu+\theta)(\mu+\gamma)(\mu+\sigma)\}^2(1-R_0)$

4. LOCAL STABILITY ANALYSIS

In this section, the local stability analysis of disease-free and endemic equilibrium is discussed. By using $S+E+I+V=A/\mu$, the original system (1) can be reduced to the following system:

$$\left.\begin{aligned}\frac{dS}{dt}&=\frac{A}{\mu}\{(1-p)\mu+\theta\}-\frac{\beta SI}{1+aI^2}-(\mu+\theta)S-\theta E+\{(1-q)\gamma-\theta\}I\\ \frac{dE}{dt}&=\frac{\beta SI}{1+aI^2}-(\mu+\sigma)E\\ \frac{dI}{dt}&=\sigma E-(\mu+\gamma)I\end{aligned}\right\} \tag{7}$$

where, initial conditions are $S(0),E(0),I(0)>0$.

Theorem 4.1. The disease-free equilibrium P_0 is locally asymptotically stable when the basic reproduction number is less than one and is unstable when it is greater than one.

Proof. The Jacobian matrix of the system (7) is

$$J=\begin{pmatrix}-\frac{\beta I}{1+aI^2}-(\mu+\theta) & -\theta & -\frac{\beta S(1-aI^2)}{(1+aI^2)^2}+(1-q)\gamma-\theta\\ \frac{\beta I}{1+aI^2} & -(\mu+\sigma) & \frac{\beta S(1-aI^2)}{(1+aI^2)^2}\\ 0 & \sigma & -(\mu+\gamma)\end{pmatrix} \tag{8}$$

At P_0, we have

$$J_0=\begin{pmatrix}-(\mu+\theta) & -\theta & -\beta S_0+(1-q)\gamma-\theta\\ 0 & -(\mu+\sigma) & -\beta S_0\\ 0 & \sigma & -(\mu+\gamma)\end{pmatrix}$$

The characteristic equation of this matrix is

$$\{\lambda+(\mu+\theta)\}[\lambda^2+(2\mu+\sigma+\gamma)\lambda+(\mu+\sigma)(\mu+\gamma)(1-R_0)]=0 \tag{9}$$

It is obvious from (9) that, all the eigen values of J_0 are negative, if R_0 is less than one. Hence

in this case P_0 is locally asymptotically stable. Also, two eigen values of J_0 are negative and one eigen value is positive, if R_0 is greater than one. Hence P_0 is unstable in this case.

Theorem 4.2. If the basic reproduction number is greater than one, then the endemic equilibrium point P_* is locally asymptotically stable.

Proof. At P_*, the Jacobian matrix of system (7) is

$$J_* = \begin{pmatrix} -\frac{\beta I_*}{1+aI_*^2} - (\mu+\theta) & -\theta & -\frac{\beta S_*(1-aI_*^2)}{(1+aI_*^2)^2} + (1-q)\gamma - \theta \\ \frac{\beta I_*}{1+aI_*^2} & -(\mu+\sigma) & \frac{\beta S_*(1-aI_*^2)}{(1+aI_*^2)^2} \\ 0 & \sigma & -(\mu+\gamma) \end{pmatrix}$$

The characteristic equation of this matrix is $\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$, where

$$A_1 = 3\mu + \sigma + \gamma + \theta + \frac{\beta I_*}{(1+aI_*^2)}$$

$$A_2 = 2a(\mu+\gamma)(\mu+\sigma)\frac{I_*^2}{(1+aI_*^2)} + (2\mu+\sigma+\gamma)\left(\mu+\theta+\frac{\beta I_*}{(1+aI_*^2)}\right) + \frac{\beta\theta I_*}{(1+aI_*^2)}$$

$$A_3 = \left[2a(\mu+\theta)(\mu+\gamma)(\mu+\sigma)I_* + \beta\{\theta(\mu+\gamma) + \sigma(q\gamma+\theta) + \mu(\mu+\sigma+\gamma)\}\right]\frac{I_*}{(1+aI_*^2)}$$

Also,

$$A_1A_2 - A_3 = (3\mu+\theta+\gamma+\sigma)(2\mu+\sigma+\gamma)(\mu+\theta)$$

$$+\{2a(\mu+\gamma)(\mu+\sigma)I_* + \beta(2\mu+\sigma+\gamma+\theta)\}\frac{\beta I_*^2}{(1+aI_*^2)^2} + [2a(\mu+\gamma)(\mu+\sigma)(2\mu+\sigma+\gamma)I_*$$

$$+\beta\{7\mu^2+\sigma^2+\gamma^2+\theta^2+(5\mu+2\theta)(\sigma+\gamma)+6\mu\theta+\sigma\gamma(2-q)\}]\frac{I_*}{(1+aI_*^2)},$$

It is easy to see that $A_1, A_2, A_3 > 0$ and $A_1A_2 - A_3 > 0$. Hence P_* is locally asymptotically stable (by Routh-Hurwitz Theorem).

5. GLOBAL STABILITY ANALYSIS

In this section, we observe the global stability of the two equilibria P_0 and P_*. To study the global stability of P_0, we use the method given in [14]. Rewrite the system (7) as

$$\left.\begin{aligned} \frac{dX}{dt} &= F_1(X,Z), \\ \frac{dZ}{dt} &= F_2(X,Z), \quad F_2(X,0)=0 \end{aligned}\right\}$$

where, $X=(S)\in R$ stands for the number of uninfected individuals and $Z=(E,I)\in R^2$ represents the infected population (exposed and infectious both). We denote the disease-free equilibrium by $T_0=(X_0,0)$. The following conditions $(M1)$ and $(M2)$ are essential for global stability:

$(M1)$ X_0 is globally asymptotically stable for $\frac{dX}{dt}=F_1(X,0)$

$(M2)$ $F_2(X,Z)=BZ-\bar{F}_2(X,Z)$, where $\bar{F}_2(X,Z)\geq 0$, for $(X,Z)\in\Re$,

where, $B=D_zF_2(X_0,0)$ represents an M-matrix (the off-diagonal elements of B are non-negative). Then the following result holds:

Lemma 5.1. If the basic reproduction number is less than one and the assumptions $(M1)$-$(M2)$ are satisfied, then the fixed point $T_0=(X_0,0)$ is globally asymptotically stable.

Now, we prove the following theorem:

Theorem 5.2. The disease-free equilibrium P_0 is globally asymptotically stable provided that the basic reproduction number is less than one.

Proof. Let $X=(S)$, $Z=(E,I)$ and $T_0=(X_0,0)$, where $X_0=\frac{A\{(1-p)\mu+\theta\}}{\mu(\mu+\theta)}$. Then

$$\frac{dX}{dt}=F_1(X,Z)=\frac{A\{(1-p)\mu+\theta\}}{\mu}-\frac{\beta SI}{1+aI^2}-(\mu+\theta)S-\theta E+\{(1-q)\gamma-\theta\}I$$

At $S=S_0$, we have $F_1(X,0)=0$ and $\frac{dX}{dt}=F_1(X,0)=\frac{A}{\mu}\{(1-p)\mu+\theta\}-(\mu+\theta)X$.

As $t\to\infty, X\to X_0$. Thus, $X=X_0(=S_0)$ is globally asymptotically stable. Now

$$F_2(X,Z)=\begin{bmatrix}-(\mu+\sigma) & \beta S_0\\ \sigma & -(\mu+\gamma)\end{bmatrix}\begin{bmatrix}E\\ I\end{bmatrix}-\begin{bmatrix}\beta S_0 I-\dfrac{\beta SI}{1+aI^2}\\ 0\end{bmatrix}=BZ-\bar{F}_2(X,Z)$$

where, $B=\begin{bmatrix}-(\mu+\sigma) & \beta S_0\\ \sigma & -(\mu+\gamma)\end{bmatrix}$, $Z=\begin{bmatrix}E\\ I\end{bmatrix}$ and $\bar{F}_2(X,Z)=\begin{bmatrix}\beta S_0 I-\dfrac{\beta SI}{1+aI^2}\\ 0\end{bmatrix}$.

In the system (7), total population is bounded by $(N_1)_0 \to \dfrac{A\{(1-p)\mu+\theta\}}{\mu(\mu+\theta)}$, that is, $S,E,I \le (N_1)_0$. Since $S_0 \ge (N_1)_0$, we have $S_0 \ge (N_1)_0 \ge S \ge S/(1+aI^2)$ and hence $\bar{F}_2(X,Z) \ge 0$. Obviously, B represents an M-matrix, so conditions $(M1)$ and $(M2)$ hold good and by Lemma 5.1 P_0 is globally asymptotically stable provided $R_0<1$.

Next, we investigate the global stability of P_* by application of a geometrical approach developed in [15] (briefly explained in Appendix A).

Theorem 5.3. If basic reproduction number is greater than one, then the endemic equilibrium P_* of the system (7) is globally stable in region $\Re$ provided $\sigma > \theta$ and $(1-q)\gamma \ge \theta$.

Proof. Theorem 4.1 shows the instability of P_0 when $R_0>1$. By means of the uniform persistence results proved in [16, 17], the instability of P_0 assures the uniform persistence of the system when $R_0>1$. Therefore the reduced system (7) is uniformly persistent. Now, the second additive compound matrix $J^{[2]}$ of the Jacobian matrix (8) is given by

$$J^{[2]}=\begin{pmatrix}-\dfrac{\beta I}{1+aI^2}-2\mu-\theta-\sigma & \dfrac{\beta S(1-aI^2)}{(1+aI^2)^2} & \dfrac{\beta S(1-aI^2)}{(1+aI^2)^2}-(1-q)\gamma+\theta\\ \sigma & -\dfrac{\beta I}{1+aI^2}-2\mu-\theta-\gamma & -\theta\\ 0 & \dfrac{\beta I}{1+aI^2} & -2\mu-\sigma-\gamma\end{pmatrix}$$

Choose the function $P=P(S,E,I)=\operatorname{diag}\left\{1,\dfrac{E}{I},\dfrac{E}{I}\right\}$. This shows that $P^{-1}=\operatorname{diag}\left\{1,\dfrac{I}{E},\dfrac{I}{E}\right\}$.

Also we have $P_f = \text{diag}\left\{0, \frac{E'I - EI'}{I^2}, \frac{E'I - EI'}{I^2}\right\}$. So that $P_f P^{-1} = diag\left\{0, \frac{E'}{E} - \frac{I'}{I}, \frac{E'}{E} - \frac{I'}{I}\right\}$. Then

$$B = P_f P^{-1} + PJ^{[2]}P^{-1}$$

$$= \begin{pmatrix} -\left(\frac{\beta I}{1+aI^2} + 2\mu + \theta + \sigma\right) & \frac{\beta S(1-aI^2)I}{(1+aI^2)^2 E} & \left\{\frac{\beta S(1-aI^2)}{(1+aI^2)^2} - (1-q)\gamma + \theta\right\}\frac{I}{E} \\ \sigma\frac{E}{I} & -\left(\frac{\beta I}{1+aI^2} + 2\mu + \theta + \gamma\right) + \frac{E'}{E} - \frac{I'}{I} & -\theta \\ 0 & \frac{\beta I}{1+aI^2} & -(2\mu + \sigma + \gamma) + \frac{E'}{E} - \frac{I'}{I} \end{pmatrix}$$

The matrix B can be expressed in block form as $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, Where

$$B_{11} = -\left(\frac{\beta I}{1+aI^2} + 2\mu + \theta + \sigma\right),\ B_{12} = \left[\frac{\beta S(1-aI^2)I}{(1+aI^2)^2 E}, \left\{\frac{\beta S(1-aI^2)}{(1+aI^2)^2} - (1-q)\gamma + \theta\right\}\frac{I}{E}\right],$$

and $B_{21} = \left[\frac{\sigma E}{I}, 0\right]^T$, $B_{22} = \begin{bmatrix} -\left(\frac{\beta I}{1+aI^2} + 2\mu + \theta + \gamma\right) + \frac{E'}{E} - \frac{I'}{I} & -\theta \\ \frac{\beta I}{1+aI^2} & -(2\mu + \sigma + \gamma) + \frac{E'}{E} - \frac{I'}{I} \end{bmatrix}$

Let $(u,v,w) \in R^3$ be a vector, then we define a norm in R^3 as $|(u,v,w)| = \max\{|u|, |v| + |w|\}$ and let η be the Lozinskiĭ measure with respect to this norm. We have $\eta(B) \le Sup\{g_1, g_2\}$, where $g_1 = \eta_1(B_{11}) + |B_{12}|$, $g_2 = \eta_1(B_{22}) + |B_{21}|$ and $|B_{12}|, |B_{21}|$ are matrix norms with respect to the L^1 vector norm and η_1 represent the Lozinskiĭ measure with respect to L^1 norm [18]. Then,

$$g_1 = \eta_1(B_{11}) + |B_{12}|$$

$$= -\left(\frac{\beta I}{1+aI^2} + 2\mu + \theta + \sigma\right) + \max\left\{\frac{\beta S(1-aI^2)I}{(1+aI^2)^2 E}, \frac{\beta S(1-aI^2)I}{(1+aI^2)^2 E} - \{(1-q)\gamma - \theta\}\frac{I}{E}\right\}$$

$$= -\left(\frac{\beta I}{1+aI^2} + 2\mu + \theta + \sigma\right) + \frac{\beta S(1-aI^2)I}{(1+aI^2)^2 E} \qquad [\text{when}\,(1-q)\gamma \ge \theta]$$

$$\leq -\left(\frac{\beta I}{1+aI^2}+2\mu+\theta+\sigma\right)+\frac{\beta SI}{(1+aI^2)E}$$

$$\leq \frac{E'}{E}-\mu-\theta-\frac{\beta I}{1+aI^2} \qquad [\text{from}(7)]$$

$$\leq \frac{E'}{E}-\mu$$

To find $\eta_1(B_{22})$ we add the absolute value of the off-diagonal elements to the diagonal one in each column of B_{22}, and then take the maximum value of these two sums. Thus

$$g_2=\eta_1(B_{22})+|B_{21}|$$

$$=\max\left\{\frac{E'}{E}-\frac{I'}{I}-2\mu-\theta-\gamma,\frac{E'}{E}-\frac{I'}{I}-2\mu-\sigma-\gamma+\theta\right\}+\sigma\frac{E}{I}$$

$$=\frac{E'}{E}-\frac{I'}{I}-2\mu-\gamma+\max(-\theta,\theta-\sigma)+\sigma\frac{E}{I}$$

$$=\frac{E'}{E}-\mu+\max(-\theta,\theta-\sigma) \qquad [\text{from}(7)]$$

$$\leq \frac{E'}{E}-\mu \qquad [\text{when}\ \sigma>\theta]$$

Therefore, $\eta(B)\leq Sup\{g_1,g_2\}=\frac{E'}{E}-\mu$. Integrating both sides at the same time, we get

$$\frac{1}{t}\int_0^t\eta(B)ds\leq\frac{1}{t}\log\frac{E(t)}{E(0)}-\mu,$$

this shows that

$$\tilde{q}=\limsup_{t\to\infty}\sup_{x_0\in\Omega}\frac{1}{t}\int_0^t\eta(B(x(s,x_0)))ds\leq-\mu<0.$$

6. NUMERICAL SIMULATION AND CONCLUDING REMARKS

In this paper, a vaccination model with non-monotonic incidence rate and partial temporary immunity is discussed. We have proved that if basic reproduction number R_0 is less than one, P_0 is globally asymptotically stable and if it is greater than one, P_* is globally asymptotically stable under some conditions. If basic reproduction number is equal to one then from equation

(3) we obtain the threshold value of vaccination $\bar{p} = (\mu+\theta)[\sigma\beta A - \mu(\mu+\sigma)(\mu+\gamma)]/\sigma\beta\mu A$. If this threshold $\bar{p}$ is less than p, $R_0 < 1$ and if it is greater than p, $R_0 > 1$. Thus the disease can be eradicated if the vaccinated number (p) is greater than the threshold vaccination coverage ($\bar{p}$); otherwise the disease will persist in the population.

Now, we provide some numerical simulations by using MATLAB in support of the analytical findings of previous sections.

(i) Considering the parameters in system (1) as $A = 4$, $\mu = 0.1$, $a = 2$, $\gamma = 0.2$, $\theta = 0.2$, $\sigma = 0.1$, $\beta = 0.01$, $q = 0.5$ and $p = 0.2$, we get basic reproduction number $R_0 = 0.622 < 1$. Thus the system (1) has a disease-free equilibrium $P_0 = (S_0, E_0, I_0, V_0) = (37.2, 0, 0, 2.67)$ and is globally asymptotically stable. In this case the disease disappears and dies out (figure. 1).

(ii) Considering the parameters in system (1) as $A = 4$, $\mu = 0.2$, $a = 2$, $\gamma = 0.2$, $\theta = 0.1$, $\sigma = 0.2$, $\beta = 0.2$, $q = 0.5$ and $p = 0.2$, we get basic reproduction number $R_0 = 4.3333 > 1$. Thus the system (1) has an endemic equilibrium $P_* = (S_*, E_*, I_*, V_*) = (13.6688, 2.1987, 1.0994, , 3.03312)$ and is globally asymptotically stable. Thus the disease becomes endemic and persists in the population (figure. 2).

Keeping all parameters fixed of endemic equilibrium and taking different initial conditions in system (7), the phase portrait in SEI-space is displayed. This phase diagram shows that $\lim_{t\to\infty}(S(t), E(t), I(t)) = (S_*, E_*, I_*)$ for $R_0 = 4.3333 > 1$ (figure. 3).

(iii) Taking all parameters of endemic equilibrium and varying the values of p, we see that increase in the values of p decreases the infected population. At ($p = 0$, $\theta = 0$), that is, without vaccination, the size of infected population is high. At ($p = 0.2, \theta = 0.1$) and ($p = 0.4, \theta = 0.1$), that is, increasing the vaccination coverage, the size of infected population get decreases. This shows the effect of vaccination on size of infected population (figure. 4).

(iv) Though the basic reproduction number R_0 does not depend on q , numerical results show that when the disease is endemic, the steady-state value S_* of the susceptible population decreases as q increases. This indicates that q affects the dynamics of the given system (figure. 5).

(v) On keeping all parameters fixed of endemic equilibrium and changing the values of a , we see that the steady-state values of I_* of the infectives decreases as a increases. Thus, spread of disease decreases as the protective measures for the susceptibles increases (figure. 6).

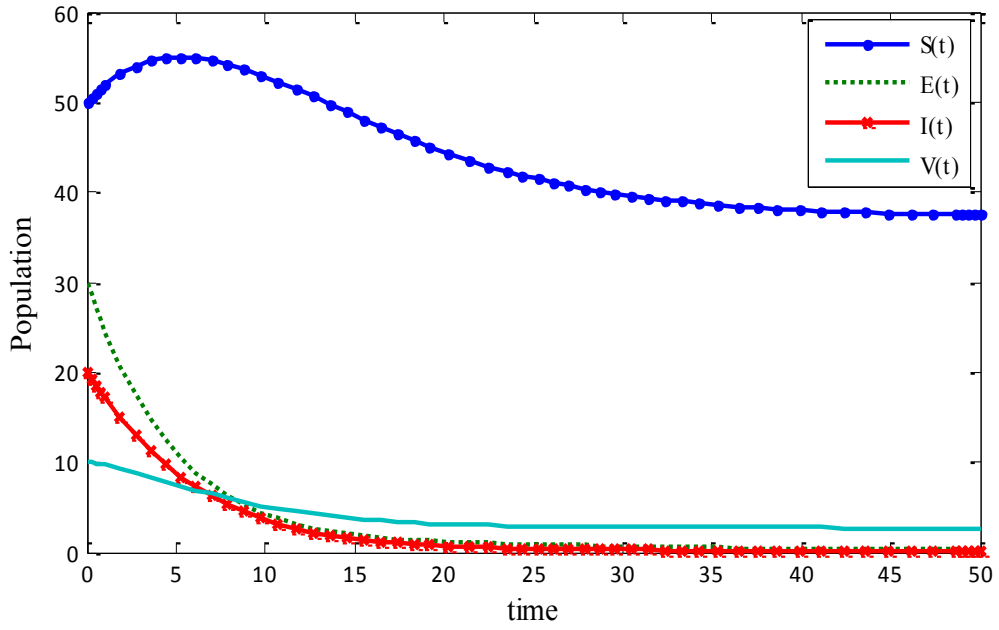

Figure. 1. Disease-free equilibrium P_0 is globally asymptotically stable and disease dies out.

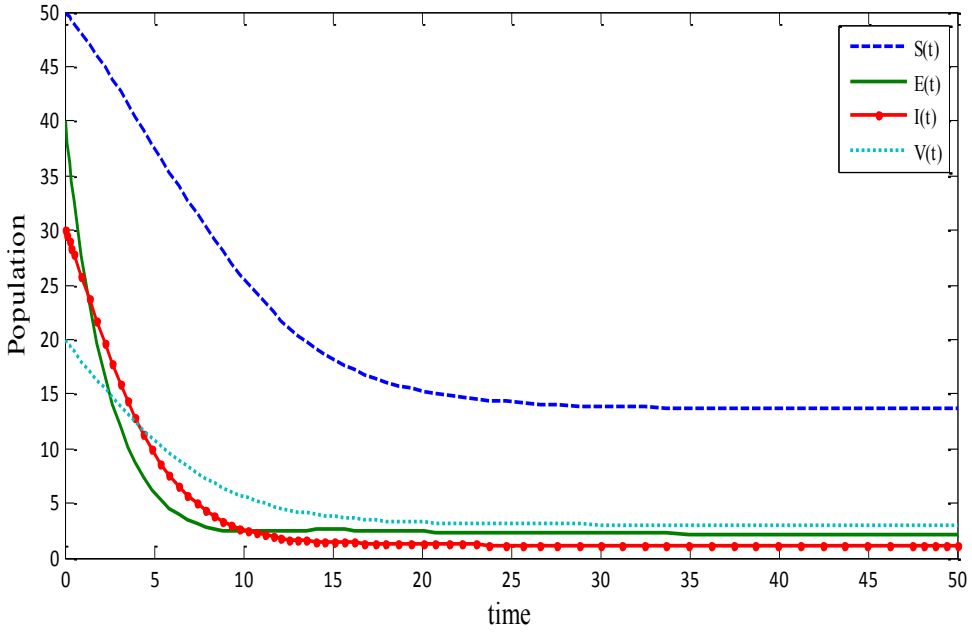

Figure. 2. Endemic equilibrium P_* is globally asymptotically stable and disease persists.

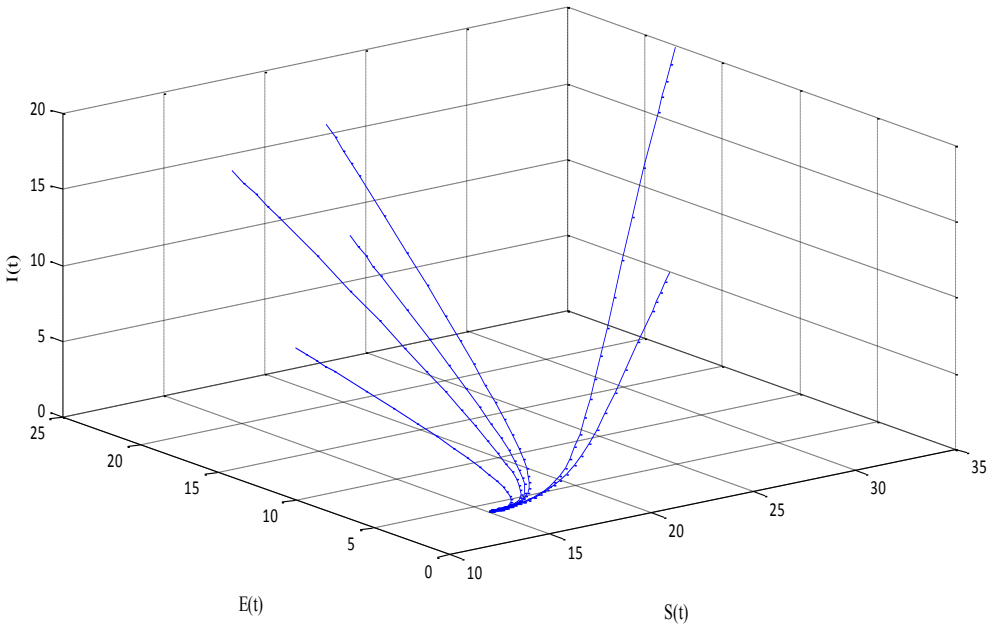

Figure. 3. Phase portrait in SEI-space for endemic equilibrium.

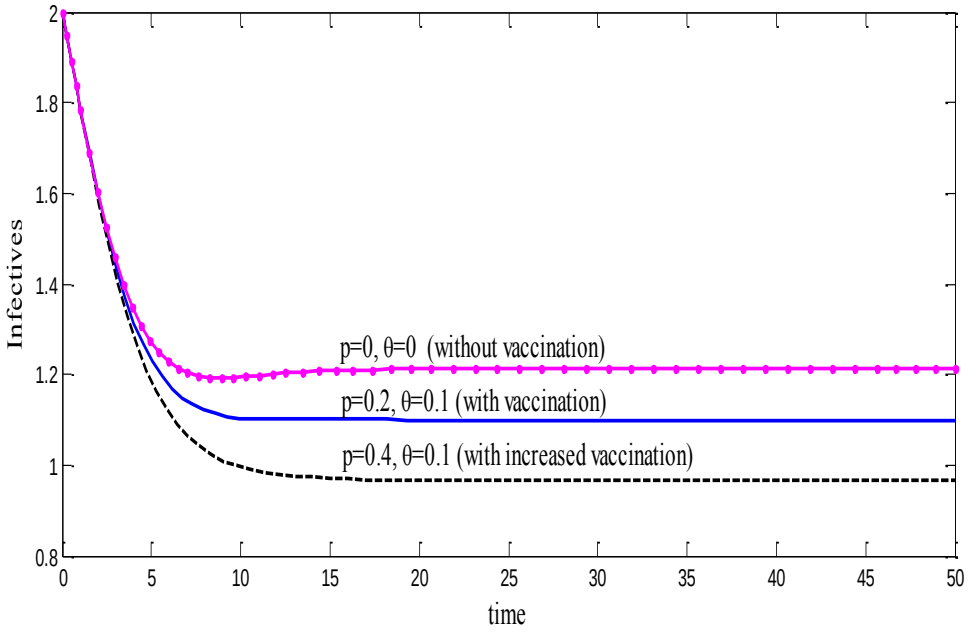

Figure. 4. Effect of vaccination coverage on size of infected population.

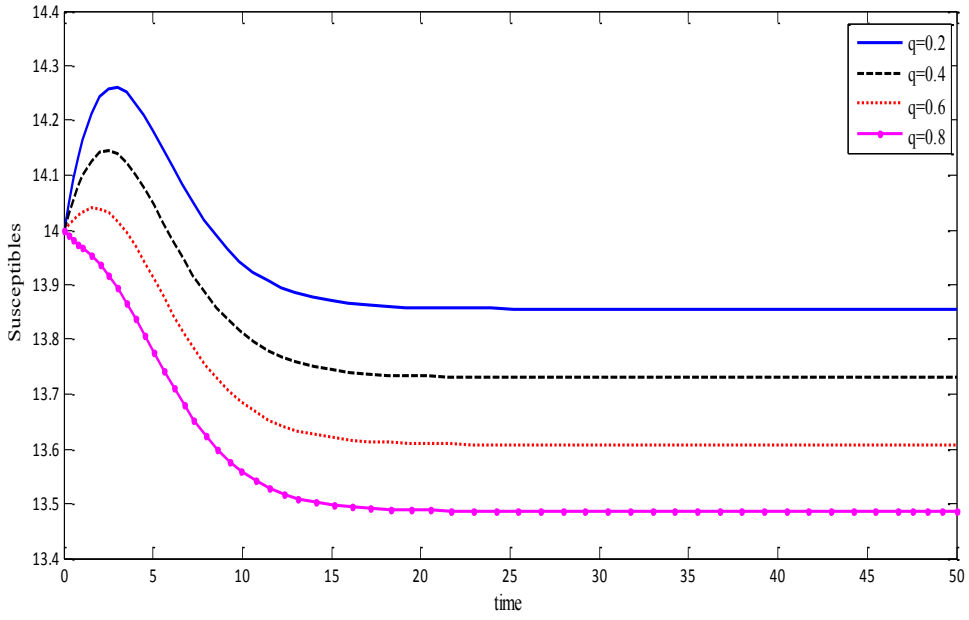

Figure. 5. Effect of q on the size of the susceptible population.

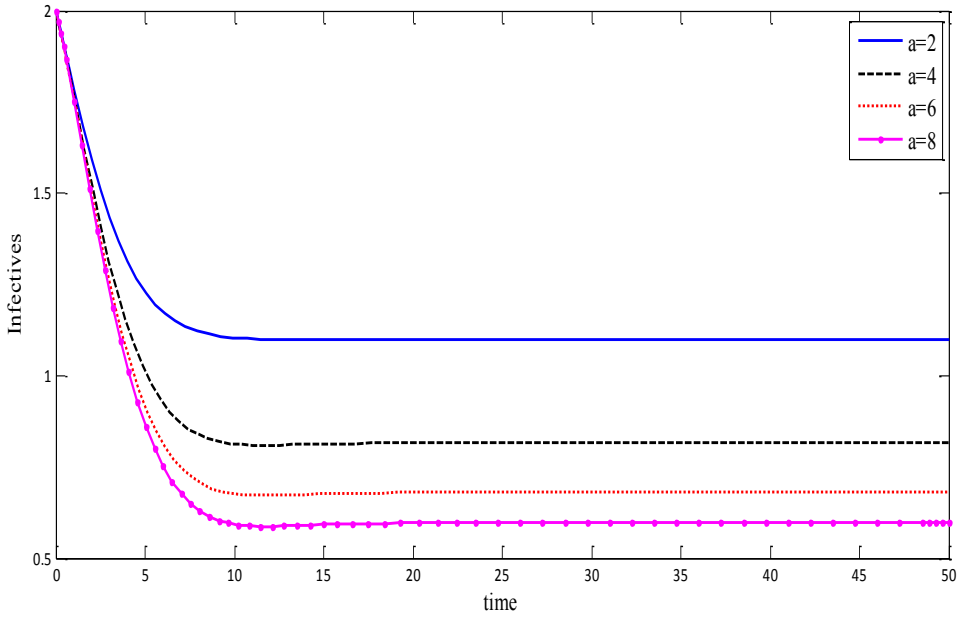

Figure. 6. The steady-state values of I_* decreases as a increases.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

APPENDIX A

To prove the global stability of endemic equilibrium, we briefly explain the geometrical approach method given by Li and Muldowney [15]. Consider the system:

$$x' = f(x) \tag{10}$$

where, $x \mapsto f(x) \in R^n$ is a C^1 function about x in $D_1 \subset R^n$. Let $x(t, x_0)$ be the solution of (10) such that $x(0, x_0) = x_0$ and x^* be an equilibrium of (10), i.e. $f(x^*) = 0$. Assume that the following hypotheses hold:

$(M3)$ There is a compact absorbing set $K \subset D_1$.

$(M4)$ Differential equation (10) has a unique equilibrium x^* in D_1.

Let $x \to P(x)$ be a $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function that is C^1 for $x \in D_1$. Suppose $P^{-1}(x)$ exists

and is continuous for $x \in K$, the compact absorbing set. We define a quantity $\tilde{q}$ as

$$\tilde{q} = \lim_{t\to\infty} \sup \sup_{x\in K} \frac{1}{t} \int_0^t \eta(B(x(s, x_0)))ds,$$

where $B = P_f P^{-1} + PJ^{[2]}P^{-1}$, the matrix P_f is obtained by substituting each element P_{ij} of P by its derivative in the direction of f and $J^{[2]}$ represents the second additive compound matrix of the Jacobian matrix J. The quantity $\eta(B)$ is the Lozinskiĭ measure of B with respect to a vector norm $|.|$ in R^N, $N = \binom{n}{2}$, given by $\eta(B) = \lim_{h\to 0^+} \frac{|I + hB| - 1}{h}$.

The following global stability analysis result is proved in Theorem 3.5 of [15].

Theorem. Suppose that D_1 is simply connected and that assumptions $(M3)$ - $(M4)$ hold, then the unique equilibrium x^* of (10) is stable in D_1 if $\tilde{q} < 0$.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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