Available online at http://scik.org
J. Math. Comput. Sci. 3 (2013), No. 1, 185-194
ISSN: 1927-5307

# CONTINUOUS LIFE ANNUITIES UNDER UNCERTAIN INTEREST RATE

SHUAI WU*, LIANG LIN, CHAO WANG

College of Science, Guilin University of Technology, Guilin Guangxi, 541004, China

**Abstract:** Actuarial science provide scientific basis and tools for raising the level of management and making strategies, and life annuity is a critical issue of actuarial science. In this paper, we focus on presenting uncertain process for modeling the force of interest, combining the survival rate to calculate various life annuities. Then we compare the net premium under uncertain interest to the traditional methods for deterministic interest and stochastic interest.

**Keywords:** life insurance, mortality, force of interest, net single premium, canonical process.

## 1. Introduction

Annuities are contracts designed to provide payments to the holder at specified intervals, usually after retirement. Traditionally, actuarial theory mainly uses deterministic interest rate to calculate annuities. It is assumed that underlying interest rate is fixed and the same for all years. However, with the rapid development of finance, interest rate changes fast with market，policy and so on. It leads some scholars begun to introspect theoretical defect of deterministic interest rate. In 1971, “the interest rate is a random variable” was first put forward by J.H.polland, who also researched some actuarial functions. After that, some scholars begun to study stochastic interest and obtained actuarial models and have some conclusions under the

* Corresponding author

Received May 31, 2012

assumptions according to the following models : Wiener process model, reflected Brownian motion model and combined model of reflected Brownian motion and Poisson process. As well as, Zhao and Gao, who characterized the force of interest function as trapezoidal fuzzy variable, established full-discrete actuarial models with fuzzy interest rate. In 2008, Liu[1] started the study of uncertain process and presented an extreme value theorem for independent increment process. After that, Liu[3] proved that the expected value of stationary independent increment process is a liner function of time, and Chen proved that the variance is proportional to the square of time. In 2009, Liu[2] modeled Brownian motion by an uncertain process called canonical process and he noted that canonical process is a stationary independent increment process with normal distribution, which method offers a new way for actuarial models with respect to interest rate.

In this paper, we will consider the force of interest rate as canonical process and we will focus on actuarial formulas of annuities during uncertainty theory. The rest of the paper is organized as follows. The next section is intended to introduce some annuities calculation with a fixed interest rate and with a random interest rate. In section 3, the formulas of annuities and level premium under canonical process interest rate are proposed and proved. Some preliminaries are given in next section. In section 5, we present an example to illustrate the superiority of Canonical interest rate. Finally, a brief summary is given in section 5.

## 2. Continuous Life Annuities with a Fixed or Random Interest Rate

First we recall basic notation used in the theory of annuities. Suppose that $i$ is the positive annual interest rate, $v$ is the annual discount factor, and $\delta$ is the force of interest.

Hence
$$v = \frac{1}{1+i} = e^{-\delta} \tag{1}$$

The survival function which represents the probability that an $x$-year old person will survive at least $t$ years is ${}_t p_x$.

Let $Y_1$ be the random variable of present value of a continuous life annuity at age $x$

of one per annum, then we have

$$Y_1 = \overline{a}_{\overline{T}|}$$

for any $T = X - x \geq 0$ , in which $X$ be the age in death.

Then the actuarial present value of a continuous life annuity is

$$\overline{a}_x = E(Y_1) = E(\overline{a}_{\overline{T}|}) = \int_0^{\infty} v^t \, {}_t p_x dt \tag{2}$$

Simultaneously, Let $Y_2$ be the random variable of present value of a continuous temporary life annuity at age $x$ of one per annum for a term of $n$ years , then we have

$$Y_2 = \begin{cases} \overline{a}_{\overline{T}|} & 0 \leq T < n \\ \overline{a}_{\overline{n}|} & T \geq n \end{cases}$$

Then the actuarial present value of continuous life annuities is

$$\overline{a}_{x:\overline{n}|} = E(Y_2) = \int_0^{n} v^t \, {}_t p_x dt \tag{3}$$

In the same way, Let $Y_3$ be the random variable of present value of a continuous $m$ –year deferred life annuity at age $x$ of one per annum, then we have

$$Y_3 = \begin{cases} 0 & 0 \leq T < m \\ \overline{a}_{\overline{T}|} - \overline{a}_{\overline{m}|} & T \geq m \end{cases}$$

Then the actuarial present value of continuous life annuities is

$$_{m|}\overline{a}_x = E(Y_3) = \int_m^{\infty} v^t \, {}_t p_x dt \tag{4}$$

When using the fixed force of interest rate, we can get the following formulas:

$$\overline{a}_x^{(f)} = E(Y_1) = E(\overline{a}_{\overline{T}|}) = \int_0^{\infty} e^{-\delta t} \, {}_t p_x dt \tag{5}$$

$$\overline{a}_{x:\overline{n}|}^{(f)} = E(Y_2) = \int_0^{n} e^{-\delta t} \, {}_t p_x dt \tag{6}$$

$$_{m|}\overline{a}_x^{(f)} = E(Y_3) = \int_m^{\infty} e^{-\delta t} \, {}_t p_x dt \tag{7}$$

When using the random force of interest rate, we suppose that a force of interest with the accumulation function is

$$\delta^{(r)}(t)=\delta t+\beta y(t)+\gamma z(t) \tag{8}$$

where $\{y(t),t\geq 0\}$ is a standard Wiener process. $\{z(t),t\geq 0\}$ is poisson process with the parameter $\lambda=0.01$. And $y(t)$ and $z(t)$ are independent. $\delta$、$\beta$ and $\gamma$ are mutually independent and no correlation with $t$.

Based on the above models, we can get the actuarial present value of continuous life annuities:

$$\bar{a}_x^{(r)}=E(Y_1)=E(\overline{a_{\overline{T|}}})=\int_0^{\infty} e^{-\delta t+\frac{1}{2}\beta^2 t+\lambda t(e^{-\gamma}-1)}\,{}_t p_x dt \tag{9}$$

$$\bar{a}_{x:\overline{n|}}^{(r)}=E(Y_2)=E(\overline{a_{\overline{T|}}})=\int_n^{\infty} e^{-\delta t+\frac{1}{2}\beta^2 t+\lambda t(e^{-\gamma}-1)}\,{}_t p_x dt \tag{10}$$

$${}_{m|}\overline{a}_x^{(r)}=E(Y_3)=E(\overline{a_{\overline{T|}}})=\int_m^{\infty} e^{-\delta t+\frac{1}{2}\beta^2 t+\lambda t(e^{-\gamma}-1)}\,{}_t p_x dt \tag{11}$$

## 3. Preliminaries

Definition 3.1 (Liu[2]) Let $T$ be an index set and let $(\Gamma,\varsigma,M)$ be an uncertainty space. An uncertain process is a measurable function from $T\times(\Gamma,\varsigma,M)$ to the set of real numbers, i.e., for each $t\in T$ and only Borel set $B$ of real numbers, the set

$$\left\{X_t\in B\right\}=\left\{\gamma\in\Gamma \,\middle|\, X_t(\gamma)\in B\right\} \tag{12}$$

is an event.

Difinition3. 2 (Liu[2]) An uncertain process $C_t$ is said to be a canonical process if

(i) $C_0=0$ and almost all sample paths are Lipschitz continuous,

(ii) $C_t$ has stationary and independent increments,

(iii) every increment $C_{s+t}-C_s$ is a normal uncertain variable with expected value 0 and variance $t^2$.

Theorem 3.1(Liu[2]) The canonical process $C_t$ is a normal uncertain variable with expected value 0 and variance $t^2$ , and has an uncertainty distribution

$$\Phi(x) = (1 + \exp(-\frac{\pi x}{\sqrt{3}t}))^{-1} \tag{13}$$

At each time $t > 0$ . That is,

$$C_t \sim N(0,t)$$

Definition 3.3(Liu[2]) Let $C_t$ be a canonical process. Then for any real numbers $\delta$ and $k$ ,

$$M_t = \delta t + kC_t \tag{14}$$

is called an arithmetic canonical process, where $\delta$ is called the drift and $k$ is called the diffusion.

Theorem 1.2(Liu[2]) At each time $t$ , the arithmetic canonical process $M_t$ is a normal uncertain variable with expected value $\delta t$ and variance $k^2 t^2$ , i.e.,

$$M_t \sim N(\delta t, kt)$$

whose uncertainty distribution is

$$\Phi(x) = (1 + \exp(\frac{\pi(\delta t - x)}{\sqrt{3}kt}))^{-1} \tag{15}$$

Definition 3.4(Liu[2]) Let $C_t$ be a canonical process. Then for any real numbers $\delta$ and $k$ ,

$$G_t = \exp(\delta t + kC_t) \tag{16}$$

is called a geometric canonical process, where $\delta$ is called the log-drift and $k$ is called the log-diffusion.

Theorem 3.3 (Liu[2]) At each time $t$ , the geometric canonical process $G_t$ is a log-normal uncertain variable, i.e.,

$$G_t \sim LogN(\delta t, kC_t)$$

whose uncertainty distribution is

$$\Phi(x) = (1 + \exp(\frac{\pi(\delta t - \ln x)}{\sqrt{3}kt}))^{-1} \tag{17}$$

Furthermore, the expected value is

$$E[G_t] = \begin{cases} \sqrt{3}kt\exp(\delta t)\csc(\sqrt{3}kt), & t < \pi / k\sqrt{3} \\ +\infty, & t \geq \pi / k\sqrt{3} \end{cases} \tag{18}$$

## 4. Continuous Life Annuities under Canonical Process Interest

Theorem 2.1 Suppose $\delta^{(c)}(t)$ be a force of interest with the accumulation function

$$\delta^{(c)}(t) = \delta t + kC_t \tag{19}$$

where $\delta t$ is risk-free force of interest and $kC_t$ is risk force of interest. $C_t$ is a canonical process with expected value 0 and variance $t^2$. Then the actuarial present values of life annuities are expressed as follow:

$$\bar{a}_x^{(c)} = E(Y_1) = E(\overline{a_{\overline{T|}}}) = \sqrt{3}k\int_0^{\infty} e^{-\delta t}\csc(\sqrt{3}kt)\,{}_t p_x dt \tag{20}$$

$$\overline{a}_{x:\overline{n|}}^{(c)} = E(Y_2) = \sqrt{3}k\int_0^{n} e^{-\delta t}\csc(\sqrt{3}kt)\,{}_t p_x dt \tag{21}$$

$${}_{m|}\bar{a}_x^{(c)} = E(Y_3) = \sqrt{3}k\int_m^{\infty} e^{-\delta t}\csc(\sqrt{3}kt)\,{}_t p_x dt \tag{22}$$

for any $t < \pi / k\sqrt{3}$.

Proof: Note that $C_t$ is a canonical process with expected value 0 and variance $t^2$. For any $t > 0$, it follows from the definition 3.3 that $\delta^{(c)}(t)$ is an arithmetic canonical process with expected value $\delta t$ and variance $k^2t^2$.

During interest theory , we get the discount factor

$$v_t = e^{-\delta^{(c)}(t)} = e^{-\delta t - kC_t}$$

which can be easily known a geometric canonical process with the expected value

$$E[v_t] = \begin{cases} \sqrt{3}kt\exp(-\delta t)\csc(\sqrt{3}kt), & t < \pi / k\sqrt{3} \\ +\infty, & t \geq \pi / k\sqrt{3} \end{cases}$$

Let $Y_1$ be the uncertain variable of present value of a continuous life annuity at age $x$ of one per annum, then we have

$$Y_1 = \overline{a}_{\overline{T}|}$$

for any $T = X - x \geq 0$ , in which $X$ be the age in death.

Then the actuarial present value of a continuous life annuity is

$$\bar{a}_x^{(c)} = E(Y_1) = E(\overline{a}_{\overline{T}|}) = \int_0^{\infty} v^t \, {}_t p_x dt = \int_0^{\infty} E(v_t) \, {}_t p_x dt$$

$$= \sqrt{3}k \int_0^{\infty} e^{-\delta t} \csc(\sqrt{3}kt) \, {}_t p_x dt$$

Simultaneously, Let $Y_2$ be the random variable of present value of a continuous temporary life annuity at age $x$ of one per annum for a term of $n$ years , then we have

$$Y_2 = \begin{cases} \overline{a}_{\overline{T}|} & 0 \leq T < n \\ \overline{a}_{\overline{n}|} & T \geq n \end{cases}$$

Then the actuarial present value of continuous life annuities is

$$\bar{a}_{x:\overline{n}|}^{(c)} = E(Y_2) = \int_0^{n} v^t \, {}_t p_x dt = \int_0^{n} E(v_t) \, {}_t p_x dt$$

$$= \sqrt{3}k \int_0^{n} e^{-\delta t} \csc(\sqrt{3}kt) \, {}_t p_x dt$$

In the same way, let $Y_3$ be the random variable of present value of a continuous $m$ –year deferred life annuity at age $x$ of one per annum, then we have

$$Y_3 = \begin{cases} 0 & 0 \leq T < m \\ \overline{a}_{\overline{T}|} - \overline{a}_{\overline{m}|} & T \geq m \end{cases}$$

Then the actuarial present value of continuous life annuities is

$${}_{m|}\bar{a}_x^{(c)} = E(Y_3) = \int_m^{\infty} v^t \, {}_t p_x dt = \int_m^{\infty} E(v_t) \, {}_t p_x dt$$

$$= \sqrt{3}k \int_m^{\infty} e^{-\delta t} \csc(\sqrt{3}kt) \, {}_t p_x dt$$

The theorem is verified.

## 5. Example

A 42-year-old purchases a continuous 28-year deferred life annuities one per annum. And he/she defrays certain payments continuously for 10 years. Assume that death probability obeys uniform distribution in the region $[0,100]$. Then we get ${}_t p_x = 1 - t\,{}_t q_x = 1 - \frac{t}{100-x}$. Level premium can be obtained by three different ways.

In traditional theory of insurance, we suppose the force of interest $\delta = 0.05$. The formula of level premium is

$$P^{(f)} = \frac{{}_{28|}\bar{a}_{42}^{(f)}}{\bar{a}_{42:\overline{10}|}^{(f)}} = \frac{\int_{60}^{100} e^{-\delta t}\,{}_t p_x dt}{\int_{0}^{10} e^{-\delta t}\,{}_t p_x dt} = \frac{\int_{60}^{100} e^{-0.05t}(1-\frac{t}{100-42})dt}{\int_{0}^{10} e^{-0.05t}(1-\frac{t}{100-42})dt} = 0.4393$$

In random interest, we suppose the force of interest with the accumulation function is

$$\delta^{(r)}(t) = \delta t + \beta y(t) + \gamma z(t)$$

where $\{y(t), t \geq 0\}$ is a standard Wiener process. $\{z(t), t \geq 0\}$ is poisson process with the parameter $\lambda = 0.01$. And $y(t)$ and $z(t)$ are independent. $\delta = 0.05$、$\beta = 0.1$ and $\gamma = 0.05$ are mutually independent and no correlation with $t$. Then the calculation formula of level premium is

$$P^{(r)} = \frac{{}_{28|}\bar{a}_{42}^{(r)}}{\bar{a}_{42:\overline{10}|}^{(r)}} = \frac{\int_{60}^{100} e^{-\delta t + \frac{1}{2}\beta^2 t + \lambda t(e^{-\gamma}-1)}\,{}_t p_x dt}{\int_{0}^{10} e^{-\delta t + \frac{1}{2}\beta^2 t + \lambda t(e^{-\gamma}-1)}\,{}_t p_x dt} = \frac{\int_{60}^{100} e^{-0.05t + \frac{1}{2}\times 0.1^2 t + 0.01\times(e^{-0.04}-1)t}(1-\frac{t}{58})dt}{\int_{0}^{10} e^{-0.05t + \frac{1}{2}\times 0.1^2 t + 0.01\times(e^{-0.04}-1)t}(1-\frac{t}{58})dt} = 0.4878$$

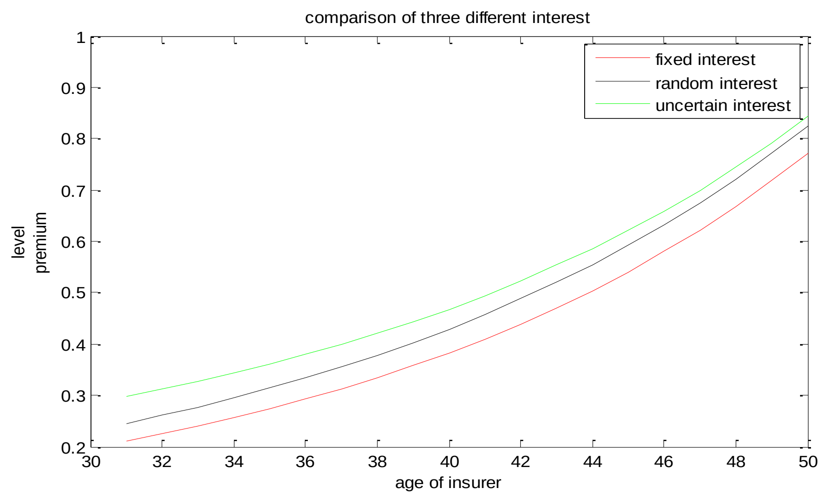
In Canonical Process Interest, we suppose the force of interest with the accumulation function is $\delta^{(c)}(t) = \delta t + kC_t$ where $C_t$ is a canonical process with expected value 0 and variance $t^2$ and $\delta = 0.05$, $k = 0.02$. Then $\pi / k\sqrt{3} = 90.6900 > 58$, the level premium is expressed as follow:

$$P^{(c)}=\frac{{}_{28|}\bar{a}_{42}^{(c)}}{\bar{a}_{42:\overline{10|}}^{(c)}}=\frac{=\sqrt{3}k\int_{60}^{100}e^{-\delta t}\csc(\sqrt{3}kt)\,{}_tp_x dt}{=\sqrt{3}k\int_{0}^{10}e^{-\delta t}\csc(\sqrt{3}kt)\,{}_tp_x dt}=\frac{\int_{60}^{100}e^{-0.05t}\csc(\sqrt{3}\times 0.02t)(1-\frac{t}{58})dt}{\int_{0}^{10}e^{-0.05t}\csc(\sqrt{3}\times 0.02t)(1-\frac{t}{58})dt}=0.5227$$

This result demonstrates that, the accuracy of level premiums in assumption of Canonical rate is higher than that in fixed and random rate. Then we take the general case into account .

A $x$ -year-old ( $30\le x\le 50$ ) purchases a continuous $(60-x)$ -year deferred life annuities one per annum. And he/she defrays certain payments continuously for 10 years. Assume that death probability obeys uniform distribution in the region $[0,100]$.

We can get the level premiums figures of three ways with MATLAB as following:




From the comparison graph, we obtain that level premiums of uncertain interest are highest in the three ways. This is important in calculating retirement pay and annuity payment.

## 6. Conclusion

In this paper, the interest rate is considered as an uncertain variable, and we suppose it satisfies canonical process. Under this hypothesis, we draw actuarial models of three continuous life annuities. During comparing them with models of fixed and random interest, we find uncertain interest has superiority to compute level premiums of retirement insurance. This research, in which we apply uncertain process to interest theory, is an additional development for traditional actuarial science.

## REFERENCES

[1] Liu, B., Fuzzy process, hybrid process and uncertain process [J], Journal of Uncertain Systems, 2008, 2(1): 3-16.

[2] Liu, B., Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty [M], Springer-Verlag, Berlin, 2010.

[3] Liu, B., Extreme value theorems of uncertain process with application to insurance risk models [J].

[4] Wang, L., Study on actuarial theory and method of life insurance under random rates of interest [D], 2004.

[5] Zhang, S., The Actuarial Mathematics of Net Single Premium Under Random Interest Rate[J], Journal of Shanghai University of Electric Power, 2007, (3):279-280.