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DENSE ENOUGH NON-STANDARD REALS

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Abstract. Non-standard analysis is a branch of Mathematics introduced by Abraham Robinson in 1966 [1]. In 1977, Edward Nelson gave an axiomatic approach to Non-standard analysis [2]. One of the main features of these approaches is the reduction of infiniteness to finiteness. In this paper we have come up with functions in the non-standard setting, especially in the extended Real number system consisting of infinitesimals and infinite numbers, that are analogous with functions in classical analysis and show that non-standard numbers are dense enough to facilitate continuity of functions.

Keywords: standard; non-standard; S-continuity; infinitesimals.

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1. INTRODUCTION

Abraham Robinson constructed a superstructure to work in any given structure like the Euclidean spaces, topological spaces, algebraic structures (rings, fields etc.,), graphs and so on. Instead, Edward Nelson restructured the axiomatics of set theory by introducing three new principles (IST) - Idealization, Standardization, Transfer - to the Zermelo Fraenkel set of axioms with the axiom of choice (ZFC). Nelson proved the consistency of the new

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system (IST + ZFC)[3]. This allows standard and non-standard elements to work within the sets.

A New Term 'Standard'

Henceforth whatever we refer to as 'classical' is anything we have come across in Mathematics so far. For instance, sets, cartesian products of sets, relations and functions studied so far, all axioms, mathematical structures and results in classical set theory (ZFC) still hold in our extended analysis - namely Non-standard analysis. Like the binary predicate ' \in ' (belongs to) and its governing rules in classical set theory, Nelson introduces a unary predicate 'standard' and gives its governing rules in the form of three axioms (IST). It is not defined by any construction within ZFC, since this new predicate is part of the new theory. This new term is applied to any element, set, relation, function etc., that is to any mathematical object[4].

Infinitesimals

The German Mathematician Leibniz treated infinitesimals as ideal numbers which are smaller in absolute value than any ordinary positive real number but obeyed all the usual laws of arithmetic. Leibniz regarded infinitesimals as a useful fiction which facilitated mathematical computation and invention. It was not until the late 19th century that an adequate definition of limit replaced the calculus of infinitesimals and provided a rigorous foundation for Analysis. Following this development, the use of infinitesimals gradually faded or rather used only as an intuitive aid to conceptualization. There the matter stood until Robinson gave a proper platform for the use of infinitesimals. Thus more than just being infinitesimal analysis, the theory developed as Non-standard analysis[5][6].

2. PRELIMINARIES

The IST axioms are presented as in [3].

Idealization axiom: Let $R = R(x, y)$ be a classical binary relation. The following two properties are equivalent :

1. For every standard and finite set F there is an $x = x_F$ such that $R(x, y)$ holds for all $y \in F$.
2. There exists an x such that $R(x, y)$ holds for all standard y .

An immediate consequence of this axiom is as follows :

Let X be any set. There exists a finite subset of X containing all standard elements of X .

In connection with this, we wish to state that we are lead into 'finiteness' argument in many a situations dealing with the infinite. For instance, for any linear space there exists a finite dimensional subspace containing all standard elements. For a sample we may have a look at a proof of this:

Let V be a linear space. Consider 'belongs to' (\in) relation on $V \times P_{fd}(V)$, where $P_{fd}(V)$ is the set of all finite dimensional subspaces of V .

For every standard finite subset $\{x_1, x_2, \dots, x_n\}$ of V , there exists

$F = \langle x_1, x_2, \dots, x_n \rangle \in P_{fd}(V)$ such that $x_i \in F$ for $i = 1, 2, \dots, n$.

By Idealization axiom, there exists $\tilde{V} \in P_{fd}(V)$ such that $x \in \tilde{V}$ for all standard x in V , completing the proof.

Analogous 'finiteness' is true of Banach spaces, Hilbert spaces etc., reducing certain questions in the infinite dimensional case to the finite one.

Standardization axiom: Let E be a standard set and P be any property(classical or not). Then there is a (unique) standard subset $A = A_P \subseteq E$ having for standard elements precisely the standard elements $x \in E$ satisfying $P(x)$.

Transfer axiom: Let A, B, \dots, L be parameters of a classical formula F having standard values. Then $\forall^s x, F(x, A, B, \dots, L) \Leftrightarrow \forall x, F(x, A, B, \dots, L)$. Consequently we have the dual statement $\exists x, F(x, A, B, \dots, L) \Leftrightarrow \exists^s x, F(x, A, B, \dots, L)$.

Definition 2.1. A real number x is called finite/limited if there exists a real number $y \in R$ such that $|x| \leq y$.

Definition 2.2. We say $x \in R$ is an infinitesimal if for every standard $y > 0$, $|x| < y$.

Definition 2.3. Two reals x and y are said to be infinitely near/close if $x - y$ is an infinitesimal and we write it as $x \simeq y$. In other words, $x - y \simeq 0$.

Definition 2.4. A real number x is called infinite/illimited if $|x| > y$ for every positive real y .

The following result will be needed in the sequel.

Result: Given any finite $x \in R$, there exists a standard real number y and an infinitesimal ε such that $x = y + \varepsilon$.

Definition 2.5. A map $f : E \rightarrow F$ is standard when the sets E, F and the graph $Gr(f)$ are standard sets.

Definition 2.6. A function $f : R \rightarrow R$ is said to be S -continuous at $x \in R$ if $y \in R$ and $y \simeq x \Rightarrow f(y) \simeq f(x)$. A function is S -continuous if it is S -continuous at all points in the domain.

3. EXAMPLES

Let us look at some examples for a better understanding. All the functions considered in the examples are from R to R and these examples are available in literature. For instance, see[3].

Example 3.1. For $0 \neq a \in R$, let $f : R \rightarrow R$ be a function defined by $f_a(x) = \frac{a}{a^2 + x^2}$.

Clearly f is continuous at all points for all $a \neq 0$. Now consider an infinitesimal a . Then $a \simeq 0$. But $f_a(a) = \frac{1}{2a}$ and $f_a(0) = \frac{1}{a}$ are both illimited and hence is $f_a(0) - f_a(a) = \frac{1}{2a}$. Therefore $a \simeq 0$ but $f_a(a)$ is not infinitely close to $f_a(0)$, which proves that f is not S -Continuous.

Example 3.2. Consider the function $f(x) = x^n$. For an illimited n , $n^{\frac{1}{n}} \simeq 1$ but $f(n^{\frac{1}{n}}) = n$ is not infinitely close to $f(1) = 1$. Hence f is continuous but not S -continuous, since f fails to be S -continuous at illimited integers.

Example 3.3. We know $sgn(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$

Let the function be $f_\varepsilon(x) = \varepsilon \cdot sgn(x)$. Clearly f_ε is discontinuous at origin when $\varepsilon \neq 0$. $|f_\varepsilon(y) - f_\varepsilon(x)| \leq 2\varepsilon$ for all x, y . If ε is an infinitesimal, then f_ε is S -continuous at all points.

Example 3.4. For a positive infinitesimal ε , the function $f(x) = \lfloor \frac{x}{\varepsilon} \rfloor \varepsilon$ is also an example which is S-continuous at all points but not continuous.

4. ANALOGOUS FUNCTIONS

In classical analysis, there exists a function f on R given by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

which is discontinuous everywhere. Now in the non standard setting we are able to construct an analogous function f on R satisfying

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is standard irrational} \\ 1 & \text{if } x \text{ is standard rational} \end{cases}$$

which happens to be continuous everywhere. This may seem to be a paradox (but not actually!), since the explicit rationals and irrationals we know are the same in both the cases. In layman's language, one can say that the later function becomes continuous because of the microscopic gaps filled thanks to the non standard reals. For the sake of completion we give the construction of the later function.

Construction : Let F be a finite subset of R containing all standard reals. Let $F = \{x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n\}$ where x_i s are rationals and y_j s are irrationals.

$$\begin{aligned} \text{Consider } f(x) = & \frac{(x-x_2)(x-x_3)\dots(x-x_m)(x-y_1)(x-y_2)\dots(x-y_n)}{(x_1-x_2)(x_1-x_3)\dots(x_1-x_m)(x_1-y_1)(x_1-y_2)\dots(x_1-y_n)} \\ & + \frac{(x-x_1)(x-x_3)\dots(x-x_m)(x-y_1)(x-y_2)\dots(x-y_n)}{(x_2-x_1)(x_2-x_3)\dots(x_2-x_m)(x_2-y_1)(x_2-y_2)\dots(x_2-y_n)} \\ & + \dots \\ & + \frac{(x-x_1)(x-x_2)\dots(x-x_{m-1})(x-y_1)(x-y_2)\dots(x-y_n)}{(x_m-x_2)(x_m-x_3)\dots(x_m-x_{m-1})(x_m-y_1)(x_m-y_2)\dots(x_m-y_n)}. \end{aligned}$$

f is a continuous function since it is a polynomial. Also f takes the value 0 at standard irrationals and the value 1 at standard rationals.

Similarly, in classical mathematics we have the famous Thomae's function $f:(0,1) \rightarrow R$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q} & \text{if } x = \frac{p}{q}, \text{ a rational in its lowest form} \end{cases}$$

which is discontinuous at all rationals and continuous at all irrationals. We now proceed in the direction of finding an analogous function (as above) in the extended system. Like the major division in the usual real line \mathbb{R} comes with rationals and irrationals, the major division in the non standard setting comes with standard and non standard real numbers. We now find a function, like Thomae's, whose values depend on whether the real number is standard or non standard. However the function is continuous everywhere, unlike Thomae's. Infact we have something more.

The function $f:(0,1) \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is standard} \\ \varepsilon & \text{if } x = y + \varepsilon \text{ with } y \text{ standard; } \varepsilon \text{ an infinitesimal} \end{cases}$$

is both S-Continuous and uniformly continuous.

Proof.

We note that there are no infinite numbers in our domain and hence every non standard number in $(0,1)$ is either an infinitesimal or of the form $y + \varepsilon$, where y is a standard real in $(0,1)$ and ε is an infinitesimal.

First we prove S-Continuity of the given function.

Let x_0 be standard. Then $f(x_0) = 0$. Let $x \simeq x_0$. Now $f(x) = 0$ or ε depending on whether x is standard or non standard ($x = y + \varepsilon$) respectively. If x is standard, $f(x) = 0 = f(x_0)$. If x is non standard, $f(x) = \varepsilon \simeq 0 = f(x_0)$. In either case, $f(x) \simeq f(x_0)$.

Let x_0 be a non standard number given by $x_0 = y_0 + \varepsilon_0$. Then $f(x_0) = \varepsilon_0$. Let $x \simeq x_0$. Now $f(x) = 0$ or ε depending on whether x is standard or non standard ($x = y + \varepsilon$) respectively. If x is standard, $f(x) = 0 \simeq \varepsilon_0 = f(x_0)$. If x is non standard $f(x) = \varepsilon \simeq \varepsilon_0 = f(x_0)$. In either case, $f(x) \simeq f(x_0)$.

Therefore f is S-Continuous.

Now we prove Uniform Continuity of f .

Let $\varepsilon > 0$ be given.

Let x_0 be standard. Note that $f(x_0) = 0$. Take $\delta = \varepsilon$.

$x \in (x_0 - \varepsilon, x_0 + \varepsilon) \Rightarrow f(x) \in (-\varepsilon, \varepsilon)$, since $f(x) = 0$ or ε' depending on whether x is standard or non standard ($x = y + \varepsilon'$) respectively with $|\varepsilon'| < \varepsilon$. Therefore $|f(x) - f(x_0)| \leq |\varepsilon'| < \varepsilon$.

Let x_0 be non standard given by $x_0 = y_0 + \varepsilon_0$ where y_0 is standard and ε_0 is an infinitesimal.

Take $\delta = \varepsilon$. For standard x , $f(x) = 0$. Hence $|f(x) - f(x_0)| = |\varepsilon_0| < \varepsilon$, by definition of an infinitesimal. For non standard x , we have $x = y_0 + \varepsilon'$ where ε' is an infinitesimal.

$|f(x) - f(x_0)| = |\varepsilon' - \varepsilon_0| < \varepsilon$, since $\varepsilon', \varepsilon_0$ are both infinitesimals.

Thus f is uniformly continuous. □

Remarks:

1. The natural way of defining f in the above proof is $f(x) = \varepsilon$, where $x = y + \varepsilon$ with x standard and ε an infinitesimal. However we preferred to write it in a superfluous manner to keep in analogy with the Thomae function.
2. Rationals are not close enough to patch up discontinuity in Thomae's function, whereas the above example shows that the nonstandard numbers in the extended setting are closely packed to warrant continuity.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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