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SOME APPLICATIONS VIA COMMON QUADRUPLE FIXED POINT THEOREMS IN G- METRIC SPACES

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Abstract. In this paper, we establish the existence of common quadruple fixed point results for (ψ,ϕ)- type contraction of six mappings in G-metric spaces. Some interesting consequences of our results are achieved. Moreover, we give an illustration which presents the applicability of the achieved results.

Keywords: quadruple fixed point; coincidence point; (ψ,ϕ)-type contraction; ω-compatible and completeness.

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1. Introduction

Fixed point theory is one of the most fruitful role in nonlinear analysis because of its wide applications in Homotopy theory, integral, integro-differential, impulsive differential equations, Approximation theory and has been studied in various metric spaces.

In 2006, Mustafa and Sims [1] initiated the concept of G-metric spaces and gave variant related fixed point results. Afterwards, many authors have developed various fixed point results on the setting of G-metric spaces ([2]-[10]).

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Very recently, E.Karapinar[11] used the notion of quadruple fixed point and proved some quadruple fixed results in partially ordered metric spaces. Subsequently, many investigators ([12]-[19]) developed quadruple fixed theorems in various metric spaces.

In this manuscript, we aim to provide some common quadruple fixed point results in G-metric spaces by using (ψ,ϕ)-type contraction. Also, we give example, applications to Integral equations and Homotopy theory.

2. Preliminaries

First, let's review the important concepts of G-metric spaces.

Definition 2.1:([1]) Let $\mathscr{P}$ be a non-empty set and let $G:\mathscr{P}\times\mathscr{P}\times\mathscr{P}\to[0,\infty)$ be a function satisfying the following properties :

- (B_0) $G(p,q,r)=0$ if $p=q=r$;
- (B_1) $0<G(p,p,q)$ for any $p,q\in\mathscr{P}$ with $p\neq q$;
- (B_2) if $G(p,p,q)\leq G(p,q,r)$ for all $p,q,r\in\mathscr{P}$ with $q\neq r$;
- (B_3) $G(p,q,r)=G(P[p,q,r])$, where P is a permutation of p,q,r (symmetry);
- (B_4) $G(p,q,r)\leq G(p,x,x)+G(x,q,r)$ for all $p,q,r,x\in\mathscr{P}$ (rectangle inequality)

then G is said to be a G-metric on $\mathscr{P}$ and pair $(\mathscr{P},G)$ is said to be a G-metric space.

Definition 2.2:([1]) A G- metric space $(\mathscr{P},G)$ is said to be symmetric if
$G(p,q,q)=G(q,p,p)$ for all $p,q\in\mathscr{P}$.

Definition 2.3:([1]) Let $\mathscr{P}$ be a G-metric space. A sequence $\{p_n\}$ in $\mathscr{P}$ is called:

- (a) G-Cauchy sequence if for every $\varepsilon>0$, there is an integer $n_0\in\mathbf{Z}^+$ such that for all $n,m,l\geq n_0$, $G(p_n,p_m,p_l)<\varepsilon$.
- (b) G-convergent to a point $p\in\mathscr{P}$ if for each $\varepsilon>0$, there is an integer $n_0\in\mathbf{Z}^+$ such that for all $n,m\geq n_0$, $G(p_n,p_m,p)<\varepsilon$.

A G-metric space on $\mathscr{P}$ is said to be G-complete if every G-Cauchy sequence in $\mathscr{P}$ is G-convergent in $\mathscr{P}$.

For more properties of a G-metric we refer the reader to ([1]).

Definition 2.4:([11]) Let $\mathscr{P}$ be a nonempty set and let $F:\mathscr{P}^4\to\mathscr{P}$ be a mapping.
If $F(p,q,r,s)=p$, $F(q,r,s,p)=q$, $F(r,s,p,q)=r$ and $F(s,p,q,r)=s$ for $p,q,r,s\in\mathscr{P}$ then (p,q,r,s) is called a Quadruple fixed point of F.

Definition 2.5: ([13]) Let $F:\mathscr{P}^4\to\mathscr{P}$ and $f:\mathscr{P}\to\mathscr{P}$ be two mappings. An element (p,q,r,s) is said to be a quadruple coincident point of F and f if

$$F(p,q,r,s)=fp,\;\; F(q,r,s,p)=fq,\;\; F(r,s,p,q)=fr \text{ and } F(s,p,q,r)=fs.$$

Definition 2.6:([13]) Let $F:\mathscr{P}^4\to\mathscr{P}$ and $f:\mathscr{P}\to\mathscr{P}$ be two mappings. An element (p,q,r,s) is said to be a quadruple common point of F and f if

$$\begin{aligned}&F(p,q,r,s)=fp=p,\;\; F(q,r,s,p)=fq=q,\\ &F(r,s,p,q)=fr=r \;\text{ and } F(s,p,q,r)=fs=s.\end{aligned}$$

Definition 2.7:([13]) Let $(\mathscr{P},G)$ be a G-metric space. A pair (F,f) is called weakly compatible if $f(F(p,q,r,s))=F(fp,fq,fr,fs)$ whenever for all $p,q,r,s\in\mathscr{P}$ such that

$$F(p,q,r,s)=fp,\;\; F(q,r,s,p)=fq,\;\; F(r,s,p,q)=fr \text{ and } F(s,p,q,r)=fs.$$

Let $\Phi=\{\phi:\phi:[0,\infty)\to[0,\infty)\}$ be a function satisfy that $\lim_{t\to a}\phi(t)>0$ for all $a>0$ and $\lim_{t\to 0^+}\phi(t)=0$.
Let $\Psi=\{\psi:\psi:[0,\infty)\to[0,\infty)\}$ be a function which satisfy

(i) $\psi(t)=0$ iff $t=0$,

(ii) ψ is continuous and nondecreasing,

(iii) $\psi(s+t)\le\psi(s)+\psi(t)$ for all $t,s\in[0,\infty)$.

Examples of typical functions ϕ and ψ are given in ([20]). The aim of this paper is to prove the following theorem.

3. MAIN RESULTS

Theorem 3.1: Let $(\mathscr{P},G)$ be a G-metric space. Suppose that $R,S,T:\mathscr{P}^4\to\mathscr{P}$ and $f,g,h:\mathscr{P}\to\mathscr{P}$ be a six mappings. Assume also that there exist $\phi\in\Phi$ and $\psi\in\Psi$ such that

$$(1)\qquad \begin{aligned}&\psi\left(G\left(R(a,b,c,d),S(x,y,z,w),T\left(p,q,r,s\right)\right)\right)\\ &\qquad\le\frac{1}{4}\psi\left(G(ga,hx,fp)+G(gb,hy,fq)+G(gc,hz,fr)+G(gd,hw,fs)\right)\\ &\qquad-\phi\left(G(ga,hx,fp)+G(gb,hy,fq)+G(gc,hz,fr)+G(gd,hw,fs)\right).\end{aligned}$$

for all $a,b,c,d,x,y,z,w,p,q,r,s\in\mathscr{P}$.

$a)$ $R(\mathscr{P}^4)\subseteq f(\mathscr{P})$, $S(\mathscr{P}^4)\subseteq g(\mathscr{P})$ and $T(\mathscr{P}^4)\subseteq h(\mathscr{P})$

$b)$ Either (R,g) or (S,h) or (T,f) are ω-compatible.

$c)$ one of $g(\mathscr{P})$, $h(\mathscr{P})$ or $f(\mathscr{P})$ is complete.

Then there is a unique common quadruple fixed point of R,S,T,f,g and h in $\mathscr{P}$.

Proof. Let $a,b,c,d\in\mathscr{P}$ be arbitrary, and from (a), we construct the sequences $\{a_{3n}\},\{b_{3n}\}$, $\{c_{3n}\},\{d_{3n}\}$, $\{\alpha_{3n}\},\{\beta_{3n}\},\{\gamma_{3n}\},\{\omega_{3n}\}$, in $\mathscr{P}$ as

$$R\left(a_{3n},b_{3n},c_{3n},d_{3n}\right)=fa_{3n+1}=\alpha_{3n},\quad R\left(b_{3n},c_{3n},d_{3n},a_{3n}\right)=fb_{3n+1}=\beta_{3n},$$

$$R\left(c_{3n},d_{3n},a_{3n},b_{3n}\right)=fc_{3n+1}=\gamma_{3n},\quad R\left(d_{3n},a_{3n},b_{3n},c_{3n},\right)=fd_{3n+1}=\omega_{3n},$$

$$S\left(a_{3n+1},b_{3n+1},c_{3n+1},d_{3n+1}\right)=ga_{3n+2}=\alpha_{3n+1},$$

$$S\left(b_{3n+1},c_{3n+1},d_{3n+1},a_{3n+1}\right)=gb_{3n+2}=\beta_{3n+1},$$

$$S\left(c_{3n+1},d_{3n+1},a_{3n+1},b_{3n+1}\right)=gc_{3n+2}=\gamma_{3n+1},$$

$$S\left(d_{3n+1},a_{3n+1},b_{3n+1},c_{3n+1},\right)=gd_{3n+2}=\omega_{3n+1},$$

$$T\left(a_{3n+2},b_{3n+2},c_{3n+2},d_{3n+2}\right)=ha_{3n+3}=\alpha_{3n+2},$$

$$T\left(b_{3n+2},c_{3n+2},d_{3n+2},a_{3n+2}\right)=hb_{3n+3}=\beta_{3n+2},$$

$$T\left(c_{3n+2},d_{3n+2},a_{3n+2},b_{3n+2}\right)=hc_{3n+3}=\gamma_{3n+2},$$

$$T\left(d_{3n+2},a_{3n+2},b_{3n+2},c_{3n+2}\right)=hd_{3n+3}=\omega_{3n+2}\text{ where }n=0,1,2,\ldots.$$

Then from (1), we can get

$$\begin{aligned}
&\psi\left(G\left(\alpha_{3n},\alpha_{3n+1},\alpha_{3n+2}\right)\right)\\
= \; &\psi\left[G\left(\begin{array}{c} R\left(a_{3n},b_{3n},c_{3n},d_{3n}\right),S\left(a_{3n+1},b_{3n+1},c_{3n+1},d_{3n+1}\right),\\ T\left(a_{3n+2},b_{3n+2},c_{3n+2},d_{3n+2}\right)\end{array}\right)\right]\\
\leq \; &\frac{1}{4}\psi\left[\begin{array}{c} G(ga_{3n},ha_{3n+1},fa_{3n+2})+G(gb_{3n},hb_{3n+1},fb_{3n+2})\\ +G(gc_{3n},hc_{3n+1},fc_{3n+2})+G(gd_{3n},hd_{3n+1},fd_{3n+2})\end{array}\right]\\
&-\phi\left[\begin{array}{c} G(ga_{3n},ha_{3n+1},fa_{3n+2})+G(gb_{3n},hb_{3n+1},fb_{3n+2})\\ +G(gc_{3n},hc_{3n+1},fc_{3n+2})+G(gd_{3n},hd_{3n+1},fd_{3n+2})\end{array}\right]
\end{aligned}$$

$$\begin{aligned}
\leq \; &\frac{1}{4}\psi\left[\begin{array}{c} G\left(\alpha_{3n-1},\alpha_{3n},\alpha_{3n+1}\right)+G\left(\beta_{3n-1},\beta_{3n},\beta_{3n+1}\right)\\ +G\left(\gamma_{3n-1},\gamma_{3n},\gamma_{3n+1}\right)+G\left(\omega_{3n-1},\omega_{3n},\omega_{3n+1}\right)\end{array}\right]\\
&-\phi\left[\begin{array}{c} G\left(\alpha_{3n-1},\alpha_{3n},\alpha_{3n+1}\right)+G\left(\beta_{3n-1},\beta_{3n},\beta_{3n+1}\right)\\ +G\left(\gamma_{3n-1},\gamma_{3n},\gamma_{3n+1}\right)+G\left(\omega_{3n-1},\omega_{3n},\omega_{3n+1}\right)\end{array}\right], \qquad (2)
\end{aligned}$$

and

$$\begin{aligned}
&\psi\left(G\left(\beta_{3n},\beta_{3n+1},\beta_{3n+2}\right)\right)\\
= \; &\psi\left[G\left(\begin{array}{c} R\left(b_{3n},c_{3n},d_{3n},a_{3n}\right),S\left(b_{3n+1},c_{3n+1},d_{3n+1},a_{3n+1}\right),\\ T\left(b_{3n+2},c_{3n+2},d_{3n+2},a_{3n+2}\right)\end{array}\right)\right]\\
\leq \; &\frac{1}{4}\psi\left[\begin{array}{c} G(gb_{3n},hb_{3n+1},fb_{3n+2})+G(gc_{3n},hc_{3n+1},fc_{3n+2})\\ +G(gd_{3n},hd_{3n+1},fd_{3n+2})+G(ga_{3n},ha_{3n+1},fa_{3n+2})\end{array}\right]\\
&-\phi\left[\begin{array}{c} G(gb_{3n},hb_{3n+1},fb_{3n+2})+G(gc_{3n},hc_{3n+1},fc_{3n+2})\\ +G(gd_{3n},hd_{3n+1},fd_{3n+2})+G(ga_{3n},ha_{3n+1},fa_{3n+2})\end{array}\right]\\
\leq \; &\frac{1}{4}\psi\left[\begin{array}{c} G\left(\beta_{3n-1},\beta_{3n},\beta_{3n+1}\right)+G\left(\gamma_{3n-1},\gamma_{3n},\gamma_{3n+1}\right)\\ +G\left(\omega_{3n-1},\omega_{3n},\omega_{3n+1}\right)+G\left(\alpha_{3n-1},\alpha_{3n},\alpha_{3n+1}\right)\end{array}\right]\\
&-\phi\left[\begin{array}{c} G\left(\beta_{3n-1},\beta_{3n},\beta_{3n+1}\right)+G\left(\gamma_{3n-1},\gamma_{3n},\gamma_{3n+1}\right)\\ +G\left(\omega_{3n-1},\omega_{3n},\omega_{3n+1}\right)+G\left(\alpha_{3n-1},\alpha_{3n},\alpha_{3n+1}\right)\end{array}\right]. \qquad (3)
\end{aligned}$$

Similarly, we can prove

$$\begin{aligned} &\psi(G(\gamma_{3n},\gamma_{3n+1},\gamma_{3n+2})) \\ \leq\ & \frac{1}{4}\psi\left[\begin{array}{c} G(\gamma_{3n-1},\gamma_{3n},\gamma_{3n+1})+G(\omega_{3n-1},\omega_{3n},\omega_{3n+1}) \\ +G(\alpha_{3n-1},\alpha_{3n},\alpha_{3n+1})+G(\beta_{3n-1},\beta_{3n},\beta_{3n+1}) \end{array}\right] \\ & -\phi\left[\begin{array}{c} G(\gamma_{3n-1},\gamma_{3n},\gamma_{3n+1})+G(\omega_{3n-1},\omega_{3n},\omega_{3n+1}) \\ +G(\alpha_{3n-1},\alpha_{3n},\alpha_{3n+1})+G(\beta_{3n-1},\beta_{3n},\beta_{3n+1}) \end{array}\right], \end{aligned} \tag{4}$$

also,

$$\begin{aligned} &\psi(G(\omega_{3n},\omega_{3n+1},\omega_{3n+2})) \\ \leq\ & \frac{1}{4}\psi\left[\begin{array}{c} G(\omega_{3n-1},\omega_{3n},\omega_{3n+1})+G(\alpha_{3n-1},\alpha_{3n},\alpha_{3n+1}) \\ +G(\beta_{3n-1},\beta_{3n},\beta_{3n+1})+G(\gamma_{3n-1},\gamma_{3n},\gamma_{3n+1}) \end{array}\right] \\ & -\phi\left[\begin{array}{c} G(\omega_{3n-1},\omega_{3n},\omega_{3n+1})+G(\alpha_{3n-1},\alpha_{3n},\alpha_{3n+1}) \\ +G(\beta_{3n-1},\beta_{3n},\beta_{3n+1})+G(\gamma_{3n-1},\gamma_{3n},\gamma_{3n+1}) \end{array}\right]. \end{aligned} \tag{5}$$

Due to $(2)-(5)$, we conclude that

$$\begin{aligned} &\psi(G(\alpha_{3n},\alpha_{3n+1},\alpha_{3n+2}))+\psi(G(\beta_{3n},\beta_{3n+1},\beta_{3n+2})) \\ &+\psi(G(\gamma_{3n},\gamma_{3n+1},\gamma_{3n+2}))+\psi(G(\omega_{3n},\omega_{3n+1},\omega_{3n+2})) \\ \leq\ & \psi\left[\begin{array}{c} G(\alpha_{3n-1},\alpha_{3n},\alpha_{3n+1})+G(\beta_{3n-1},\beta_{3n},\beta_{3n+1}) \\ +G(\gamma_{3n-1},\gamma_{3n},\gamma_{3n+1})+G(\omega_{3n-1},\omega_{3n},\omega_{3n+1}) \end{array}\right] \\ & -4\phi\left[\begin{array}{c} G(\alpha_{3n-1},\alpha_{3n},\alpha_{3n+1})+G(\beta_{3n-1},\beta_{3n},\beta_{3n+1}) \\ +G(\gamma_{3n-1},\gamma_{3n},\gamma_{3n+1})+G(\omega_{3n-1},\omega_{3n},\omega_{3n+1}) \end{array}\right]. \end{aligned} \tag{6}$$

From the property (iii) of ψ, we have

$$\begin{aligned} &\psi\left(\begin{array}{c} G(\alpha_{3n},\alpha_{3n+1},\alpha_{3n+2})+G(\beta_{3n},\beta_{3n+1},\beta_{3n+2}) \\ +G(\gamma_{3n},\gamma_{3n+1},\gamma_{3n+2})+G(\omega_{3n},\omega_{3n+1},\omega_{3n+2}) \end{array}\right) \\ \leq\ & \psi(G(\alpha_{3n},\alpha_{3n+1},\alpha_{3n+2}))+\psi(G(\beta_{3n},\beta_{3n+1},\beta_{3n+2})) \\ & +\psi(G(\gamma_{3n},\gamma_{3n+1},\gamma_{3n+2}))+\psi(G(\omega_{3n},\omega_{3n+1},\omega_{3n+2})). \end{aligned} \tag{7}$$

Combining with (6) and (7), we get that

$$\psi\begin{pmatrix} G(\alpha_{3n},\alpha_{3n+1},\alpha_{3n+2})+G(\beta_{3n},\beta_{3n+1},\beta_{3n+2}) \\ +G(\gamma_{3n},\gamma_{3n+1},\gamma_{3n+2})+G(\omega_{3n},\omega_{3n+1},\omega_{3n+2}) \end{pmatrix}$$

$$\le \psi\begin{bmatrix} G(\alpha_{3n-1},\alpha_{3n},\alpha_{3n+1})+G(\beta_{3n-1},\beta_{3n},\beta_{3n+1}) \\ +G(\gamma_{3n-1},\gamma_{3n},\gamma_{3n+1})+G(\omega_{3n-1},\omega_{3n},\omega_{3n+1}) \end{bmatrix}$$

$$-4\phi\begin{bmatrix} G(\alpha_{3n-1},\alpha_{3n},\alpha_{3n+1})+G(\beta_{3n-1},\beta_{3n},\beta_{3n+1}) \\ +G(\gamma_{3n-1},\gamma_{3n},\gamma_{3n+1})+G(\omega_{3n-1},\omega_{3n},\omega_{3n+1}) \end{bmatrix}. \tag{8}$$

Set

$$\delta_{3n}=\begin{pmatrix} G(\alpha_{3n},\alpha_{3n+1},\alpha_{3n+2})+G(\beta_{3n},\beta_{3n+1},\beta_{3n+2}) \\ +G(\gamma_{3n},\gamma_{3n+1},\gamma_{3n+2})+G(\omega_{3n},\omega_{3n+1},\omega_{3n+2}) \end{pmatrix}.$$

Then we have

$$\psi(\delta_{3n})\le\psi(\delta_{3n-1})-4\phi(\delta_{3n-1}) \text{ for all } n, \tag{9}$$

which yields that

$$\psi(\delta_{3n})\le\psi(\delta_{3n-1}) \text{ for all } n.$$

Obviously, for any $n\in\mathbf{N}$, we obtain

$$\psi(\delta_n)\le\psi(\delta_{n-1}).$$

Since ψ is non-decreasing, we get that $\delta_n\le\delta_{n-1}$ for all n. Hence $\{\delta_n\}$ is a non-increasing sequence. Since it is bounded below from 0, there is some $\delta>0$ such that

$$\lim_{n\to\infty}\delta_n=\delta. \tag{10}$$

We will show that $\delta=0$. Suppose, on contrary, that $\delta>0$. Letting $n\to\infty$ in (9) and having in mind that we suppose that $\lim_{t\to\zeta}\phi(t)>0$ for all $\zeta>0$ and $\lim_{t\to 0^+}\phi(t)=0$, we have

$$\psi(\delta)\le\psi(\delta)-4\phi(\delta)<\psi(\delta),$$

which is a contraction. Thus, $\delta = 0$, that is

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \begin{pmatrix} G(\alpha_{3n}, \alpha_{3n+1}, \alpha_{3n+2}) + G(\beta_{3n}, \beta_{3n+1}, \beta_{3n+2}) \\ +G(\gamma_{3n}, \gamma_{3n+1}, \gamma_{3n+2}) + G(\omega_{3n}, \omega_{3n+1}, \omega_{3n+2}) \end{pmatrix} = 0. \tag{11}$$

Now, we shall show that $\{\alpha_{3n}\}, \{\beta_{3n}\}, \{\gamma_{3n}\}, \{\omega_{3n}\}$ are Cauchy sequences in the G-metric space $(\mathscr{P}, G)$. Assume the contrary, that is, one of the sequences $\{\alpha_{3n}\}, \{\beta_{3n}\}, \{\gamma_{3n}\}$ or $\{\omega_{3n}\}$ is not a Cauchy sequence, that is

$$\lim_{n,m\to\infty} G(\alpha_{3m}, \alpha_{3n}, \alpha_{3n}) \neq 0 \text{ or } \lim_{n,m\to\infty} G(\beta_{3m}, \beta_{3n}, \beta_{3n}) \neq 0$$
$$\lim_{n,m\to\infty} G(\gamma_{3m}, \gamma_{3n}, \gamma_{3n}) \neq 0 \text{ or } \lim_{n,m\to\infty} G(\omega_{3m}, \omega_{3n}, \omega_{3n}) \neq 0.$$

This means that there exists $\varepsilon > 0$, for which we can find subsequences $\{\alpha_{3n_k}\}$, $\{\alpha_{3m_k}\}$ of $\{\alpha_{3n}\}$, $\{\beta_{3n_k}\}$, $\{\beta_{3m_k}\}$ of $\{\beta_{3n}\}$, $\{\gamma_{3n_k}\}$, $\{\gamma_{3m_k}\}$ of $\{\gamma_{3n}\}$ and $\{\omega_{3n_k}\}$, $\{\omega_{3m_k}\}$ of $\{\omega_{3n}\}$ with $n_k > m_k \geq k$ such that

$$\begin{aligned} & G\left(\alpha_{3m_k}, \alpha_{3n_k}, \alpha_{3n_k}\right) + G\left(\beta_{3m_k}, \beta_{3n_k}, \beta_{3n_k}\right) \\ & +G\left(\gamma_{3m_k}, \gamma_{3n_k}, \gamma_{3n_k}\right) + G\left(\omega_{3m_k}, \omega_{3n_k}, \omega_{3n_k}\right) \geq \varepsilon. \end{aligned} \tag{12}$$

In addition, by virtue of m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k \geq k$ satisfying (12). It follows that

$$\begin{aligned} & G\left(\alpha_{3m_k}, \alpha_{3n_{k-1}}, \alpha_{3n_{k-1}}\right) + G\left(\beta_{3m_k}, \beta_{3n_{k-1}}, \beta_{3n_{k-1}}\right) \\ & +G\left(\gamma_{3m_k}, \gamma_{3n_{k-1}}, \gamma_{3n_{k-1}}\right) + G\left(\omega_{3m_k}, \omega_{3n_{k-1}}, \omega_{3n_{k-1}}\right) < \varepsilon. \end{aligned} \tag{13}$$

By use of the rectangle inequality, we have

$$\begin{aligned} & G\left(\alpha_{3m_k}, \alpha_{3n_k}, \alpha_{3n_k}\right) \leq G\left(\alpha_{3m_k}, \alpha_{3n_{k-1}}, \alpha_{3n_{k-1}}\right) + G\left(\alpha_{3n_{k-1}}, \alpha_{3n_k}, \alpha_{3n_k}\right), \\ & G\left(\beta_{3m_k}, \beta_{3n_k}, \beta_{3n_k}\right) \leq G\left(\beta_{3m_k}, \beta_{3n_{k-1}}, \beta_{3n_{k-1}}\right) + G\left(\beta_{3n_{k-1}}, \beta_{3n_k}, \beta_{3n_k}\right), \\ & G\left(\gamma_{3m_k}, \gamma_{3n_k}, \gamma_{3n_k}\right) \leq G\left(\gamma_{3m_k}, \gamma_{3n_{k-1}}, \gamma_{3n_{k-1}}\right) + G\left(\gamma_{3n_{k-1}}, \gamma_{3n_k}, \gamma_{3n_k}\right), \\ & G\left(\omega_{3m_k}, \omega_{3n_k}, \omega_{3n_k}\right) \leq G\left(\omega_{3m_k}, \omega_{3n_{k-1}}, \omega_{3n_{k-1}}\right) + G\left(\omega_{3n_{k-1}}, \omega_{3n_k}, \omega_{3n_k}\right). \end{aligned}$$

(14)

Adding both sides to (14) and using (12) and (13), we have that

$$\begin{aligned}
\varepsilon &\leq G\left(\alpha_{3m_k},\alpha_{3n_k},\alpha_{3n_k}\right)+G\left(\beta_{3m_k},\beta_{3n_k},\beta_{3n_k}\right)\\
&\quad +G\left(\gamma_{3m_k},\gamma_{3n_k},\gamma_{3n_k}\right)+G\left(\omega_{3m_k},\omega_{3n_k},\omega_{3n_k}\right)\\
&\leq G\left(\alpha_{3m_k},\alpha_{3n_{k-1}},\alpha_{3n_{k-1}}\right)+G\left(\alpha_{3n_{k-1}},\alpha_{3n_k},\alpha_{3n_k}\right)\\
&\quad +G\left(\beta_{3m_k},\beta_{3n_{k-1}},\beta_{3n_{k-1}}\right)+G\left(\beta_{3n_{k-1}},\beta_{3n_k},\beta_{3n_k}\right)\\
&\quad +G\left(\gamma_{3m_k},\gamma_{3n_{k-1}},\gamma_{3n_{k-1}}\right)+G\left(\gamma_{3n_{k-1}},\gamma_{3n_k},\gamma_{3n_k}\right)\\
&\quad +G\left(\omega_{3m_k},\omega_{3n_{k-1}},\omega_{3n_{k-1}}\right)+G\left(\omega_{3n_{k-1}},\omega_{3n_k},\omega_{3n_k}\right)\\
&\leq \varepsilon+G\left(\alpha_{3n_{k-1}},\alpha_{3n_k},\alpha_{3n_k}\right)+G\left(\beta_{3n_{k-1}},\beta_{3n_k},\beta_{3n_k}\right)\\
&\quad +G\left(\gamma_{3n_{k-1}},\gamma_{3n_k},\gamma_{3n_k}\right)+G\left(\omega_{3n_{k-1}},\omega_{3n_k},\omega_{3n_k}\right).
\end{aligned}$$

Letting $k\to\infty$ and by use of (11), we get

$$\lim_{k\to\infty}\zeta_k=\lim_{k\to\infty}\left(\begin{array}{c} G\left(\alpha_{3m_k},\alpha_{3n_k},\alpha_{3n_k}\right)+G\left(\beta_{3m_k},\beta_{3n_k},\beta_{3n_k}\right)\\ +G\left(\gamma_{3m_k},\gamma_{3n_k},\gamma_{3n_k}\right)+G\left(\omega_{3m_k},\omega_{3n_k},\omega_{3n_k}\right)\end{array}\right)=\varepsilon.$$

Again, by the rectangle inequality, we have

$$\begin{aligned}
\zeta_k &= G\left(\alpha_{3m_k},\alpha_{3n_k},\alpha_{3n_k}\right)+G\left(\beta_{3m_k},\beta_{3n_k},\beta_{3n_k}\right)\\
&\quad +G\left(\gamma_{3m_k},\gamma_{3n_k},\gamma_{3n_k}\right)+G\left(\omega_{3m_k},\omega_{3n_k},\omega_{3n_k}\right)\\
&\leq G\left(\alpha_{3m_k},\alpha_{3m_{k+1}},\alpha_{3m_{k+1}}\right)+G\left(\alpha_{3m_{k+1}},\alpha_{3n_{k+1}},\alpha_{3n_{k+1}}\right)+G\left(\alpha_{3n_{k+1}},\alpha_{3n_k},\alpha_{3n_k}\right)\\
&\quad +G\left(\beta_{3m_k},\beta_{3m_{k+1}},\beta_{3m_{k+1}}\right)+G\left(\beta_{3m_{k+1}},\beta_{3n_{k+1}},\beta_{3n_{k+1}}\right)+G\left(\beta_{3n_{k+1}},\beta_{3n_k},\beta_{3n_k}\right)\\
&\quad +G\left(\gamma_{3m_k},\gamma_{3m_{k+1}},\gamma_{3m_{k+1}}\right)+G\left(\gamma_{3m_{k+1}},\gamma_{3n_{k+1}},\gamma_{3n_{k+1}}\right)+G\left(\gamma_{3n_{k+1}},\gamma_{3n_k},\gamma_{3n_k}\right)\\
&\quad +G\left(\omega_{3m_k},\omega_{3m_{k+1}},\omega_{3m_{k+1}}\right)+G\left(\omega_{3m_{k+1}},\omega_{3n_{k+1}},\omega_{3n_{k+1}}\right)+G\left(\omega_{3n_{k+1}},\omega_{3n_k},\omega_{3n_k}\right)\\
&\leq \delta_{3m_k}+\delta_{3n_k}+G\left(\alpha_{3m_{k+1}},\alpha_{3n_{k+1}},\alpha_{3n_{k+1}}\right)+G\left(\beta_{3m_{k+1}},\beta_{3n_{k+1}},\beta_{3n_{k+1}}\right)\\
&\quad +G\left(\gamma_{3m_{k+1}},\gamma_{3n_{k+1}},\gamma_{3n_{k+1}}\right)+G\left(\omega_{3m_{k+1}},\omega_{3n_{k+1}},\omega_{3n_{k+1}}\right). \qquad (15)
\end{aligned}$$

Since $n_k\geq m_k$, then

$$\alpha_{3m_k}\leq\alpha_{3n_k}\ \ \beta_{3m_k}\leq\beta_{3n_k}\ \ \gamma_{3m_k}\leq\gamma_{3n_k}\ \ \omega_{3m_k}\leq\omega_{3n_k}. \qquad (16)$$

Hence from (1) and (15), we get

$$\begin{aligned}
&\psi\left(G\left(\alpha_{3m_{k+1}},\alpha_{3n_{k+1}},\alpha_{3n_{k+1}}\right)\right)\\
&= \psi\left[G\left(\begin{array}{c} R\left(a_{3m_{k+1}},b_{3m_{k+1}},c_{3mk+1},d_{3m_{k+1}}\right),S\left(a_{3n_{k+1}},b_{3n_{k+1}},c_{3n_{k+1}},d_{3n_{k+1}}\right),\\ T\left(a_{3n_{k+1}},b_{3n_{k+1}},c_{3n_{k+1}},d_{3n_{k+1}}\right)\end{array}\right)\right]\\
&\leq \frac{1}{4}\psi\left[\begin{array}{c} G(ga_{3m_{k+1}},ha_{3n_{k+1}},fa_{3n_{k+1}})+G(gb_{3m_{k+1}},hb_{3n_{k+1}},fb_{3n_{k+1}})\\ +G(gc_{3m_{k+1}},hc_{3n_{k+1}},fc_{3n_{k+1}})+G(gd_{3m_{k+1}},hd_{3n_{k+1}},fd_{3n_{k+1}})\end{array}\right]\\
&\quad -\phi\left[\begin{array}{c} G(ga_{3m_{k+1}},ha_{3n_{k+1}},fa_{3n_{k+1}})+G(gb_{3m_{k+1}},hb_{3n_{k+1}},fb_{3n_{k+1}})\\ +G(gc_{3m_{k+1}},hc_{3n_{k+1}},fc_{3n_{k+1}})+G(gd_{3m_{k+1}},hd_{3n_{k+1}},fd_{3n_{k+1}})\end{array}\right]\\
&\leq \frac{1}{4}\psi\left[\begin{array}{c} G\left(\alpha_{3m_k},\alpha_{3n_k},\alpha_{3n_k}\right)+G\left(\beta_{3m_k},\beta_{3n_k},\beta_{3n_k}\right)\\ +G\left(\gamma_{3m_k},\gamma_{3n_k},\gamma_{3n_k}\right)+G\left(\omega_{3m_k},\omega_{3n_k},\omega_{3n_k}\right)\end{array}\right]\\
&\quad -\phi\left[\begin{array}{c} G\left(\alpha_{3m_k},\alpha_{3n_k},\alpha_{3n_k}\right)+G\left(\beta_{3m_k},\beta_{3n_k},\beta_{3n_k}\right)\\ +G\left(\gamma_{3m_k},\gamma_{3n_k},\gamma_{3n_k}\right)+G\left(\omega_{3m_k},\omega_{3n_k},\omega_{3n_k}\right)\end{array}\right].
\end{aligned}\tag{17}$$

Similarly, we can prove

$$\begin{aligned}
&\psi\left(G\left(\beta_{3m_{k+1}},\beta_{3n_{k+1}},\beta_{3n_{k+1}}\right)\right)\\
&\leq \frac{1}{4}\psi\left[\begin{array}{c} G\left(\beta_{3m_k},\beta_{3n_k},\beta_{3n_k}\right)+G\left(\gamma_{3m_k},\gamma_{3n_k},\gamma_{3n_k}\right)\\ +G\left(\omega_{3m_k},\omega_{3n_k},\omega_{3n_k}\right)+G\left(\alpha_{3m_k},\alpha_{3n_k},\alpha_{3n_k}\right)\end{array}\right]\\
&\quad -\phi\left[\begin{array}{c} G\left(\beta_{3m_k},\beta_{3n_k},\beta_{3n_k}\right)+G\left(\gamma_{3m_k},\gamma_{3n_k},\gamma_{3n_k}\right)\\ +G\left(\omega_{3m_k},\omega_{3n_k},\omega_{3n_k}\right)+G\left(\alpha_{3m_k},\alpha_{3n_k},\alpha_{3n_k}\right)\end{array}\right],
\end{aligned}\tag{18}$$

$$\begin{aligned}
&\psi\left(G\left(\gamma_{3m_{k+1}},\gamma_{3n_{k+1}},\gamma_{3n_{k+1}}\right)\right)\\
&\leq \frac{1}{4}\psi\left[\begin{array}{c} G\left(\gamma_{3m_k},\gamma_{3n_k},\gamma_{3n_k}\right)+G\left(\omega_{3m_k},\omega_{3n_k},\omega_{3n_k}\right)\\ +G\left(\alpha_{3m_k},\alpha_{3n_k},\alpha_{3n_k}\right)+G\left(\beta_{3m_k},\beta_{3n_k},\beta_{3n_k}\right)\end{array}\right]\\
&\quad -\phi\left[\begin{array}{c} G\left(\gamma_{3m_k},\gamma_{3n_k},\gamma_{3n_k}\right)+G\left(\omega_{3m_k},\omega_{3n_k},\omega_{3n_k}\right)\\ +G\left(\alpha_{3m_k},\alpha_{3n_k},\alpha_{3n_k}\right)+G\left(\beta_{3m_k},\beta_{3n_k},\beta_{3n_k}\right)\end{array}\right],
\end{aligned}\tag{19}$$

$$\begin{aligned}
&\psi\left(G\left(\omega_{3m_{k+1}},\omega_{3n_{k+1}},\omega_{3n_{k+1}}\right)\right)\\
\leq\ & \frac{1}{4}\psi\begin{bmatrix} G\left(\omega_{3m_k},\omega_{3n_k},\omega_{3n_k}\right)+G\left(\alpha_{3m_k},\alpha_{3n_k},\alpha_{3n_k}\right)\\ +G\left(\beta_{3m_k},\beta_{3n_k},\beta_{3n_k}\right)+G\left(\gamma_{3m_k},\gamma_{3n_k},\gamma_{3n_k}\right)\end{bmatrix}\\
&-\phi\begin{bmatrix} G\left(\omega_{3m_k},\omega_{3n_k},\omega_{3n_k}\right)+G\left(\alpha_{3m_k},\alpha_{3n_k},\alpha_{3n_k}\right)\\ +G\left(\beta_{3m_k},\beta_{3n_k},\beta_{3n_k}\right)+G\left(\gamma_{3m_k},\gamma_{3n_k},\gamma_{3n_k}\right)\end{bmatrix}. \quad (20)
\end{aligned}$$

Combining (15) and $(17)-(20)$

$$\begin{aligned}
\psi(\zeta_k) &\leq \psi\begin{pmatrix} \delta_{3m_k}+\delta_{3n_k}\\ +G\left(\alpha_{3m_{k+1}},\alpha_{3n_{k+1}},\alpha_{3n_{k+1}}\right)+G\left(\beta_{3m_{k+1}},\beta_{3n_{k+1}},\beta_{3n_{k+1}}\right)\\ +G\left(\gamma_{3m_{k+1}},\gamma_{3n_{k+1}},\gamma_{3n_{k+1}}\right)+G\left(\omega_{3m_{k+1}},\omega_{3n_{k+1}},\omega_{3n_{k+1}}\right)\end{pmatrix}.\\
&\leq \psi(\delta_{3m_k})+\psi(\delta_{3n_k})+\psi(\zeta_k)-4\phi(\zeta_k).
\end{aligned}$$

Letting $k\to\infty$, we get a contradiction. This shows that $\{\alpha_{3n}\},\{\beta_{3n}\},\{\gamma_{3n}\},\{\omega_{3n}\}$ are Cauchy sequences in the G-metric space $(\mathscr{P},G)$. Suppose $g(\mathscr{P})$ is complete subspace of $(\mathscr{P},G)$, then the sequences $\{\alpha_{3n}\},\{\beta_{3n}\},\{\gamma_{3n}\},\{\omega_{3n}\}$ are convergence to α,β,γ and ω respectively in $g(\mathscr{P})$. Thus, there exist $a,b,c,d\in g(\mathscr{P})$ such that

$$\begin{aligned}
&\lim_{n\to\infty}\alpha_{3n}=\alpha=ga \quad \lim_{n\to\infty}\beta_{3n}=\beta=gb\\
&\lim_{n\to\infty}\gamma_{3n}=\gamma=gc \quad \lim_{n\to\infty}\omega_{3n}=\omega=gd. \quad (21)
\end{aligned}$$

We claim that $R(a,b,c,d)=\alpha, R(b,c,d,a)=\beta, R(c,d,a,b)=\gamma, R(d,a,b,c)=\omega$. By using (1), we have

$$\begin{aligned}
&\psi\left(G\left(R(a,b,c,d),\alpha_{3n+1},\alpha_{3n+2}\right)\right)\\
=\ &\psi\left(G\left(R(a,b,c,d),S(a_{3n+1},b_{3n+1},c_{3n+1},d_{3n+1}),T(a_{3n+2},b_{3n+2},c_{3n+2},d_{3n+2})\right)\right)
\end{aligned}$$

$$\leq \frac{1}{4}\psi\left[\begin{array}{c} G(ga,ha_{3n+1},fa_{3n+2})+G(gb,hb_{3n+1},fb_{3n+2}) \\ +G(gc,hc_{3n+1},fc_{3n+2})+G(gd,hd_{3n+1},fd_{3n+2}) \end{array}\right]$$

$$-\phi\left[\begin{array}{c} G(ga,ha_{3n+1},fa_{3n+2})+G(gb,hb_{3n+1},fb_{3n+2}) \\ +G(gc,hc_{3n+1},fc_{3n+2})+G(gd,hd_{3n+1},fd_{3n+2}) \end{array}\right]$$

$$\leq \frac{1}{4}\psi\left[\begin{array}{c} G(ga,\alpha_{3n},\alpha_{3n+1})+G(gb,\beta_{3n},\beta_{3n+1}) \\ +G(gc,\gamma_{3n},\gamma_{3n+1})+G(gd,\omega_{3n},\omega_{3n+1}) \end{array}\right]$$

$$-\phi\left[\begin{array}{c} G(ga,\alpha_{3n},\alpha_{3n+1})+G(gb,\beta_{3n},\beta_{3n+1}) \\ +G(gc,\gamma_{3n},\gamma_{3n+1})+G(gd,\omega_{3n},\omega_{3n+1}) \end{array}\right]. \tag{22}$$

Similarly, we can prove that

$$\psi(G(R(b,c,d,a),\beta_{3n+1},\beta_{3n+2}))$$

$$\leq \frac{1}{4}\psi\left[\begin{array}{c} G(gb,\beta_{3n},\beta_{3n+1})+G(gc,\gamma_{3n},\gamma_{3n+1}) \\ +G(gd,\omega_{3n},\omega_{3n+1})+G(ga,\alpha_{3n},\alpha_{3n+1}) \end{array}\right]$$

$$-\phi\left[\begin{array}{c} G(gb,\beta_{3n},\beta_{3n+1})+G(gc,\gamma_{3n},\gamma_{3n+1}) \\ +G(gd,\omega_{3n},\omega_{3n+1})+G(ga,\alpha_{3n},\alpha_{3n+1}) \end{array}\right]. \tag{23}$$

$$\psi(G(R(c,d,a,b),\gamma_{3n+1},\gamma_{3n+2}))$$

$$\leq \frac{1}{4}\psi\left[\begin{array}{c} G(gc,\gamma_{3n},\gamma_{3n+1})+G(gd,\omega_{3n},\omega_{3n+1}) \\ +G(ga,\alpha_{3n},\alpha_{3n+1})+G(gb,\beta_{3n},\beta_{3n+1}) \end{array}\right]$$

$$-\phi\left[\begin{array}{c} G(gc,\gamma_{3n},\gamma_{3n+1})+G(gd,\omega_{3n},\omega_{3n+1}) \\ +G(ga,\alpha_{3n},\alpha_{3n+1})+G(gb,\beta_{3n},\beta_{3n+1}) \end{array}\right]. \tag{24}$$

$$\psi(G(R(d,a,b,c),\omega_{3n+1},\omega_{3n+2}))$$

$$\leq \frac{1}{4}\psi\left[\begin{array}{c} G(gd,\omega_{3n},\omega_{3n+1})+G(ga,\alpha_{3n},\alpha_{3n+1}) \\ +G(gb,\beta_{3n},\beta_{3n+1})+G(gc,\gamma_{3n},\gamma_{3n+1}) \end{array}\right]$$

$$-\phi\left[\begin{array}{c} G(gd,\omega_{3n},\omega_{3n+1})+G(ga,\alpha_{3n},\alpha_{3n+1}) \\ +G(gb,\beta_{3n},\beta_{3n+1})+G(gc,\gamma_{3n},\gamma_{3n+1}) \end{array}\right]. \tag{25}$$

By using property of ψ and combining $(22)-(25)$, we get

$$
\begin{aligned}
&\psi\left(\begin{array}{c} G(R(a,b,c,d),\alpha_{3n+1},\alpha_{3n+2})+G(R(b,c,d,a),\beta_{3n+1},\beta_{3n+2}) \\ +G(R(c,d,a,b),\gamma_{3n+1},\gamma_{3n+2})+G(R(d,a,b,c),\omega_{3n+1},\omega_{3n+2}) \end{array}\right) \\
\leq\ & \psi(G(R(a,b,c,d),\alpha_{3n+1},\alpha_{3n+2}))+\psi(G(R(b,c,d,a),\beta_{3n+1},\beta_{3n+2})) \\
& +\psi(G(R(c,d,a,b),\gamma_{3n+1},\gamma_{3n+2}))+\psi(G(R(d,a,b,c),\omega_{3n+1},\omega_{3n+2})) \\
\leq\ & \psi\left[\begin{array}{c} G(ga,\alpha_{3n},\alpha_{3n+1})+G(gb,\beta_{3n},\beta_{3n+1}) \\ +G(gc,\gamma_{3n},\gamma_{3n+1})+G(gd,\omega_{3n},\omega_{3n+1}) \end{array}\right] \\
& -4\phi\left[\begin{array}{c} G(ga,\alpha_{3n},\alpha_{3n+1})+G(gb,\beta_{3n},\beta_{3n+1}) \\ +G(gc,\gamma_{3n},\gamma_{3n+1})+G(gd,\omega_{3n},\omega_{3n+1}) \end{array}\right].
\end{aligned}
$$

Taking the limit as $n\to\infty$ in the above inequality, we obtain

$$
\begin{aligned}
&\psi\left(\begin{array}{c} G(R(a,b,c,d),\alpha,\alpha)+G(R(b,c,d,a),\beta,\beta) \\ +G(R(c,d,a,b),\gamma,\gamma)+G(R(d,a,b,c),\omega,\omega) \end{array}\right) \\
\leq\ & \psi\left[\begin{array}{c} G(ga,\alpha,\alpha)+G(gb,\beta,\beta) \\ +G(gc,\gamma,\gamma)+G(gd,\omega,\omega) \end{array}\right]-4\phi\left[\begin{array}{c} G(ga,\alpha,\alpha)+G(gb,\beta,\beta) \\ +G(gc,\gamma,\gamma)+G(gd,\omega,\omega) \end{array}\right] \\
\leq\ & \psi(0)-4\phi(0)=\psi(0)
\end{aligned}
$$

which implies $R(a,b,c,d)=\alpha, R(b,c,d,a)=\beta, R(c,d,a,b)=\gamma, R(d,a,b,c)=\omega$. Therefore, it follows that $R(a,b,c,d)=\alpha=ga, R(b,c,d,a)=\beta=gb$, $R(c,d,a,b)=\gamma=gc, R(d,a,b,c)=\omega=gd$. Since $\{R,g\}$ is weakly compatible pair, we have $R(\alpha,\beta,\gamma,\omega)=g\alpha$, $R(\beta,\gamma,\omega,\alpha)=g\beta$, $R(\gamma,\omega,\alpha,\beta)=g\gamma$ and $R(\omega,\alpha,\beta,\gamma)=g\omega$. Now we prove that $g\alpha=\alpha$, $g\beta=\beta$, $g\gamma=\gamma$ and $g\omega=\omega$.

$$
\begin{aligned}
&\psi(G(g\alpha,\alpha_{3n+1},\alpha_{3n+2})) \\
=\ & \psi(G(R(\alpha,\beta,\gamma,\omega),S(a_{3n+1},b_{3n+1},c_{3n+1},d_{3n+1}),T(a_{3n+2},b_{3n+2},c_{3n+2},d_{3n+2}))) \\
\leq\ & \frac{1}{4}\psi\left[\begin{array}{c} G(g\alpha,\alpha_{3n},\alpha_{3n+1})+G(g\beta,\beta_{3n},\beta_{3n+1}) \\ +G(g\gamma,\gamma_{3n},\gamma_{3n+1})+G(g\omega,\omega_{3n},\omega_{3n+1}) \end{array}\right] \\
& -\phi\left[\begin{array}{c} G(g\alpha,\alpha_{3n},\alpha_{3n+1})+G(g\beta,\beta_{3n},\beta_{3n+1}) \\ +G(g\gamma,\gamma_{3n},\gamma_{3n+1})+G(g\omega,\omega_{3n},\omega_{3n+1}) \end{array}\right]
\end{aligned}
$$

By using property of ψ and we get

$$\begin{aligned}
&\psi\begin{pmatrix} G(g\alpha,\alpha_{3n+1},\alpha_{3n+2})+G(g\beta,\beta_{3n+1},\beta_{3n+2}) \\ +G(g\gamma,\gamma_{3n+1},\gamma_{3n+2})+G(g\omega,\omega_{3n+1},\omega_{3n+2}) \end{pmatrix} \\
\leq\ & \psi(G(g\alpha,\alpha_{3n+1},\alpha_{3n+2}))+\psi(G(g\beta,\beta_{3n+1},\beta_{3n+2})) \\
&+\psi(G(g\gamma,\gamma_{3n+1},\gamma_{3n+2}))+\psi(G(g\omega,\omega_{3n+1},\omega_{3n+2})) \\
\leq\ & \psi\begin{bmatrix} G(g\alpha,\alpha_{3n},\alpha_{3n+1})+G(g\beta,\beta_{3n},\beta_{3n+1}) \\ +G(g\gamma,\gamma_{3n},\gamma_{3n+1})+G(g\omega,\omega_{3n},\omega_{3n+1}) \end{bmatrix} \\
&-4\phi\begin{bmatrix} G(g\alpha,\alpha_{3n},\alpha_{3n+1})+G(g\beta,\beta_{3n},\beta_{3n+1}) \\ +G(g\gamma,\gamma_{3n},\gamma_{3n+1})+G(g\omega,\omega_{3n},\omega_{3n+1}) \end{bmatrix}.
\end{aligned}$$

Taking the limit as $n\to\infty$ in the above inequality, we obtain

$$\begin{aligned}
\psi\begin{pmatrix} G(g\alpha,\alpha,\alpha)+G(g\beta,\beta,\beta) \\ +G(g\gamma,\gamma,\gamma)+G(g\omega,\omega,\omega) \end{pmatrix} \leq\ & \psi\begin{bmatrix} G(g\alpha,\alpha,\alpha)+G(g\beta,\beta,\beta) \\ +G(g\gamma,\gamma,\gamma)+G(g\omega,\omega,\omega) \end{bmatrix} \\
&-4\phi\begin{bmatrix} G(g\alpha,\alpha,\alpha)+G(g\beta,\beta,\beta) \\ +G(g\gamma,\gamma,\gamma)+G(g\omega,\omega,\omega) \end{bmatrix}. \\
<\ & \psi\begin{bmatrix} G(g\alpha,\alpha,\alpha)+G(g\beta,\beta,\beta) \\ +G(g\gamma,\gamma,\gamma)+G(g\omega,\omega,\omega) \end{bmatrix}
\end{aligned}$$

which possibility holds only $G(g\alpha,\alpha,\alpha)=0$, $G(g\beta,\beta,\beta)=0$, $G(g\gamma,\gamma,\gamma)=0$ and $G(g\omega,\omega,\omega)=0$ implies that $g\alpha=\alpha, g\beta=\beta$, $g\gamma=\gamma$ and$g\omega=\omega$. Therefore, $R(\alpha,\beta,\gamma,\omega)=g\alpha=\alpha$, $R(\beta,\gamma,\omega,\alpha)=g\beta=\beta$, $R(\gamma,\omega,\alpha,\beta)=g\gamma=\gamma$ and $R(\omega,\alpha,\beta,\gamma)=g\omega=\omega$. Thus $(\alpha,\beta,\gamma,\omega)$ is quadruple fixed point of R and g.

Since $R(\mathscr{P}^4)\subseteq f(\mathscr{P})$, so there exist $p,q,r,s\in\mathscr{P}$ such that $R(\alpha,\beta,\gamma,\omega)=\alpha=fp$, $R(\beta,\gamma,\omega,\alpha)=\beta=fq$, $R(\gamma,\omega,\alpha,\beta)=\gamma=fr$ and $R(\omega,\alpha,\beta,\gamma)=\omega=fs$.

$$
\begin{aligned}
&\psi(G(\alpha_{3n+1},\alpha_{3n+2},T(p,q,r,s),))\\
=\ &\psi(G(R(a_{3n+1},b_{3n+1},c_{3n+1},d_{3n+1}),S(a_{3n+2},b_{3n+2},c_{3n+2},d_{3n+2}),T(p,q,r,s),))\\
\leq\ &\frac{1}{4}\psi\left[\begin{array}{c} G(ga_{3n+1},ha_{3n+2}fp)+G(gb_{3n+1},hb_{3n+2},fq)\\ +G(gc_{3n+1},hc_{3n+2},fr)+G(gd_{3n+1},hd_{3n+2},fs)\end{array}\right]\\
&-\phi\left[\begin{array}{c} G(ga_{3n+1},ha_{3n+2}fp)+G(gb_{3n+1},hb_{3n+2},fq)\\ +G(gc_{3n+1},hc_{3n+2},fr)+G(gd_{3n+1},hd_{3n+2},fs)\end{array}\right]\\
\leq\ &\frac{1}{4}\psi\left[\begin{array}{c} G(\alpha_{3n},\alpha_{3n+1},fp)+G(\beta_{3n},\beta_{3n+1},fq)\\ +G(\gamma_{3n},\gamma_{3n+1},fr)+G(\omega_{3n},\omega_{3n+1})\end{array}\right]\\
&-\phi\left[\begin{array}{c} G(\alpha_{3n},\alpha_{3n+1},fp)+G(\beta_{3n},\beta_{3n+1},fq)\\ +G(\gamma_{3n},\gamma_{3n+1},fr)+G(\omega_{3n},\omega_{3n+1},fs)\end{array}\right].
\end{aligned}
$$

By using the properties of ψ and we get

$$
\begin{aligned}
&\psi\left(\begin{array}{c} G(\alpha_{3n+1},\alpha_{3n+2},T(p,q,r,s),)+G(\beta_{3n+1},\beta_{3n+2},T(q,r,s,p),)\\ +G(\gamma_{3n+1},\gamma_{3n+2},T(r,s,p,q),)+G(\omega_{3n+1},\omega_{3n+2},T(s,p,q,r),)\end{array}\right)\\
\leq\ &\psi\left[\begin{array}{c} G(\alpha_{3n},\alpha_{3n+1},fp)+G(\beta_{3n},\beta_{3n+1},fq)\\ +G(\gamma_{3n},\gamma_{3n+1},fr)+G(\omega_{3n},\omega_{3n+1})\end{array}\right]\\
&-4\phi\left[\begin{array}{c} G(\alpha_{3n},\alpha_{3n+1},fp)+G(\beta_{3n},\beta_{3n+1},fq)\\ +G(\gamma_{3n},\gamma_{3n+1},fr)+G(\omega_{3n},\omega_{3n+1})\end{array}\right].
\end{aligned}
$$

Letting $n\to\infty$ in above inequality, we get

$$
\begin{aligned}
&\psi\left(\begin{array}{c} G(\alpha,\alpha,T(p,q,r,s),)+G(\beta,\beta,T(q,r,s,p),)\\ +G(\gamma,\gamma,T(r,s,p,q),)+G(\omega,\omega,T(s,p,q,r))\end{array}\right)\\
\leq\ &\psi\left[\begin{array}{c} G(\alpha,\alpha,fp)+G(\beta,\beta,fq)\\ +G(\gamma,\gamma,fr)+G(\omega,\omega,fs)\end{array}\right]-4\phi\left[\begin{array}{c} G(\alpha,\alpha,fp)+G(\beta,\beta,fq)\\ +G(\gamma,\gamma,fr)+G(\omega,\omega,fs)\end{array}\right]\\
\leq\ &\psi(0).
\end{aligned}
$$

Therefore, $T(p,q,r,s)=\alpha, T(q,r,s,p)=\beta$, $T(r,s,p,q)=\gamma$ and $T(s,p,q,r)=\omega$.

Since $\{T,f\}$ weakly compatible pair, we have

$T(\alpha,\beta,\gamma,\omega) = f\alpha, T(\beta,\gamma,\omega,\alpha) = f\beta$, $T(\gamma,\omega,\alpha,\beta) = f\gamma$ and $T(\omega,\alpha,\beta,\gamma) = f\omega$.
Now we prove that $f\alpha = \alpha, f\beta = \beta, f\gamma = \gamma$ and $f\omega = \omega$.

$$\begin{aligned}
&\psi(G(\alpha,\alpha_{3n+1},f\alpha)) \\
= \;& \psi(G(R(\alpha,\beta,\gamma,\omega),S(a_{3n+1},b_{3n+1},c_{3n+1},d_{3n+1}),T(\alpha,\beta,\gamma,\omega))) \\
\leq \;& \frac{1}{4}\psi\left[\begin{array}{c} G(\alpha,\alpha_{3n},f\alpha)+G(\beta,\beta_{3n},f\beta) \\ +G(\gamma,\gamma_{3n},f\gamma)+G(\omega,\omega_{3n},f\omega) \end{array}\right] - \phi\left[\begin{array}{c} G(\alpha,\alpha_{3n},f\alpha)+G(\beta,\beta_{3n},f\beta) \\ +G(\gamma,\gamma_{3n},f\gamma)+G(\omega,\omega_{3n},f\omega) \end{array}\right]
\end{aligned}$$

and hence

$$\begin{aligned}
&\psi\left(\begin{array}{c} G(\alpha,\alpha_{3n+1},f\alpha)+G(\beta,\beta_{3n+1},f\beta) \\ +G(\gamma,\gamma_{3n+1},f\gamma)+G(\omega,\omega_{3n+1},f\omega) \end{array}\right) \\
\leq \;& \psi\left[\begin{array}{c} G(\alpha,\alpha_{3n},f\alpha)+G(\beta,\beta_{3n},f\beta) \\ +G(\gamma,\gamma_{3n},f\gamma)+G(\omega,\omega_{3n},f\omega) \end{array}\right] - 4\phi\left[\begin{array}{c} G(\alpha,\alpha_{3n},f\alpha)+G(\beta,\beta_{3n},f\beta) \\ +G(\gamma,\gamma_{3n},f\gamma)+G(\omega,\omega_{3n},f\omega) \end{array}\right].
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we obtain

$$\begin{aligned}
&\psi\left(\begin{array}{c} G(\alpha,\alpha,f\alpha)+G(\beta,\beta,f\beta) \\ +G(\gamma,\gamma,f\gamma)+G(\omega,\omega,f\omega) \end{array}\right) \\
\leq \;& \psi\left[\begin{array}{c} G(\alpha,\alpha,f\alpha)+G(\beta,\beta,f\beta) \\ +G(\gamma,\gamma,f\gamma)+G(\omega,\omega,f\omega) \end{array}\right] - 4\phi\left[\begin{array}{c} G(\alpha,\alpha,f\alpha)+G(\beta,\beta,f\beta) \\ +G(\gamma,\gamma,f\gamma)+G(\omega,\omega,f\omega) \end{array}\right].
\end{aligned}$$

which holds only $G(\alpha,\alpha,f\alpha) = 0$, $G(\beta,\beta,f\beta) = 0$, $G(\gamma,\gamma,f\gamma) = 0$, $G(\omega,\omega,f\omega) = 0$ implies that $f\alpha = \alpha, f\beta = \beta$, $f\gamma = \gamma$ and$f\omega = \omega$. Therefore, $T(\alpha,\beta,\gamma,\omega) = f\alpha = \alpha$, $T(\beta,\gamma,\omega,\alpha) = f\beta = \beta$, $T(\gamma,\omega,\alpha,\beta) = f\gamma = \gamma$ and $T(\omega,\alpha,\beta,\gamma) = f\omega = \omega$. Thus $(\alpha,\beta,\gamma,\omega)$ is quadruple fixed point of R,T,g and f. Since $T(\mathscr{P}^4) \subseteq h(\mathscr{P})$, so there exist $x,y,z,w \in \mathscr{P}$ such that $T(\alpha,\beta,\gamma,\omega) = \alpha = hx$, $T(\beta,\gamma,\omega,\alpha) = \beta = hy$, $T(\gamma,\omega,\alpha,\beta) = \gamma = hz$ and

$T(\omega,\alpha,\beta,\gamma)=\omega=hw.$

$$\begin{aligned}
& \psi(G(\alpha_{3n+1},S(x,y,z,w),\alpha_{3n+2})) \\
= & \psi(G(R(a_{3n+1},b_{3n+1},c_{3n+1},d_{3n+1}),S(x,y,z,w),T(a_{3n+2},b_{3n+2},c_{3n+2},d_{3n+2}))) \\
\leq & \frac{1}{4}\psi\left[\begin{array}{c} G(ga_{3n+1},hx,fa_{3n+2})+G(gb_{3n+1},hy,fb_{3n+2}) \\ +G(gc_{3n+1},hz,fc_{3n+2})+G(gd_{3n+1},hw,fd_{3n+2}) \end{array}\right] \\
& -\phi\left[\begin{array}{c} G(ga_{3n+1},hx,fa_{3n+2})+G(gb_{3n+1},hy,fb_{3n+2}) \\ +G(gc_{3n+1},hz,fc_{3n+2})+G(gd_{3n+1},hw,fd_{3n+2}) \end{array}\right] \\
\leq & \frac{1}{4}\psi\left[\begin{array}{c} G(\alpha_{3n},hx,\alpha_{3n+1})+G(\beta_{3n},hy,\beta_{3n+1}) \\ +G(\gamma_{3n},hz,\gamma_{3n+1})+G(\omega_{3n},hw,\omega_{3n+1}) \end{array}\right] \\
& -\phi\left[\begin{array}{c} G(\alpha_{3n},hx,\alpha_{3n+1})+G(\beta_{3n},hy,\beta_{3n+1}) \\ +G(\gamma_{3n},hz,\gamma_{3n+1})+G(\omega_{3n},hw,\omega_{3n+1}) \end{array}\right].
\end{aligned}$$

and hence

$$\begin{aligned}
& \psi\left(\begin{array}{c} G(\alpha_{3n+1},S(x,y,z,w),\alpha_{3n+2})+G(\beta_{3n+1},S(y,z,w,x),\beta_{3n+2}) \\ +G(\gamma_{3n+1},S(z,w,x,y),\gamma_{3n+2})+G(\omega_{3n+1},S(w,x,y,z),\omega_{3n+2}) \end{array}\right) \\
\leq & \psi\left[\begin{array}{c} G(\alpha_{3n},hx,\alpha_{3n+1})+G(\beta_{3n},hy,\beta_{3n+1}) \\ +G(\gamma_{3n},hz,\gamma_{3n+1})+G(\omega_{3n},hw,\omega_{3n+1}) \end{array}\right] \\
& -4\phi\left[\begin{array}{c} G(\alpha_{3n},hx,\alpha_{3n+1})+G(\beta_{3n},hy,\beta_{3n+1}) \\ +G(\gamma_{3n},hz,\gamma_{3n+1})+G(\omega_{3n},hw,\omega_{3n+1}) \end{array}\right].
\end{aligned}$$

Letting $n\to\infty$ in above inequality, we get

$$\begin{aligned}
& \psi\left(\begin{array}{c} G(\alpha,S(x,y,z,w),\alpha)+G(\beta,S(y,z,w,x),\beta) \\ +G(\gamma,S(z,w,x,y),\gamma)+G(\omega,S(w,x,y,z),\omega) \end{array}\right) \\
\leq & \psi\left[\begin{array}{c} G(\alpha,hx,\alpha)+G(\beta,hy,\beta) \\ +G(\gamma,hz,\gamma)+G(\omega,hw,\omega) \end{array}\right] \\
& -4\phi\left[\begin{array}{c} G(\alpha,hx,\alpha)+G(\beta,hy,\beta) \\ +G(\gamma,hz,\gamma)+G(\omega,hw,\omega) \end{array}\right] \leq \psi(0).
\end{aligned}$$

Therefore,

$S(x,y,z,w)=\alpha, S(y,z,w,x)=\beta,\ S(z,w,x,y)=\gamma$ and $S(w,x,y,z)=\omega$. Since $\{S,h\}$ weakly compatible pair, we have

$S(\alpha,\beta,\gamma,\omega)=h\alpha, S(\beta,\gamma,\omega,\alpha)=h\beta$, $S(\gamma,\omega,\alpha,\beta)=h\gamma$ and $S(\omega,\alpha,\beta,\gamma)=h\omega$.

Now we prove that $h\alpha=\alpha, h\beta=\beta, h\gamma=\gamma$ and $h\omega=\omega$.

$$\begin{aligned}
&\psi(G(\alpha,h\alpha,\alpha))\\
=\ &\psi(G(R(\alpha,\beta,\gamma,\omega),S(\alpha,\beta,\gamma,\omega),T(\alpha,\beta,\gamma,\omega)))\\
\leq\ &\frac{1}{4}\psi\left[\begin{array}{c} G(\alpha,h\alpha,\alpha)+G(\beta,h\beta,\beta)\\ +G(\gamma,h\gamma,\gamma)+G(\omega,h\omega,\omega)\end{array}\right]-\phi\left[\begin{array}{c} G(\alpha,h\alpha,\alpha)+G(\beta,h\beta,\beta)\\ +G(\gamma,h\gamma,\gamma)+G(\omega,h\omega,\omega)\end{array}\right]
\end{aligned}$$

and hence

$$\begin{aligned}
&\psi\left(\begin{array}{c} G(\alpha,h\alpha,\alpha)+G(\beta,h\beta,\beta)\\ +G(\gamma,h\gamma,\gamma)+G(\omega,h\omega,\omega)\end{array}\right)\\
\leq\ &\psi\left[\begin{array}{c} G(\alpha,h\alpha,\alpha)+G(\beta,h\beta,\beta)\\ +G(\gamma,h\gamma,\gamma)+G(\omega,h\omega,\omega)\end{array}\right]-4\phi\left[\begin{array}{c} G(\alpha,h\alpha,\alpha)+G(\beta,h\beta,\beta)\\ +G(\gamma,h\gamma,\gamma)+G(\omega,h\omega,\omega)\end{array}\right]
\end{aligned}$$

which holds only $G(\alpha,h\alpha,\alpha)=0$, $G(\beta,h\beta,\beta)=0$, $G(\gamma,h\gamma,\gamma)=0$ and $G(\omega,h\omega,\omega)=0$ implies that $h\alpha=\alpha, h\beta=\beta$, $h\gamma=\gamma$ and $h\omega=\omega$. Therefore, $S(\alpha,\beta,\gamma,\omega)=h\alpha=\alpha$, $S(\beta,\gamma,\omega,\alpha)=h\beta=\beta$, $S(\gamma,\omega,\alpha,\beta)=h\gamma=\gamma$ and $S(\omega,\alpha,\beta,\gamma)=h\omega=\omega$. Thus $(\alpha,\beta,\gamma,\omega)$ is quadruple fixed point of R,T,S,g,f and h. In the following we will show the uniqueness of common quadruple fixed point in $\mathscr{P}$. For this purpose, assume that there is another quadruple fixed point $(\alpha',\beta',\gamma',\omega')$ of R,S,T,g,h,f. Then

$$\begin{aligned}
&\psi(G(\alpha,\alpha,\alpha'))\\
=\ &\psi(G(R(\alpha,\beta,\gamma,\omega),S(\alpha,\beta,\gamma,\omega),T(\alpha',\beta',\gamma',\omega')))\\
\leq\ &\frac{1}{4}\psi\left[\begin{array}{c} G(\alpha,\alpha,\alpha')+G(\beta,\beta,\beta')\\ +G(\gamma,\gamma,\gamma')+G(\omega,\omega,\omega')\end{array}\right]-\phi\left[\begin{array}{c} G(\alpha,\alpha,\alpha')+G(\beta,\beta,\beta')\\ +G(\gamma,\gamma,\gamma')+G(\omega,\omega,\omega')\end{array}\right]
\end{aligned}$$

and hence

$$\psi\left(\begin{array}{c} G(\alpha,\alpha,\alpha')+G(\beta,\beta,\beta') \\ +G(\gamma,\gamma,\gamma')+G(\omega,\omega,\omega') \end{array}\right)$$
$$\leq \psi\left[\begin{array}{c} G(\alpha,\alpha,\alpha')+G(\beta,\beta,\beta') \\ +G(\gamma,\gamma,\gamma')+G(\omega,\omega,\omega') \end{array}\right]-4\phi\left[\begin{array}{c} G(\alpha,\alpha,\alpha')+G(\beta,\beta,\beta') \\ +G(\gamma,\gamma,\gamma')+G(\omega,\omega,\omega') \end{array}\right]$$

hence, we get $\alpha=\alpha',\beta=\beta',\gamma=\gamma'$ and $\omega=\omega'$. Therefore, $(\alpha,\beta,\gamma,\omega)$ is a unique quadruple common fixed point of R,S,T,g,h and f.

$$\begin{aligned} & \psi(G(\alpha,\alpha,\beta)) \\ = \ & \psi(G(R(\alpha,\beta,\gamma,\omega),S(\alpha,\beta,\gamma,\omega),T(\beta,\gamma,\omega,\alpha))) \\ \leq \ & \frac{1}{4}\psi\left[\begin{array}{c} G(\alpha,\alpha,\beta)+G(\beta,\beta,\gamma) \\ +G(\gamma,\gamma,\omega)+G(\omega,\omega,\alpha) \end{array}\right]-\phi\left[\begin{array}{c} G(\alpha,\alpha,\beta)+G(\beta,\beta,\gamma) \\ +G(\gamma,\gamma,\omega)+G(\omega,\omega,\alpha) \end{array}\right] \end{aligned}$$

and hence

$$\psi\left(\begin{array}{c} G(\alpha,\alpha,\beta)+G(\beta,\beta,\gamma) \\ +G(\gamma,\gamma,\omega)+G(\omega,\omega,\alpha) \end{array}\right)$$
$$\leq \psi\left[\begin{array}{c} G(\alpha,\alpha,\beta)+G(\beta,\beta,\gamma) \\ +G(\gamma,\gamma,\omega)+G(\omega,\omega,\alpha) \end{array}\right]-4\phi\left[\begin{array}{c} G(\alpha,\alpha,\beta)+G(\beta,\beta,\gamma) \\ +G(\gamma,\gamma,\omega)+G(\omega,\omega,\alpha) \end{array}\right]$$

hence, we get $\alpha=\beta=\gamma=\omega$. Which means that R,S,T,g,h and f have a unique common fixed point.

Corollary 3.2: Let $(\mathscr{P},G)$ be a G-metric space. Suppose that $R,S,T:\mathscr{P}^4\to\mathscr{P}$ and $f,g,h:\mathscr{P}\to\mathscr{P}$ be a six mappings. Assume also that there exist $\phi\in\Phi$ and $\psi\in\Psi$ such that

$$\begin{aligned} & \psi(G(R(a,b,c,d),S(x,y,z,w),T(p,q,r,s))) \\ & \qquad \leq \frac{1}{4}\psi(\max\{G(ga,hx,fp),G(gb,hy,fq),G(gc,hz,fr),G(gd,hw,fs)\}) \\ & \qquad -\phi(G(ga,hx,fp)+G(gb,hy,fq)+G(gc,hz,fr)+G(gd,hw,fs)). \end{aligned}$$

for all $a,b,c,d,x,y,z,w,p,q,r,s\in\mathscr{P}$.

$a)$ $R(\mathscr{P}^4)\subseteq f(\mathscr{P})$, $S(\mathscr{P}^4)\subseteq g(\mathscr{P})$ and $T(\mathscr{P}^4)\subseteq h(\mathscr{P})$,

$b)$ Either (R,g) or (S,h) or (T,f) are ω-compatible,

$c)$ one of $g(\mathscr{P})$, $h(\mathscr{P})$ or $f(\mathscr{P})$ is complete.

Then there is a unique common quadruple fixed point of R,S,T,f,g and h in $\mathscr{P}$.

Proof. Since

$$\begin{aligned}
&\max\{G(ga,hx,fp),G(gb,hy,fq),G(gc,hz,fr),G(gd,hw,fs)\} \\
\leq\ & G(ga,hx,fp)+G(gb,hy,fq)+G(gc,hz,fr)+G(gd,hw,fs),
\end{aligned}$$

then we apply Theorem (3.1), since ψ is assumed to be nondecreasing.

Corollary 3.3: Let $(\mathscr{P},G)$ be a complete G-metric space. Suppose that $R:\mathscr{P}^4\to\mathscr{P}$ be a mappings. Assume also that there exist $\phi\in\Phi$ and $\psi\in\Psi$ such that

$$\begin{aligned}
&\psi(G(R(a,b,c,d),R(x,y,z,w),R(p,q,r,s))) \\
&\qquad\leq \frac{1}{4}\psi(\max\{G(a,x,p),G(b,y,q),G(c,z,r),G(d,w,s)\}) \\
&\qquad\ -\phi(\max\{G(a,x,p),G(b,y,q),G(c,z,r),G(d,w,s)\}).
\end{aligned}$$

for all $a,b,c,d,x,y,z,w,p,q,r,s\in\mathscr{P}$. Then there is a unique quadruple fixed point of R in $\mathscr{P}$.

Corollary 3.4: Let $(\mathscr{P},G)$ be a G-metric space. Suppose that $R,S,T:\mathscr{P}^4\to\mathscr{P}$ and $f,g,h:\mathscr{P}\to\mathscr{P}$ be a six mappings. Assume also that there exist $\kappa\in[0,1)$ such that

$$G(R(a,b,c,d),S(x,y,z,w),T(p,q,r,s))\leq\frac{\kappa}{4}\begin{pmatrix} G(ga,hx,fp)+G(gb,hy,fq) \\ +G(gc,hz,fr)+G(gd,hw,fs)\end{pmatrix}$$

for all $a,b,c,d,x,y,z,w,p,q,r,s\in\mathscr{P}$.

$a)$ $R(\mathscr{P}^4)\subseteq f(\mathscr{P})$, $S(\mathscr{P}^4)\subseteq g(\mathscr{P})$ and $T(\mathscr{P}^4)\subseteq h(\mathscr{P})$;

$b)$ Either (R,g) or (S,h) or (T,f) are ω-compatible;

$c)$ one of $g(\mathscr{P})$, $h(\mathscr{P})$ or $f(\mathscr{P})$ is complete.

Then there is a unique common quadruple fixed point of R,S,T,f,g and h in $\mathscr{P}$.

Proof. It is sufficient to set $\psi(t)=t$ and $\phi(t)=\frac{1-\kappa}{4}t$ in Theorem (3.1).

Example: Let $\mathscr{P}=[0,\infty)$ and $G(x,y,z,w)=|x-y|+|y-z|+|z-w|$, $(\mathscr{P},G)$ is a complete G-metric spaces. Let $R,S,T:\mathscr{P}^4\to\mathscr{P}$ and $g,h,f:\mathscr{P}\to\mathscr{P}$ be given by $g(x)=8x$, $h(x)=2x$, $f(x)=x$ and $R(x,y,z,w)=\frac{x+y+z+w}{2}$, $S(x,y,z,w)=\frac{x+y+z+w}{8}$, $T(x,y,z,w)=\frac{x+y+z+w}{16}$ also, $\psi(t)=\frac{t}{3}$ and $\phi(t)=\frac{t}{16}$ for all $t\in[0,\infty)$. Then obviously, $R(\mathscr{P}^4)\subseteq f(\mathscr{P})$,

$S(\mathscr{P}^4)\subseteq g(\mathscr{P})$, $T(\mathscr{P}^4)\subseteq h(\mathscr{P})$ and the pairs $(R,g),(S,h)$, (T,f) are ω-compatible.

Now we have

$$\begin{aligned}
&\psi(G(R(a,b,c,d),S(x,y,z,w),T(p,q,r,s)))\\
=\ & \frac{1}{3}\left[\begin{array}{c} |\frac{a+b+c+d}{2}-\frac{x+y+z+w}{8}|+|\frac{x+y+z+w}{8}-\frac{p+q+r+s}{16}| \\ +|\frac{p+q+r+s}{16}-\frac{a+b+c+d}{2}| \end{array}\right]\\
\leq\ & \frac{1}{3}\left[\begin{array}{c} (|\frac{a}{2}-\frac{x}{8}|+|\frac{x}{8}-\frac{p}{16}|+|\frac{p}{16}-\frac{a}{2}|)+(|\frac{b}{2}-\frac{y}{8}|+|\frac{y}{8}-\frac{q}{16}|+|\frac{q}{16}-\frac{b}{2}|) \\ +(|\frac{c}{2}-\frac{z}{8}|+|\frac{z}{8}-\frac{r}{16}|+|\frac{r}{16}-\frac{c}{2}|)+(|\frac{d}{2}-\frac{w}{8}|+|\frac{w}{8}-\frac{s}{16}|+|\frac{s}{16}-\frac{d}{2}|) \end{array}\right]\\
\leq\ & \frac{1}{48}\left[\begin{array}{c} (|8a-2x|+|2x-p|+|p-8a|)+(|8b-2y|+|2y-q|+|q-8b|) \\ +(|8c-2z|+|2z-r|+|r-8c|)+(|8d-2w|+|2w-s|+|s-8d|) \end{array}\right]\\
\leq\ & \frac{1}{4}\psi(G(ga,hx,fp)+G(gb,hy,fq)+G(gc,hz,fr)+G(gd,hw,fs))\\
&-\ \phi(G(ga,hx,fp)+G(gb,hy,fq)+G(gc,hz,fr)+G(gd,hw,fs)).
\end{aligned}$$

Thus all the conditions of the Theorem (3.1) are satisfied and $(0,0,0,0)$ is unique fixed point.

3.1. APPLICATION TO INTEGRAL EQUATIONS.

this section, we study the existence of an unique solution to an initial value problem, as an application to Corollary (3.3).

Theorem 3.1.1: Consider the initial value problem

(26) $$x^1(t)=T(t,(x,y,z,w)(t)),\ t\in I=[0,1],\ (x,y,z,w)(0)=(x_0,y_0,z_0,w_0)$$

where $T:I\times\mathbb{R}^4\to\mathbb{R}$ and $x_0,y_0,z_0,w_0\in\mathbb{R}$. Then there exists unique solution in $C(I,\mathbb{R})$ for the initial value problem (26).

Proof. The integral equation corresponding to initial value problem (26) is

$$x(t)=x_0+8\int_0^t T(s,(x,y,z,w)(s))ds.$$

Let $\mathscr{P}=C(I,\mathbb{R})$ and $G(x,y,z)=|x-y|+|y-z|+|z-x|\ for\ x,y,z\in\mathscr{P}$.

Define $\psi,\phi:[0,\infty)\to[0,\infty)$ by $\psi(t)=t$, $\phi(t)=\frac{t}{8}$ and define $R:\mathscr{P}^4\to\mathscr{P}$ by

$$R(x,y,z,w)(t)\ =\ \frac{x_0}{8}+\int_0^t T(s,(x,y,z,w)(s))ds.$$

Now

$$\begin{aligned}
&\psi(G(R(x,y,z,w)(t),R(a,b,c,d)(t),R(l,m,n,o)(t)))\\
= & |R(x,y,z,w)(t)-R(a,b,c,d)(t)|+|R(a,b,c,d)(t)-R(l,m,n,o)(t)|\\
&+|R(l,m,n,o)(t)-R(x,y,z,w)(t)|\\
= & |\frac{x_0}{4}+\int_0^t T(s,(x,y,z,w)(s))ds-\frac{a_0}{4}+\int_0^t T(s,(a,b,c,d)(s))ds|\\
&+|\frac{a_0}{4}+\int_0^t T(s,(a,b,c,d)(s))ds-\frac{l_0}{4}+\int_0^t T(s,(l,m,n,o)(s))ds|\\
&+|\frac{l_0}{4}+\int_0^t T(s,(l,m,n,o)(s))ds-\frac{x_0}{4}+\int_0^t T(s,(x,y,z,w)(s))ds|\\
= & \frac{1}{8}(|x(t)-a(t)|+|a(t)-l(t)|+|l(t)-x(t)|)=\frac{1}{8}G(x,a,l)\\
\leq & \frac{1}{4}\psi(\max\{G(x,a,l),G(y,b,m),G(z,c,n),G(w,d,o)\})\\
&-\phi(\max\{G(x,a,l),G(y,b,m),G(z,c,n),G(w,d,o)\}).
\end{aligned}$$

It follows from Corollary (3.3), we conclude that R has a unique fixed point in $\mathscr{P}$.

3.2. APPLICATION TO HOMOTOPY.

In this section, we study the existence of an unique solution to Homotopy theory.

Theorem 3.2.1: Let $(\mathscr{P},G)$ be complete G-metric space, U and $\overline{U}$ be an open and closed subset of $\mathscr{P}$ such that $U\subseteq\overline{U}$. Suppose $H:\overline{U}^4\times[0,1]\to\mathscr{P}$ be an operator with following conditions are satisfying,

τ_0) $x\neq H(x,y,z,w,\kappa)$, $y\neq H(y,z,w,x,\kappa)$, $z\neq H(z,w,x,z,\kappa)$ and $w\neq H(w,x,y,z,\kappa)$ for each $x,y,z,w\in\partial U$ and $\kappa\in[0,1]$ (Here ∂U is boundary of U in $\mathscr{P}$);

τ_1) for all $x,y,z,w,a,b,c,d\in\overline{U}$ and $\psi\in\Psi,\phi\in\Phi$, $\kappa\in[0,1]$ such that

$$\begin{aligned}
&\psi(G(H(x,y,z,w,\kappa),H(x,y,z,w,\kappa),H(a,b,c,d,\kappa)))\\
&\qquad\leq\frac{1}{4}\psi(G(x,x,a)+G(y,y,b)+G(z,z,c)+G(w,w,d))\\
&\qquad-\phi(G(x,x,a)+G(y,y,b)+G(z,z,c)+G(w,w,d)),
\end{aligned}$$

$\tau_2)$ $\exists\, M \geq 0 \ni G(H(x,y,z,w,\kappa),H(x,y,z,w,\kappa),H(a,b,c,d,\zeta)) \leq M|\kappa-\zeta|$
for every $x,y,z,w,a,b,c,d \in \overline{U}$ and $\kappa,\zeta \in [0,1]$.
Then $H(.,0)$ has a quadruple fixed point $\iff$ $H(.,1)$ has a quadruple fixed point.

Proof. Let the set

$$X = \left\{ \begin{array}{c} \kappa \in [0,1] : H(x,y,z,w,\kappa) = x, H(y,z,w,x,\kappa) = y \\ H(z,w,x,y,\kappa) = z, H(w,x,y,z,w,\kappa) = w, \text{ for some } x,y,z,w, \in U \end{array} \right\}.$$

Since $H(.,0)$ has a quadruple fixed point in U^4, we have that $(0,0,0,0) \in X^4$. So that X is non-empty set. Now we show that X is both closed and open in $[0,1]$ and hence by the connectedness $X=[0,1]$. As a result, $H(.,1)$ has a quadruple fixed point in U^4. First we show that X closed in $[0,1]$. To see this, Let $\{\kappa_n\}_{n=1}^{\infty} \subseteq X$ with $\kappa_n \to \kappa \in [0,1]$ as $n \to \infty$. We must show that $\kappa \in X$. Since $\kappa_n \in X$ for $n=0,1,2,3,\cdots$, there exists sequences $\{x_n\},\{y_n\},\{z_n\},\{w_n\}$ with
$x_{n+1} = H(x_n,y_n,z_n,w_n,\kappa_n)$, $y_{n+1} = H(y_n,z_n,w_n,x_n,\kappa_n)$, $z_{n+1} = H(z_n,w_n,x_n,y_n,\kappa_n)$
and $w_{n+1} = H(w_n,x_n,y_n,z_n,\kappa_n)$
Consider

$$\begin{aligned}
& G(x_{n+1},x_{n+1},x_{n+2}) \\
= \;& G(H(x_n,y_n,z_n,w_n,\kappa_n),H(x_n,y_n,z_n,w_n,\kappa_n),H(x_{n+1},y_{n+1},z_{n+1},w_{n+1},\kappa_{n+1})) \\
\leq \;& G\left(\begin{array}{c} H(x_n,y_n,z_n,w_n,\kappa_n),H(x_n,y_n,z_n,w_n,\kappa_n), \\ H(x_{n+1},y_{n+1},z_{n+1},w_{n+1},\kappa_n) \end{array} \right) \\
& +G\left(\begin{array}{c} H(x_{n+1},y_{n+1},z_{n+1},w_{n+1},\kappa_n),H(x_{n+1},y_{n+1},z_{n+1},w_{n+1},\kappa_n), \\ H(x_{n+1},y_{n+1},z_{n+1},w_{n+1},\kappa_{n+1}) \end{array} \right) \\
\leq \;& G(H(x_n,y_n,z_n,w_n,\kappa_n),H(x_n,y_n,z_n,w_n,\kappa_n),H(x_{n+1},y_{n+1},z_{n+1},w_{n+1},\kappa_n)) \\
& +M|\kappa_n-\kappa_{n+1}|.
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}
& \lim_{n\to\infty} G(x_{n+1},x_{n+1},x_{n+2}) \\
\leq \;& \lim_{n\to\infty} G(H(x_n,y_n,z_n,w_n,\kappa_n),H(x_n,y_n,z_n,w_n,\kappa_n),H(x_{n+1},y_{n+1},z_{n+1},w_{n+1},\kappa_n))
\end{aligned}$$

Since ψ is continuous and non-decreasing, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\psi(G(x_{n+1},x_{n+1},x_{n+2}))\\
\leq\ &\lim_{n\to\infty}\psi(G(H(x_n,y_n,z_n,w_n,\kappa_n),H(x_n,y_n,z_n,w_n,\kappa_n),H(x_{n+1},y_{n+1},z_{n+1},w_{n+1},\kappa_n)))\\
\leq\ &\lim_{n\to\infty}\frac{1}{4}\psi\begin{pmatrix} G(x_n,x_n,x_{n+1})+G(y_n,y_n,y_{n+1})\\ +G(z_n,z_n,z_{n+1})+G(w_n,w_n,w_{n+1})\end{pmatrix}\\
&-\lim_{n\to\infty}\phi\begin{pmatrix} G(x_n,x_n,x_{n+1})+G(y_n,y_n,y_{n+1})\\ +G(z_n,z_n,z_{n+1})+G(w_n,w_n,w_{n+1})\end{pmatrix}
\end{aligned}$$

By using the property of ψ, we get

$$\begin{aligned}
&\lim_{n\to\infty}\psi\begin{pmatrix} G(x_{n+1},x_{n+1},x_{n+2})+G(y_{n+1},y_{n+1},y_{n+2})+G(z_{n+1},z_{n+1},z_{n+2})\\ +G(w_{n+1},w_{n+1},w_{n+2})\end{pmatrix}\\
\leq\ &\lim_{n\to\infty}\psi\left(G(x_n,x_n,x_{n+1})+G(y_n,y_n,y_{n+1})+G(z_n,z_n,z_{n+1})+G(w_n,w_n,w_{n+1})\right)\\
&-\lim_{n\to\infty}4\phi\left(G(x_n,x_n,x_{n+1})+G(y_n,y_n,y_{n+1})+G(z_n,z_n,z_{n+1})+G(w_n,w_n,w_{n+1})\right)
\end{aligned}$$

Set

$$\delta_{n+1}=\begin{matrix} G(x_{n+1},x_{n+1},x_{n+2})+G(y_{n+1},y_{n+1},y_{n+2})+G(z_{n+1},z_{n+1},z_{n+2})\\ +G(w_{n+1},w_{n+1},w_{n+2})\end{matrix}$$

Then we have

$$\lim_{n\to\infty}\psi(\delta_{n+1})\leq\lim_{n\to\infty}\psi(\delta_n)-\lim_{n\to\infty}4\phi(\delta_n)\text{ for all }n,$$

Since ψ is non-decreasing, we get that $\delta_{n+1}\leq\delta_n$ for all n. Hence $\{\delta_n\}$ is a non-increasing sequence. Since it is bounded below from 0, there is some $\delta>0$ such that
$\psi(\delta)\leq\psi(\delta)-4\phi(\delta)<\psi(\delta)$, which is a contraction. Thus, $\delta=0$, that is

$$\begin{aligned}
\lim_{n\to\infty}\delta_{n+1} &= \lim_{n\to\infty}\begin{pmatrix} G(x_{n+1},x_{n+1},x_{n+2})+G(y_{n+1},y_{n+1},y_{n+2})+G(z_{n+1},z_{n+1},z_{n+2})\\ +G(w_{n+1},w_{n+1},w_{n+2})\end{pmatrix}\\
&= 0
\end{aligned}$$

Now, we shall show that $\{x_n\},\{y_n\},\{z_n\},\{w_n\}$ are Cauchy sequences in the G-metric space $(\mathscr{P},G)$. Assume the contrary, that is, one of the sequences$\{x_n\},\{y_n\},\{z_n\},\{w_n\}$ is not a

Cauchy sequence, there exists $\varepsilon > 0$ and monotone increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$,

$$\begin{aligned} & G(x_{m_k}, x_{n_k}, x_{n_k}) + G(y_{m_k}, y_{n_k}, y_{n_k}) \\ & + G(z_{m_k}, z_{n_k}, z_{n_k}) + G(w_{m_k}, w_{n_k}, w_{n_k}) \geq \varepsilon. \end{aligned} \tag{27}$$

and

$$\begin{aligned} & G\left(x_{m_k}, x_{n_{k-1}}, x_{n_{k-1}}\right) + G\left(y_{m_k}, y_{n_{k-1}}, y_{n_{k-1}}\right) \\ & + G\left(z_{m_k}, z_{n_{k-1}}, z_{n_{k-1}}\right) + G\left(w_{m_k}, w_{n_{k-1}}, w_{n_{k-1}}\right) < \varepsilon. \end{aligned} \tag{28}$$

By use of the rectangle inequality and (27), (28), we have

$$\begin{aligned} \varepsilon & \leq G(x_{m_k}, x_{n_k}, x_{n_k}) + G(y_{m_k}, y_{n_k}, y_{n_k}) \\ & \quad + G(z_{m_k}, z_{n_k}, z_{n_k}) + G(w_{m_k}, w_{n_k}, w_{n_k}) \\ & \leq G\left(x_{m_k}, x_{n_{k-1}}, x_{n_{k-1}}\right) + G\left(x_{n_{k-1}}, x_{n_k}, x_{n_k}\right) \\ & \qquad + G\left(y_{m_k}, y_{n_{k-1}}, y_{n_{k-1}}\right) + G\left(y_{n_{k-1}}, y_{n_k}, y_{n_k}\right) \\ & \qquad + G\left(z_{m_k}, z_{n_{k-1}}, z_{n_{k-1}}\right) + G\left(z_{n_{k-1}}, z_{n_k}, z_{n_k}\right) \\ & \qquad + G\left(w_{m_k}, w_{n_{k-1}}, w_{n_{k-1}}\right) + G\left(w_{n_{k-1}}, w_{n_k}, w_{n_k}\right) \\ & \leq \varepsilon + G\left(x_{n_{k-1}}, x_{n_k}, x_{n_k}\right) + G\left(y_{n_{k-1}}, y_{n_k}, y_{n_k}\right) \\ & \qquad + G\left(z_{n_{k-1}}, z_{n_k}, z_{n_k}\right) + G\left(w_{n_{k-1}}, w_{n_k}, w_{n_k}\right). \end{aligned}$$

Letting $k \to \infty$, we get

$$\lim_{k\to\infty} \zeta_k = \lim_{k\to\infty} \begin{pmatrix} G(x_{m_k}, x_{n_k}, x_{n_k}) + G(y_{m_k}, y_{n_k}, y_{n_k}) \\ + G(z_{m_k}, z_{n_k}, z_{n_k}) + G(w_{m_k}, w_{n_k}, w_{n_k}) \end{pmatrix} = \varepsilon.$$

Again, by the rectangle inequality, we have

$$\begin{aligned}
\zeta_k &= G(x_{m_k},x_{n_k},x_{n_k})+G(y_{m_k},y_{n_k},y_{n_k}) \\
&\quad +G(z_{m_k},z_{n_k},z_{n_k})+G(w_{m_k},w_{n_k},w_{n_k}) \\
&\leq G\left(x_{m_k},x_{m_{k+1}},x_{m_{k+1}}\right)+G\left(x_{m_{k+1}},x_{n_{k+1}},x_{n_{k+1}}\right)+G\left(x_{n_{k+1}},x_{n_k},x_{n_k}\right) \\
&\quad +G\left(y_{m_k},y_{m_{k+1}},y_{m_{k+1}}\right)+G\left(y_{m_{k+1}},y_{n_{k+1}},y_{n_{k+1}}\right)+G\left(y_{n_{k+1}},y_{n_k},y_{n_k}\right) \\
&\quad +G\left(z_{m_k},z_{m_{k+1}},z_{m_{k+1}}\right)+G\left(z_{m_{k+1}},z_{n_{k+1}},z_{n_{k+1}}\right)+G\left(z_{n_{k+1}},z_{n_k},z_{n_k}\right) \\
&\quad +G\left(w_{m_k},w_{m_{k+1}},w_{m_{k+1}}\right)+G\left(w_{m_{k+1}},w_{n_{k+1}},w_{n_{k+1}}\right)+G\left(w_{n_{k+1}},w_{n_k},w_{n_k}\right) \\
&\leq \delta_{m_k}+\delta_{n_k}+G\left(x_{m_{k+1}},x_{n_{k+1}},x_{n_{k+1}}\right)+G\left(y_{m_{k+1}},y_{n_{k+1}},y_{n_{k+1}}\right) \\
&\quad +G\left(z_{m_{k+1}},z_{n_{k+1}},z_{n_{k+1}}\right)+G\left(w_{m_{k+1}},w_{n_{k+1}},w_{n_{k+1}}\right).
\end{aligned}$$

Letting $k\to\infty$, and applying ψ on both side, we get

$$\begin{aligned}
\lim_{k\to\infty}\psi(\zeta_k) &\leq \lim_{k\to\infty}\psi\left(\begin{array}{c} \delta_{m_k}+\delta_{n_k} \\ +G\left(x_{m_{k+1}},x_{n_{k+1}},x_{n_{k+1}}\right)+G\left(y_{m_{k+1}},y_{n_{k+1}},y_{n_{k+1}}\right) \\ +G\left(z_{m_{k+1}},z_{n_{k+1}},z_{n_{k+1}}\right)+G\left(w_{m_{k+1}},w_{n_{k+1}},w_{n_{k+1}}\right) \end{array}\right). \\
&\leq \lim_{k\to\infty}\psi(\delta_{m_k})+\lim_{k\to\infty}\psi(\delta_{n_k})+\lim_{k\to\infty}\psi(\zeta_k)-\lim_{k\to\infty}4\phi(\zeta_k)<\lim_{k\to\infty}\psi(\zeta_k).
\end{aligned}$$

we get a contradiction. This shows that $\{x_n\},\{y_n\},\{z_n\},\{w_n\}$ are Cauchy sequences in the G-metric space $(\mathscr{P},G)$ and by completeness of $(\mathscr{P},G)$, there exist $a,b,c,d\in\mathscr{P}$ with

$$\begin{aligned}
\lim_{n\to\infty}x_{n+1}=a \quad \lim_{n\to\infty}y_{n+1}=b \\
\lim_{n\to\infty}z_{n+1}=c \quad \lim_{n\to\infty}w_{n+1}=d.
\end{aligned}$$

By using (τ_1) and property of ψ, we have

$$\psi\left(\begin{array}{c} G(H(a,b,c,d,\kappa),H(a,b,c,d,\kappa),a)+G(H(b,c,d,a,\kappa),H(b,c,d,a,\kappa),b) \\ +G(H(c,d,a,b,\kappa),H(c,d,a,b,\kappa),c)+G(H(d,a,b,c,\kappa),H(d,a,b,c,\kappa),d) \end{array}\right)$$

$$= \lim_{n\to\infty} \psi \begin{pmatrix} G(H(a,b,c,d,\kappa),H(a,b,c,d,\kappa),H(x_{n+1},y_{n+1},z_{n+1},w_{n+1},\kappa)) \\ +G(H(b,c,d,a,\kappa),H(b,c,d,a,\kappa),H(y_{n+1},z_{n+1},w_{n+1},x_{n+1},\kappa)) \\ +G(H(c,d,a,b,\kappa),H(c,d,a,b,\kappa),H(z_{n+1},w_{n+1},x_{n+1},y_{n+1},\kappa)) \\ +G(H(d,a,b,c,\kappa),H(d,a,b,c,\kappa),H(w_{n+1},x_{n+1},y_{n+1},z_{n+1},\kappa)) \end{pmatrix}$$

$$\leq \lim_{n\to\infty} \psi \begin{bmatrix} G(a,a,x_{n+1})+G(b,b,y_{n+1}) \\ +G(c,c,z_{n+1})+G(d,d,w_{n+1}) \end{bmatrix}$$

$$-\lim_{n\to\infty} 4\phi \begin{bmatrix} G(a,a,x_{n+1})+G(b,b,y_{n+1}) \\ +G(c,c,z_{n+1})+G(d,d,w_{n+1}) \end{bmatrix} = 0.$$

It follows that $H(a,b,c,d,\kappa)=a, H(b,c,d,a,\kappa)=b, H(c,d,a,b,\kappa)=c$ and $H(d,a,b,c,\kappa)=d$. Thus $\kappa \in X$. Hence X is closed in $[0,1]$. Let $\kappa_0 \in X$, then there exist $x_0,y_0,z_0,w_0 \in U$ with $x_0 = H(x_0,y_0,z_0,w_0,\kappa_0)$, $y_0 = H(y_0,z_0,w_0,x_0,\kappa_0)$, $z_0 = H(z_0,w_0,x_0,y_0,\kappa_0)$ and $w_0 = H(w_0,x_0,y_0,z_0,\kappa_0)$. Since U is open, then there exist $r>0$ such that $B_G(x_0,x_0,r) \subseteq U$. Choose $\kappa \in (\kappa_0-\varepsilon,\kappa_0+\varepsilon)$ such that
$|\kappa-\kappa_0| \leq \frac{1}{M^n} < \frac{\varepsilon}{2}$, then for $x \in \overline{B_G(x_0,x_0,r)} = \{x \in X/G(x,x,x_0) \leq r+G(x_0,x_0,x_0)\}$, Also

$$\begin{aligned} & G(H(x,y,z,w,\kappa),H(x,y,z,w,\kappa),x_0) \\ = \; & G(H(x,y,z,w,\kappa),H(x,y,z,w,\kappa),H(x_0,y_0,z_0,w_0,\kappa_0)) \\ \leq \; & G(H(x,y,z,w,\kappa),H(x,y,z,w,\kappa),H(x,y,z,w,\kappa_0)) \\ & +G(H(x,y,z,w,\kappa_0),H(x,y,z,w,\kappa_0),H(x_0,y_0,z_0,w_0,\kappa_0)) \\ \leq \; & M|\kappa-\kappa_0| + G(H(x,y,z,w,\kappa_0),H(x,y,z,w,\kappa_0),H(x_0,y_0,z_0,w_0,\kappa_0)) \\ \leq \; & \frac{1}{M^{n-1}} + G(H(x,y,z,w,\kappa_0),H(x,y,z,w,\kappa_0),H(x_0,y_0,z_0,w_0,\kappa_0)). \end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned} & G(H(x,y,z,w,\kappa),H(x,y,z,w,\kappa),x_0) \\ & \leq G(H(x,y,z,w,\kappa_0),H(x,y,z,w,\kappa_0),H(x_0,y_0,z_0,w_0,\kappa_0)). \end{aligned}$$

Since ψ is continuous and non-decreasing, we have

$$\begin{aligned}
&\psi(G(H(x,y,z,w,\kappa),H(x,y,z,w,\kappa),x_0)) \\
\leq\ & \psi(G(H(x,y,z,w,\kappa_0),H(x,y,z,w,\kappa_0),H(x_0,y_0,z_0,w_0,\kappa_0))) \\
\leq\ & \frac{1}{4}\psi\Big(G(x,x,x_0)+G(y,y,y_0)+G(z,z,z_0)+G(w,w,w_0) \Big) \\
& -\phi\Big(G(x,x,x_0)+G(y,y,y_0)+G(z,z,z_0)+G(w,w,w_0) \Big)
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \psi\left(\begin{array}{c} G(H(x,y,z,w,\kappa),H(x,y,z,w,\kappa),x_0)+G(H(y,z,w,x,\kappa),H(y,z,w,x,\kappa),y_0) \\ +G(H(z,w,x,y,\kappa),H(z,w,x,y,\kappa),z_0)+G(H(w,x,y,w,\kappa),H(w,x,y,w,\kappa),w_0) \end{array} \right) \\
\leq\ & \psi\Big(G(x,x,x_0)+G(y,y,y_0)+G(z,z,z_0)+G(w,w,w_0) \Big) \\
& -4\phi\Big(G(x,x,x_0)+G(y,y,y_0)+G(z,z,z_0)+G(w,w,w_0) \Big) \\
\leq\ & \psi\Big(G(x,x,x_0)+G(y,y,y_0)+G(z,z,z_0)+G(w,w,w_0) \Big).
\end{aligned}$$

Since ψ is non-decreasing, we have

$$\begin{aligned}
& G(H(x,y,z,w,\kappa),H(x,y,z,w,\kappa),x_0)+G(H(y,z,w,x,\kappa),H(y,z,w,x,\kappa),y_0) \\
& +G(H(z,w,x,y,\kappa),H(z,w,x,y,\kappa),z_0)+G(H(w,x,y,w,\kappa),H(w,x,y,w,\kappa),w_0) \\
\leq\ & G(x,x,x_0)+G(y,y,y_0)+G(z,z,z_0)+G(w,w,w_0) \\
\leq\ & 4r+G(x_0,x_0,x_0)+G(y_0,y_0,y_0)+G(z_0,z_0,z_0)+G(w_0,w_0,w_0).
\end{aligned}$$

Thus for each fixed $\kappa \in (\kappa_0-\varepsilon, \kappa_0+\varepsilon)$, $H(.,\kappa) : \overline{B_G(x_0,x_0,r)} \to \overline{B_G(x_0,x_0,r)}$, $H(.,\kappa) : \overline{B_G(y_0,y_0,r)} \to \overline{B_G(y_0,y_0,r)}$, $H(.,\kappa) : \overline{B_G(z_0,z_0,r)} \to \overline{B_G(z_0,z_0,r)}$ and $H(.,\kappa) : \overline{B_G(w_0,w_0,r)} \to \overline{B_G(w_0,w_0,r)}$. Then all conditions of Theorem (3.2.1) are satisfied. Thus we conclude that $H(.,\kappa)$ has a quadruple fixed point in $\overline{U}^4$. But this must be in U^4. Since (τ_0) holds. Thus, $\kappa \in X$ for any $\kappa \in (\kappa_0-\varepsilon, \kappa_0+\varepsilon)$. Hence $(\kappa_0-\varepsilon, \kappa_0+\varepsilon) \subseteq X$. Clearly X is open in [0, 1].

For the reverse implication, we use the same strategy.

4. Conclusion

We ensured the existence and uniqueness of a common fixed point for six mappings in the class of G-metric spaces via (ψ,ϕ)-type contractions. Two illustrated applications have been provided.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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