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NEIGHBORHOOD PSEUDO CHROMATIC POLYNOMIAL OF A PATH

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Abstract. Let $N(v) = \{x : vx \in E(G)\}$ be the open neighborhood and $N[v] = N(v) \cup \{v\}$ be the closed neighborhood of a vertex $v \in V$. A neighborhood pseudo coloring of a connected graph $G(V,E)$ is a function $c : V \to \{1,2,\cdots,k\}$ such that for each vertex $v \in V$, there exists at least two vertices $u,w \in N[v]$ with $c(u) = c(w)$. A neighborhood pseudo coloring $c : V \to \{1,2,\cdots,k\}$ which is surjective, is called neighborhood pseudo $k-$coloring and the maximum k for which G admits a neighborhood pseudo $k-$coloring is called the neighborhood pseudo chromatic number of G, denoted by $\psi_{nhd}(G)$. Chromatic polynomials are defined and studied for various types of proper coloring. In this paper, we initiate the study of neighborhood pseudo chromatic polynomial of a graph G as a polynomial in λ to count the number of distinct ways to neighborhood pseudo color G with atmost λ colors. Neighborhood pseudo chromatic polynomial of a path P_n is determined with a recurrence relation for its coefficients. Further an efficient algorithm is developed for its evaluation.

Keywords: coloring; neighborhood; neighborhood pseudo coloring; chromatic polynomial; neighborhood pseudo chromatic polynomial.

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1. INTRODUCTION

Let $G(V,E)$ be a graph of order n. For a given positive integer $k \leq n$, a vertex k-coloring of G is a surjection $c : V \to \{1,2,3,\cdots,k\}$. A vertex k-coloring of G is said to be a proper vertex k-coloring of G if $c(u) \neq c(v)$, whenever u and v are adjacent in G. The smallest integer such that the graph G admits a proper vertex k-coloring is called a chromatic number of G and is denoted by $\chi(G)$.

The chromatic polynomial of a graph is a polynomial in λ which counts the number of distinct ways G admits proper coloring with atmost λ colors. George David Birkhoff introduced the chromatic polynomial in 1912 as an attempt to prove four color theorem [4]. Some properties of chromatic polynomials are derived, coefficients and bounds of the chromatic polynomial are studied in [1, 6, 7]. Some of the results on chromatic polynomial are found in [3]. standard terminology terms not defined here may be found in [2, 5]. The graph coloring and chromatic polynomial problem has huge number of applications like resource allocation, image segmentation, networking etc.

A pseudo k-coloring of graph G is a k-coloring of G in which adjacent vertices can receive the same color. In 2014, B. Sooryanarayana et. al [8] introduced a neighborhood pseudo k-coloring of a graph G. Let $G(V,E)$ be a simple, non trivial, connected and undirected graph. A neighborhood pseudo coloring is a function $c : V \to \{1,2,\cdots,k\}$ such that for each vertex $v \in V$, there exists atleast two vertices $u,w \in N[v]$ with $c(u) = c(w)$. A neighborhood pseudo coloring $c : V \to \{1,2,\cdots,k\}$ which is surjective, is called neighborhood pseudo $k-$coloring and the maximum k for which G admits a neighborhood pseudo $k-$coloring is called the neighborhood pseudo chromatic number of G, denoted by $\psi_{nhd}(G)$. Also from [8], $\psi_{nhd}(P_n) = \lfloor \frac{n}{2} \rfloor$ for $n \geq 2$ and $1 \leq \psi_{nhd}(G) \leq n-1$, we consider this result for the immediate reference.

The purpose of this paper is to introduce the new notion of pseudo chromatic polynomial for the case of pseudo coloring similar to one which is well established as chromatic polynomial for the purpose of proper coloring. Further, we obtained the neighborhood pseudo chromatic polynomial of a path P_n.

2. NEIGHBORHOOD PSEUDO CHROMATIC POLYNOMIAL

For given λ colors, a neighborhood pseudo chromatic polynomial of a graph G denoted by $P_{nhd}(G,\lambda)$ is a polynomial in λ which counts the number of distinct ways to neighborhood pseudo color the vertices of G with atmost λ colors and is given by $P_{nhd}(G,\lambda) = \sum_{k=1}^{\lambda} ({}^{\lambda}C_k)c_{nhd}(G,k)$ where, $c_{nhd}(G,k)$ represents possible number of distinct neighborhood pseudo coloring of $V(G)$ with exactly k colors and is called the k^{th} coefficient of neighborhood pseudo chromatic polynomial $P_{nhd}(G,\lambda)$.

Observation 2.1. *It is obvious that graph G can be neighborhood pseudo colored with one color in exactly one way and hence* $c_{nhd}(G,1) = 1$.

Observation 2.2. *By the definition of neighborhood pseudo coloring, atmost* $\psi_{nhd}(G)$ *colors can be used for any neighborhood pseudo coloring of G, which implies* $c_{nhd}(G,k) = 0$ *when* $k > \psi_{nhd}(G)$.

3. NEIGHBORHOOD PSEUDO CHROMATIC POLYNOMIAL OF A PATH

Let P_n be a path with the vertex set $\{v_1, v_2, v_3, \cdots, v_n\}$ and the edge set $\{v_1v_2, v_2v_3, , \cdots, v_{n-1}v_n\}$.

Observation 3.1. *Consecutive three vertices of* P_n *does not receive three different colors for any neighborhood pseudo coloring, for otherwise the closed neighborhood of the middle vertex contain three different colors.*

Observation 3.2. $\psi_{nhd}(P_2) = 1$ *and* $\psi_{nhd}(P_3) = 1$ *which implies that* $c_{nhd}(P_2,1) = c_{nhd}(P_3,1) = 1$ *and* $c_{nhd}(P_2,k) = c_{nhd}(P_3,k) = 0$ *for every* $k > 1$. *Hence* $P_{nhd}(P_2,\lambda) = \lambda$ *and* $P_{nhd}(P_3,\lambda) = \lambda$.

Example 3.3. $P_{nhd}(P_2,2) = {}^2C_1 \times c_{nhd}(P_2,1) + {}^2C_2 \times c_{nhd}(P_2,2) = 2 \times 1 + 1 \times 0 = 2.$

Distinct possible ways of neighborhood pseudo coloring of P_2 with $\lambda = 2$, is as shown in FIGURE 1.

Example 3.4. $P_{nhd}(P_3,2) = {}^2C_1 \times c_{nhd}(P_3,1) + {}^2C_2 \times c_{nhd}(P_3,2) = 2 \times 1 + 1 \times 0 = 2.$

FIGURE 1. All possible ways of neighborhood pseudo coloring of P_2 with atmost two colors.

FIGURE 2. All possible ways of neighborhood pseudo coloring of P_3 with atmost two colors.

Distinct possible ways of neighborhood pseudo coloring of P_3 with $\lambda = 2$, is as shown in FIGURE 2.

Theorem 3.5. *For the path P_n with $n \geq 4$, $P_{nhd}(P_n, \lambda) = 2^{n-2}$ for $\lambda = 2$.*

Proof. Let P_n, $n \geq 4$ be a path, and let the given two colors are represented by 1, 2. Then for any neighborhood pseudo coloring $c : V(P_n) \to \{1, 2\}$, $c(v_1) = c(v_2)$, $c(v_{n-1}) = c(v_n)$ and each vertex v_i, $2 \leq i \leq n-1$ can be assigned with either 1 or 2, which implies $P_{nhd}(P_n, 2) = 2^{n-2}$. □

Since $P_{nhd}(P_n, 2)$ includes the possibilities of neighborhood pseudo coloring the vertices of P_n using single color, we conclude the following lemma.

Lemma 3.6. *For the path P_n, $n \geq 4$, $c_{nhd}(P_n, 2) = 2^{n-2} - 2$.*

Lemma 3.7. *For the path P_n, $n \geq 2$,*

$$c_{nhd}(P_n, 3) = \begin{cases} 0 & \text{if} \quad 2 \leq n \leq 5 \\ a_{n-1} - 3 \times 2^{n-2} + 3 & \text{if} \quad n \geq 6 \end{cases}$$

where $a_{n-1} = 2a_{n-2} + a_{n-3}, a_3 = 9, a_4 = 21$.

Proof. $\psi_{nhd}(P_n) = \lfloor \frac{n}{2} \rfloor$ for $n \geq 2$ which imply the minimum value of n required for neighborhood pseudo coloring with exactly three colors is six and hence $c_{nhd}(P_n, 3) = 0$ for every n with $2 \leq n \leq 5$. Let $n \geq 6$ and $a_1, a_2, a_3, ..., a_n$ represents the total number of possibilities of neighborhood pseudo color the vertices $v_1, v_2, v_3, ..., v_n$ respectively, using three colors say 1, 2 and 3, assuming $v_1, v_2, v_3, ..., v_{i-1}$ are already colored while finding the possibilities to color the vertex v_i. Since for neighborhood pseudo coloring, v_1 and v_2 should receive the same color, the vertex

v_1, can color using either color 1 or 2 or 3 and so is the vertex v_2, which imply $a_1 = a_2 = 3$ and for each choice of color of v_2, v_3 can color independently using three colors and hence $a_3 = 3 \times a_2 = 3^2$. Consider the vertex v_4 with $n \neq 4$. Let us denote color of v_i as $c(v_i)$. Consider the case when $c(v_2) = 1$, then $c(v_3) = 1$ or 2 or 3 and from Observation 3.1 if $c(v_3) = 1$, then v_4 can color using either 1 or 2 or 3, if $c(v_3) = 2$ then v_4 can color using either 1 or 2 and if $c(v_3) = 3$, then v_4 can color using either 1 or 3. Hence v_4 has $3 \times (3 \times 1 + 2 \times 2) = 3 \times 7$ possibilities of coloring, that is $a_4 = 21$.

To find a_5 consider the following cases with $c(v_1) = c(v_2) = 1$ and $n \neq 5$.

Case 1: $c(v_3) = 1$.

i. If $c(v_4) = 1$ then $c(v_5) = 1$ or 2 or 3 — Three possibilities.
ii. If $c(v_4) = 2$ then $c(v_5) = 1$ or 2 — Two possibilities.
iii. If $c(v_4) = 3$ then $c(v_5) = 1$ or 2 — Two possibilities.

Case 2: $c(v_3) = 2$.

i. If $c(v_4) = 1$ then $c(v_5) = 1$ or 2 — Two possibilities.
ii. If $c(v_4) = 2$ then $c(v_5) = 1$ or 2 or 3 — Three possibilities.

Case 3: $c(v_3) = 3$.

i. If $c(v_4) = 1$ then $c(v_5) = 1$ or 3 — Two possibilities.
ii. If $c(v_4) = 3$ then $c(v_5) = 1$ or 2 or 3 — Three possibilities.

Hence v_5 has $3 \times (3 \times 3 + 2 \times 4) = 3 \times 17$ possibilities of coloring, that is $a_5 = 51 = 3 \times (3 \times 3 + 2 \times 4) = 3 \times 3^2 + 2(12) = 3 \times 3^2 + 2(21 - 9) = 3 \times a_3 + 2(a_4 - a_3) = a_3 + 2a_4$.

Further listing the possibilities, to find a_6, consider the following cases with $c(v_1) = c(v_2) = 1$ and $n \neq 6$.

Case 1: $c(v_3) = 1$.

Subcase 1: $c(v_4) = 1$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 2 or 3 — Three possibilities.
ii. If $c(v_5) = 2$ then $c(v_6) = 1$ or 2 — Two possibilities.
iii. If $c(v_5) = 3$ then $c(v_6) = 1$ or 3 — Two possibilities.

Subcase 2: $c(v_4) = 2$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 2 — Two possibilities.

ii. If $c(v_5) = 2$ then $c(v_6) = 1$ or 2 or 3 — Three possibilities.

Subcase 3: $c(v_4) = 3$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 3 — Two possibilities.

ii. If $c(v_5) = 3$ then $c(v_6) = 1$ or 2 or 3 — Three possibilities.

Case 2: $c(v_3) = 2$.

Subcase 1: $c(v_4) = 1$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 2 or 3 — Three possibilities.

ii. If $c(v_5) = 2$ then $c(v_6) = 1$ or 2 — Two possibilities.

Subcase 2: $c(v_4) = 2$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 2 — Two possibilities.

ii. If $c(v_5) = 2$ then $c(v_6) = 1$ or 2 or 3 — Three possibilities.

iii. If $c(v_5) = 3$ then $c(v_6) = 2$ or 3 — Two possibilities.

Case 3: $c(v_3) = 3$.

Subcase 1: $c(v_4) = 1$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 2 or 3 — Three possibilities.

ii. If $c(v_5) = 3$ then $c(v_6) = 1$ or 3 — Two possibilities.

Subcase 2: $c(v_4) = 3$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 3 — Two possibilities.

ii. If $c(v_5) = 2$ then $c(v_6) = 2$ or 3 — Two possibilities.

iii. If $c(v_5) = 3$ then $c(v_6) = 1$ or 2 or 3 — Three possibilities.

Hence v_6 has $3 \times (3 \times 7 + 2 \times 10) = 3 \times 41 = 123$ possibilities of coloring, that is $a_6 = 3 \times (3 \times 7 + 2 \times 10) = 3 \times 21 + 2(30) = 3 \times 21 + 2(51 - 21) = 3 \times a_4 + 2(a_5 - a_4) = a_4 + 2a_5$. In general, neighborhood pseudo coloring of any vertex v_i is effected only by $c(v_{i-1})$ and $c(v_{i-2})$, the similar pattern of coloring follows at each level and hence a_i is recursively expressed in terms of a_{i-1} and a_{i-2}. Therefore, $a_i = 2a_{i-1} + a_{i-2}$ for any i with $4 \leq i \leq n-1$. Also for path P_n the end vertex v_n should receive the same color as that of v_{n-1} for neighborhood pseudo coloring and hence $a_{n-1} = a_n = 2a_{n-2} + a_{n-3}$. Particularly, $a_5 = 51$, $a_6 = 123$ $\Rightarrow$ $a_7 = 2a_6 + a_5 = 2(123) + 51 = 297$, $a_8 = 2a_7 + a_6 = 2(297) + 123 = 717$ so on. But this a_n also include the counts of possibilities of neighborhood pseudo coloring with exactly one and exactly two colors and number of possibilities with exactly one color is in 3 ways; with exactly two colors is in ${}^3C_2 \times (2^{n-2} - 2)$ ways from Lemma 3.6. Therefore for $n \geq 6$, $c_{nhd}(P_n, 3) = 2a_{n-2} + a_{n-3} - {}^3C_2(2^{n-2} - 2) - 3 = 2a_{n-2} + a_{n-3} - 3 \times 2^{n-2} + 3$ with $a_3 = 9$, $a_4 = 21$ and $a_5 = 51$. □

Observation 3.8. *It follows from the proof of Lemma 3.7, for $n \geq 6$, the value of neighborhood pseudo chromatic polynomial of path P_n with atmost three colors is $P_{nhd}(P_n, 3) = a_{n-1} = 2a_{n-2} + a_{n-3}$ with $a_3 = 9$, $a_4 = 21$ and $a_5 = 51$.*

Example 3.9. *For the path P_6, $c_{nhd}(P_6, 3) = 2a_{6-2} + a_{6-3} - 3 \times 2^{6-2} + 3 = 2a_4 + a_3 - 3 \times 2^4 + 3 = 2(21) + 9 - 48 + 3 = 6$.*

For the path P_7, $c_{nhd}(P_7, 3) = 2a_{7-2} + a_{7-3} - 3 \times 2^{7-2} + 3 = 2a_5 + a_4 - 3 \times 2^5 + 3 = 2(51) + 21 - 96 + 3 = 30$.

Possible ways of neighborhood pseudo coloring of P_7 with exactly three colors and the program output for number of possibilities of neighborhood pseudo coloring of P_7 with exactly three colors, are shown in FIGURE 3 and FIGURE 4 respectively.

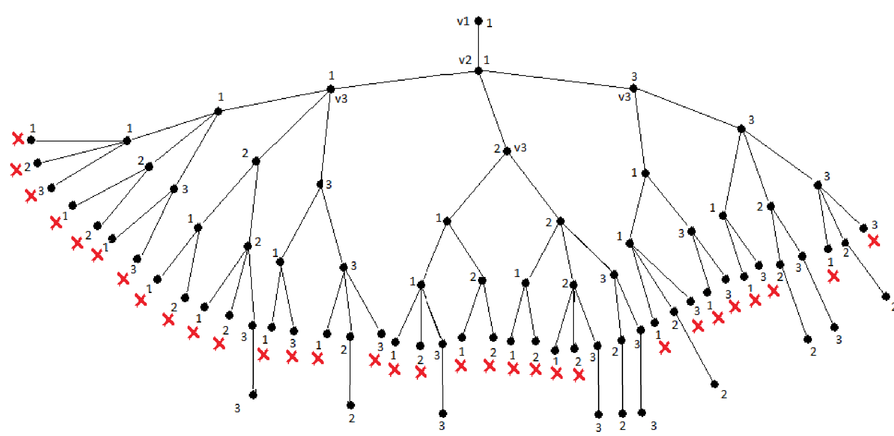

FIGURE 3. Possible ways of neighborhood pseudo coloring of P_7 with exactly three colors when $c(v_1) = 1$.

Lemma 3.10. *For the path P_n, $n \geq 2$,*

$$c_{nhd}(P_n, 4) = \begin{cases} 0 & \text{if} \quad 2 \leq n \leq 7 \\ a_{n-1} - {}^4C_3 \times c_{nhd}(P_n, 3) - {}^4C_2 \times c_{nhd}(P_n, 2) - 4, & \text{if} \quad n \geq 8 \end{cases}$$

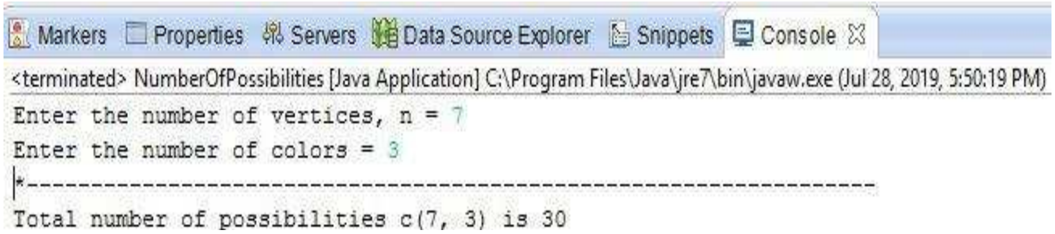

FIGURE 4. Program output for the value of $c_{nhd}(P_7, 3)$.

where $a_n = 2a_{n-2} + 2a_{n-3}, a_2 = 4, a_3 = 16$.

Proof. We have $\psi_{nhd}(P_8) = 4$, which imply $c_{nhd}(P_n, 4) = 0$ for every n with $2 \le n \le 7$. Let $n \ge 8$ and $a_1, a_2, a_3, ..., a_n$ represents the total number of possibilities of neighborhood pseudo color the vertices $v_1, v_2, v_3, ..., v_n$ respectively, using four colors say 1, 2, 3 and 4, assuming $v_1, v_2, v_3, ..., v_{i-1}$ are already colored while finding a_i. We have, $a_1 = a_2 = 4$ and v_3 can color independently using four colors and hence $a_3 = 4 \times a_2 = 4^2 = 16$. Consider the case when $c(v_1) = c(v_2) = 1$ and using Observation 3.1, v_4 can be colored in the following ways:

i. If $c(v_3) = 1$ then $c(v_4) = 1$ or 2 or 3 or 4— four possibilities.
ii. If $c(v_3) = 2$ then $c(v_4) = 1$ or 2 — two possibilities.
iii. If $c(v_3) = 3$ then $c(v_4) = 1$ or 3 — two possibilities.
iv. If $c(v_3) = 4$ then $c(v_4) = 1$ or 4 — two possibilities.

Hence v_4 has $4 \times (4 \times 1 + 2 \times 3) = 40$ possibilities of coloring.

Therefore, $a_4 = 40 = 4 \times (4 \times 1 + 2 \times 3) = (4 \times 4 + 2 \times 12) = 4 \times a_2 + 2 \times (a_3 - a_2)$.

Similarly possibilities of neighborhood pseudo coloring of the vertex v_5 are;

Case 1: $c(v_3) = 1$.

i. If $c(v_4) = 1$ then $c(v_5) = 1$ or 2 or 3 or 4 — Four possibilities.
ii. If $c(v_4) = 2$ then $c(v_5) = 1$ or 2 — Two possibilities.
iii. If $c(v_4) = 3$ then $c(v_5) = 1$ or 3 — Two possibilities.
iv. If $c(v_4) = 4$ then $c(v_5) = 1$ or 4 — Two possibilities.

Case 2: $c(v_3) = 2$.

i. If $c(v_4) = 1$ then $c(v_5) = 1$ or 2 — Two possibilities.
ii. If $c(v_4) = 2$ then $c(v_5) = 1$ or 2 or 3 or 4 — Four possibilities.

Case 3: $c(v_3) = 3$

i. If $c(v_4) = 1$ then $c(v_5) = 1$ or 3 — Two possibilities.

ii. If $c(v_4) = 3$ then $c(v_5) = 1$ or 2 or 3 or 4 — Four possibilities.

Case 4: $c(v_3) = 4$

i. If $c(v_4) = 1$ then $c(v_5) = 1$ or 4 — Two possibilities.

ii. If $c(v_4) = 4$ then $c(v_5) = 1$ or 2 or 3 or 4 — Four possibilities.

Hence v_5 has $4 \times (4 \times 4 + 2 \times 6) = 112$ possibilities of coloring.

Therefore $a_5 = 112 = 4 \times (4 \times 4 + 2 \times 6) = (4 \times 4^2 + 2 \times 24) = 4 \times a_3 + 2 \times (a_4 - a_3)$.

Similarly possibilities of neighborhood pseudo coloring of the vertex v_6 are;

Case 1: $c(v_3) = 1$

Subcase 1: $c(v_4) = 1$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 2 or 3 or 4 — Four possibilities.

ii. If $c(v_5) = 2$ then $c(v_6) = 1$ or 2 — Two possibilities.

iii. If $c(v_5) = 3$ then $c(v_6) = 1$ or 3 — Two possibilities.

iv. If $c(v_5) = 4$ then $c(v_6) = 1$ or 4 — Two possibilities.

Subcase 2: $c(v_4) = 2$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 2 — Two possibilities.

ii. If $c(v_5) = 2$ then $c(v_6) = 1$ or 2 or 3 or 4 — Four possibilities.

Subcase 3: $c(v_4) = 3$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 3 — Two possibilities.

ii. If $c(v_5) = 3$ then $c(v_6) = 1$ or 2 or 3 or 4 — Four possibilities.

Subcase 4: $c(v_4) = 4$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 4 — Two possibilities.

ii. If $c(v_5) = 4$ then $c(v_6) = 1$ or 2 or 3 or 4 — Four possibilities.

Case 2: $c(v_3) = 2$.

Subcase 1: $c(v_4) = 1$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 2 or 3 or 4 — Four possibilities.

ii. If $c(v_5) = 2$ then $c(v_6) = 1$ or 2 — Two possibilities.

Subcase 2: $c(v_4) = 2$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 2 — Two possibilities.

ii. If $c(v_5) = 2$ then $c(v_6) = 1$ or 2 or 3 or 4 — Four possibilities.

iii. If $c(v_5) = 3$ then $c(v_6) = 2$ 3 — Two possibilities.

iv. If $c(v_5) = 4$ then $c(v_6) = 2$ or 4 — Two possibilities.

Case 3: $c(v_3) = 3$.

Subcase 1: $c(v_4) = 1$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 2 or 3 or 4 — Four possibilities.

ii. If $c(v_5) = 3$ then $c(v_6) = 1$ or 3 — Two possibilities.

Subcase 2: $c(v_4) = 3$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 3 — Two possibilities.

ii. If $c(v_5) = 2$ then $c(v_6) = 2$ or 3 — Two possibilities.

iii. If $c(v_5) = 3$ then $c(v_6) = 1$ or 2 or 3 or 4 — Four possibilities.

iv. If $c(v_5) = 4$ then $c(v_6) = 4$ or 3 — Two possibilities.

Case 4: $c(v_3) = 4$.

Subcase 1: $c(v_4) = 1$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 2 or 3 or 4 — Four possibilities.

ii. If $c(v_5) = 4$ then $c(v_6) = 1$ or 4 — Two possibilities.

Subcase 2: $c(v_4) = 4$.

i. If $c(v_5) = 1$ then $c(v_6) = 1$ or 4 — Two possibilities.

ii. If $c(v_5) = 2$ then $c(v_6) = 2$ or 4 — Two possibilities.

iii. If $c(v_5) = 3$ then $c(v_6) = 3$ or 4 — Two possibilities.

iv. If $c(v_5) = 4$ then $c(v_6) = 1$ or 2 or 3 or 4 — Four possibilities.

Hence v_6 has $4 \times (4 \times 10 + 2 \times 18) = 304$ possibilities of coloring. Therefore $a_6 = 304 = (4 \times 40 + 2 \times 72) = 4 \times 40 + 2 \times (112 - 40) = 4 \times a_4 + 2 \times (a_5 - a_4)$ and the recurrence formula follows at each level, hence $a_i = 4 \times a_{i-2} + 2 \times (a_{i-1} - a_{i-2}) = 2a_{i-1} + 2a_{i-2}$ for any i with $4 \leq i \leq n-1$. Also $a_n = a_{n-1} = 4 \times a_{n-3} + 2 \times (a_{n-2} - a_{n-3}) = 2a_{n-2} + 2a_{n-3}$. Particulary $a_5 = 112$, $a_6 = 304$, $a_7 = 2a_5 + 2a_6 = 2(112) + 2(304) = 832$, $a_8 = 2a_6 + 2a_7 = 2(304) + 2(832) = 2272$.

But this a_n represents the number of possibilities of neighborhood pseudo coloring of path P_n using 4 colors which includes the coloring with exactly one, two, three colors. Therefore, when

$n \geq 8$, $c_{nhd}(P_n,4) = a_{n-1} -^4C_3(c_{nhd}(P_n,3)) -^4C_2(c_{nhd}(P_n,2)) - 4$ where $a_{n-1} = 2a_{n-2} + 2a_{n-3}$ with $a_2 = 4, a_3 = 16$. □

Observation 3.11. *It follows from the proof of Lemma 3.10, For $n \geq 8$, the value of neighborhood pseudo chromatic polynomial of path P_n with atmost four colors is $P_{nhd}(P_n,4) = a_{n-1} = 2a_{n-2} + 2a_{n-3}$ where $a_2 = 4$ and $a_3 = 16$.*

Example 3.12. *For the path P_8, $c_{nhd}(P_8,4) = a_7 -^4C_3(c_{nhd}(P_8,3)) -^4C_2(c_{nhd}(P_8,2)) - 4 = 832 - 4(108) - 6(2^6 - 2) - 4 = 24$.*

For the path P_9, $c_{nhd}(P_9,4) = a_8 -^4C_3(c_{nhd}(P_9,3)) -^4C_2(c_{nhd}(P_9,2)) - 4 = 2272 - 4(336) - 6(2^7 - 2) - 4 = 168$.

The program output for number of possibilities of neighborhood pseudo coloring of P_9 with exactly four colors, is shown in FIGURE 5.

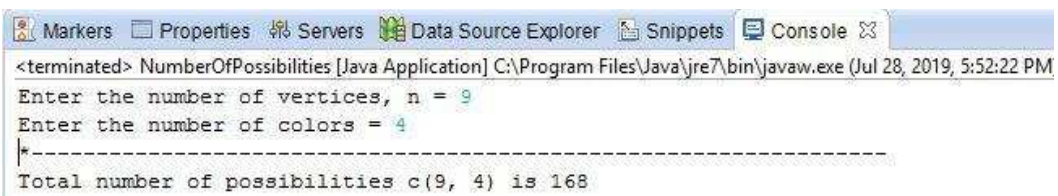

FIGURE 5. Program output for the value of $c_{nhd}(P_9,4)$

Lemma 3.13. *For the path P_n, $n \geq 2$,*

$$c_{nhd}(P_n,5) = \begin{cases} 0 & \text{if} \quad 2 \leq n \leq 9 \\ a_{n-1} -^5C_4(c_{nhd}(P_n,4)) -^5C_3(c_{nhd}(P_n,3)) -^5C_2(c_{nhd}(P_n,2)) - 5 & \text{if} \quad n \geq 10 \end{cases}$$

where $a_{n-1} = 2a_{n-2} + 3a_{n-3}, a_8 = 5465, a_9 = 16405$.

Proof. $\psi_{nbd}(P_{10}) = 5$, which imply $c_{nhd}(P_n,5) = 0$, $2 \leq n \leq 9$. Let $n \geq 10$ and $a_1, a_2, a_3, ..., a_n$ represents the total number of possibilities of neighborhood pseudo color the vertices v_1, v_2, v_3, $\ldots$, v_n respectively, using five colors say 1, 2, 3, 4 and 5, assuming $v_1, v_2, v_3, \ldots, v_{i-1}$ are already colored while finding the possibilities to color the vertex v_i. Then for neighborhood pseudo coloring, v_1 and v_2 should receive the same color, which imply $a_1 = 5$, $a_2 = 5$ and for each choice of color of v_2, v_3 can color independently using five colors and hence $a_3 = 5 \times a_2 = 5^2 =$

25. Consider the case when $c(v_1) = c(v_2) = 1$ where remaining four cases follows similarly. Then neighborhood pseudo coloring of the vertex v_4 is as follows.

i. If $c(v_3) = 1$ then $c(v_4) = 1$ or 2 or 3 or 4 or 5 — five possibilities.
ii. If $c(v_3) = 2$ then $c(v_4) = 1$ or 2 — two possibilities.
iii. If $c(v_3) = 3$ then $c(v_4) = 1$ or 3 — two possibilities.
iv. If $c(v_3) = 4$ then $c(v_4) = 1$ or 4 — two possibilities.
v. If $c(v_3) = 5$ then $c(v_4) = 1$ or 5 — two possibilities.

Hence v_4 has $5 \times (5 \times 1 + 2 \times 4) = 65$ possibilities of coloring and therefore $a_4 = 65 = 5 \times (5 \times 1 + 2 \times 4) = (5 \times 5 + 2 \times 20) = 5 \times a_2 + 2 \times (a_3 - a_2)$.

Similarly possibilities of neighborhood pseudo coloring of the vertex v_5 are as follows:

Case 1: $c(v_3) = 1$.

i. If $c(v_4) = 1$ then $c(v_5) = 1$ or 2 or 3 or 4 or 5 — five possibilities.
ii. If $c(v_4) = 2$ then $c(v_5) = 1$ or 2 — Two possibilities.
iii. If $c(v_4) = 3$ then $c(v_5) = 1$ or 3 — Two possibilities.
iv. If $c(v_4) = 4$ then $c(v_5) = 1$ or 4 — Two possibilities.
v. If $c(v_4) = 5$ then $c(v_5) = 1$ or 5 — Two possibilities.

Case 2: $c(v_3) = 2$.

i. If $c(v_4) = 1$ then $c(v_5) = 1$ or 2 — two possibilities.
ii. If $c(v_4) = 2$ then $c(v_5) = 1$ or 2 or 3 or 4 or 5 — five possibilities.

Case 3: $c(v_3) = 3$.

i. If $c(v_4) = 1$ then $c(v_5) = 1$ or 3 — two possibilities.
ii. If $c(v_4) = 3$ then $c(v_5) = 1$ or 2 or 3 or 4 or 5 — five possibilities.

Case 4: $c(v_3) = 4$.

i. If $c(v_4) = 1$ then $c(v_5) = 1$ or 4 — two possibilities.
ii. If $c(v_4) = 4$ then $c(v_5) = 1$ or 2 or 3 or 4 or 5 — five possibilities.

Case 5: $c(v_3) = 5$.

i. If $c(v_4) = 1$ then $c(v_5) = 1$ or 5 — two possibilities.
ii. If $c(v_4) = 5$ then $c(v_5) = 1$ or 2 or 3 or 4 or 5 — five possibilities.

Hence v_5 has $5 \times (5 \times 5 + 2 \times 8) = 205$ possibilities of coloring and therefore $a_5 = 205 = 5 \times (5 \times 5 + 2 \times 8) = (5 \times 5^2 + 2 \times 40) = 5 \times a_3 + 2 \times (a_4 - a_3)$ and the recurrence formula follows at each level, hence $a_i = 5 \times a_{i-2} + 2 \times (a_{i-1} - a_{i-2}) = 2a_{i-1} + 3a_{i-2}$ for any i with $4 \leq i \leq n-1$. Also, $a_n = a_{n-1} = 5 \times a_{n-3} + 2 \times (a_{n-2} - a_{n-3}) = 2a_{n-2} + 3a_{n-3}$. Particulary $a_4 = 65$, $a_5 = 205 \Rightarrow a_6 = 2a_5 + 3a_4 = 2(205) + 3(65) = 605$, $a_7 = 2a_6 + 3a_5 = 2(605) + 3(205) = 1825$, $a_8 = 2a_7 + 3a_6 = 2(1825) + 3(605) = 5465$.

But this a_n represents the number of possibilities of neighborhood pseudo coloring of path P_n using five colors which includes the coloring with exactly one, two, three, four colors. Therefore when $n \geq 10$, $c_{nhd}(P_n, 5) = a_{n-1} - {}^5C_4(c_{nhd}(P_n, 4)) - {}^5C_3(c_{nhd}(P_n, 3)) - {}^5C_2(c_{nhd}(P_n, 2)) - 5$ where $a_{n-1} = 2a_{n-2} + 3a_{n-3}$ with $a_8 = 5465, a_9 = 16405$. □

Observation 3.14. *For $n \geq 10$, value of the neighborhood pseudo chromatic polynomial of path P_n with atmost five colors is $P_{nhd}(P_n, 5) = a_{n-1} = 2a_{n-2} + 3a_{n-3}$ with $a_8 = 5465, a_9 = 16405$.*

Example 3.15. *For the path P_n, $c_{nhd}(P_{10}, 5) = a_9 - {}^5C_4(c_{nhd}(P_{10}, 4)) - {}^5C_3(c_{nhd}(P_{10}, 3)) - {}^5C_2(c_{nhd}(P_{10}, 2)) - 5 = 16405 - 5(816) - 10(966) - 10(2^8 - 2) - 5 = 120.$*

Theorem 3.16. *The k^{th} coefficient of the neighborhood pseudo chromatic polynomial of path P_n with $n \geq 2k$ and $k \geq 3$ is $c_{nhd}(P_n, k) = a_{n-1}(k) - {}^kC_{k-1}(c_{nhd}(P_n, k-1)) - {}^kC_{k-2}(c_{nhd}(P_n, k-2)) - \cdots - {}^kC_2(c_{nhd}(P_n, 2)) - k$ where $a_{n-1}(k) = 2a_{n-2}(k) + (k-2)a_{n-3}(k)$.*

Proof. From Lemma 3.7, given 3 colors, $a_1 = a_2 = 3$, $a_3 = 3^2$, $a_4 = 3 \times (3 \times 1 + 2 \times 2), \cdots, a_i = 3 \times a_{i-2} + 2 \times (a_{i-1} - a_{i-2}) = 2a_{i-1} + a_{i-2}, \cdots, a_{n-1} = 2a_{n-2} + a_{n-3}$

From Lemma 3.10, given 4 colors, $a_1 = a_2 = 4$, $a_3 = 4^2$, $a_4 = 4 \times (4 \times 1 + 2 \times 3), \cdots, a_i = 4 \times a_{i-2} + 2 \times (a_{i-1} - a_{i-2}) = 2a_{i-1} + 2a_{i-2}, \cdots, a_{n-1} = 2a_{n-2} + 2a_{n-3}$

From Lemma 3.13, given 5 colors, $a_1 = a_2 = 5$, $a_3 = 5^2$, $a_4 = 5 \times (5 \times 1 + 2 \times 4), \cdots, a_i = 5 \times a_{i-2} + 2 \times (a_{i-1} - a_{i-2}) = 2a_{i-1} + 3a_{i-2}, \cdots, a_{n-1} = 2a_{n-2} + 3a_{n-3}$

Representing $a_i(k)$ be the number of possibilities of neighborhood pseudo coloring of the vertex v_i of path P_n with k colors, then by induction on k, $a_1(k) = k$, $a_2(k) = k$, $a_3(k) = k^2$,

$$\begin{aligned}
a_4(k) &= k(k \times 1 + 2(k-1)) \\
&= k \times k + 2(k^2 - k) \\
&= k \times a_2(k) + 2(a_3(k) - a_2(k)) \\
&= 2a_3(k) + (k-2)a_2(k), \\
&\vdots \\
a_i(k) &= 2a_{i-1}(k) + (k-2)a_{i-2}(k), \\
&\vdots \\
a_n(k) &= a_{n-1}(k) = 2a_{n-2}(k) + (k-2)a_{n-3}(k).
\end{aligned}$$

Therefore, $c_{nhd}(P_n,k) = a_{n-1}(k) -^k C_{k-1}(c_{nhd}(P_n,k-1)) -^k C_{k-2}(c_{nhd}(P_n,k-2)) - \cdots -^k C_2(c_{nhd}(P_n,2)) - k$ where $a_{n-1}(k) = 2a_{n-2}(k) + (k-2)a_{n-3}(k)$. $\square$

Solving the recurrence relation $a_{n-1}(k) = 2a_{n-2}(k) + (k-2)a_{n-3}(k)$, indicated in Theorem 3.16, by taking $a_{n-1} = Am^{n-1}$, $a_{n-2} = Am^{n-2}$, $a_{n-3} = Am^{n-3}$ we get

$$a_{n-1} = A(1+\sqrt{k-1})^n + B(1-\sqrt{k-1})^n.$$

The constants A and B are found using $a_1 = a_2 = k$ and hence $a_{n-1}(k) = \frac{k}{2}[(1+\sqrt{k-1})^{n-2} + (1-\sqrt{k-1})^{n-2}]$. We record this fact together with the results in Theorem 3.5 in the form of the following theorem.

Theorem 3.17. *The neighborhood pseudo chromatic polynomial of a path P_n is given by*

$$P_{nhd}(P_n,\lambda) = \sum_{k=1}^{\lambda} {}^{\lambda}C_k c_{nhd}(P_n,k),$$

where

$$c_{nhd}(P_n,k) = a_{n-1}(k) - {}^kC_{k-1}(c_{nhd}(P_n,k-1)) - {}^kC_{k-2}(c_{nhd}(P_n,k-2)) \cdots - {}^kC_2(c_{nhd}(P_n,2)) - k$$

and

$$a_{n-1}(k) = \begin{cases} 2^{n-2} & \text{if} \quad n \geq 4 \text{ and } k = 2 \\ \frac{k}{2}[(1+\sqrt{k-1})^{n-2} + (1-\sqrt{k-1})^{n-2}] & \text{if} \quad n \geq 2k \text{ and } k \geq 3. \end{cases}$$

Example 3.18. *The value of neighborhood pseudo chromatic polynomial of a path P_5, given six colors, is* $P_{nhd}(P_5,6) = {}^6C_1 \times c_{nhd}(P_5,1) + {}^6C_2 \times c_{nhd}(P_5,2) + {}^6C_3 \times c_{nhd}(P_5,3) + {}^6C_4 \times c_{nhd}(P_5,4) + {}^6C_5 \times c_{nhd}(P_5,5) + {}^6C_6 \times c_{nhd}(P_5,6) = 6(1) + 15(2^3-2) + 20(0) + 15(0) + 6(0) + 1(0) = 96.$

Example 3.19. *The value of neighborhood pseudo chromatic polynomial of a path P_8, given four colors is*

$$\begin{aligned} P_{nhd}(P_8,4) &= {}^4C_1 \times c_{nhd}(P_8,1) + {}^4C_2 \times c_{nhd}(P_8,2) + {}^4C_3 \times c_{nhd}(P_8,3) + {}^4C_4 \times c_{nhd}(P_8,4) \\ &= 4(1) + 6(2^8-2) + 4(108) + 1(24) = 1984. \end{aligned}$$

The program outcome for the value of neighborhood pseudo chromatic polynomial of P_6 when $\lambda = 10$, is shown in the FIGURE 6.

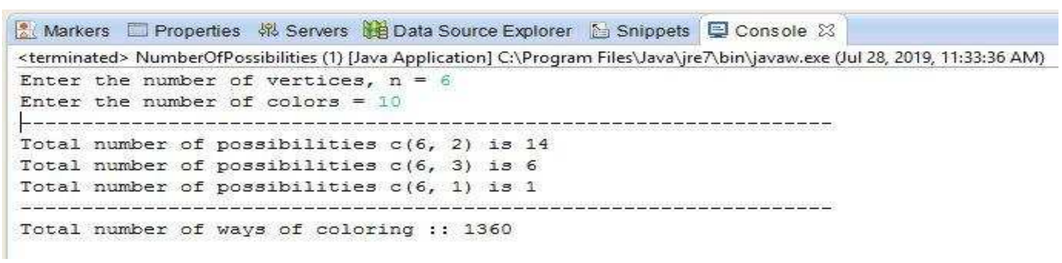

FIGURE 6. Program output for the value of $P_{nhd}(P_6, 10)$.

4. PSEUDO CODE TO COMPUTE NEIGHBORHOOD PSEUDO CHROMATIC POLYNOMIAL OF A PATH P_n

Input number of colors and vertices.

If (chromatic number of path >input Colors)

psi = input colors;

else

psi = chromatic number

For loop from 1 to psi, as i,

$\{$calculate c_n for each i value from 1 to psi $\}$

if index $= 1$

$c_n = 1$

else

valueA $= \{$ compute a_{n-1} using recursive findValueOfA(vertices -1, index) algorithm $\}$

For (i from 1 to index) do

Compute nC_r with $n =$index and $r =$(index $-i$)

Multiply computed nC_r value with previous computed c_n value $[c_n$ value of $i-1]$

Subtract result from valueA

End for loop

$c_n =$ valueA

polynomialValue $= 0$

Loop through all the calculated c_n values as pi and do,

a. extract index i value from pi

b. compute nC_r with $n =$input colors and $r = i$

c. polynomialValue $=$ polynomialValue $+$ computed $nC_r * pi$

polynomialValue is the result.

Logic to find Value of A

findValueOfA(vertices, input colors)

if vertices < 1 or input colors < 1

return "Invalid input"

else if vertices $= 1$ or vertices$= 2$

return input colors

else if vertices $= 3$

return input colors $*$ input colors

else

return $2*$ findValueOfA(vertices $+1-2$, input colors) $+$ (input colors -2) $*$findValueOfA(vertices $+1-3$, input colors)

5. Conclusion

We have derived the general formula for k^{th} coefficient of the pseudo chromatic polynomial and hence obtained the pseudo chromatic polynomial of a path P_n. The complexity of computing the coefficients and pseudo chromatic polynomial of a path P_n is reduced by solving the recurrence formula, developing the pseudo code and implemented the same through Java program.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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