Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 5, 1755-1769
https://doi.org/10.28919/jmcs/4322
ISSN: 1927-5307

VISCOSITY METHODS FOR INCLUSION AND FIXED-POINT PROBLEMS

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Abstract. We are concerned with a variational inclusion problem with two monotone mappings, and fixed-point problems of asymptotically nonexpansive mappings and pseudocontractive mappings. A viscosity method is introduced and studied for the two problems. A convergence theorem in norm of common solutions is established in the setting of infinite dimensional real Hilbert spaces.

Keywords: variational inclusion; nonexpansive mapping; zero pont; splitting method; fixed point.

2010 AMS Subject Classification: 47H09, 49J40.

1. Introduction

Let H be an infinite dimensional real Hilbert space and C be its convex and closed set. For each point $x \in H$, we know that there exists a unique nearest point in C, denoted by $P_C x$, s.t. $\|x - P_C x\| \leq \|x - y\|$ $\forall y \in C$. The operator P_C is called the metric projection of H onto C. One knows the following known tools. (i) $\langle y - z, P_C y - P_C z\rangle \geq \|P_C y - P_C z\|^2$, $\forall y, z \in H$; (ii) $\langle y - P_C y, z - P_C y\rangle \leq 0, \forall y \in H, z \in C$; (iii) $\|y - z\|^2 \geq \|y - P_C y\|^2 + \|z - P_C y\|^2, \forall y \in H, z \in C$; (iv) $\|y - z\|^2 = \|y\|^2 - \|z\|^2 - 2\langle y - z, z\rangle$, $\forall y, z \in H$; (v) $\|\lambda y + (1 - \lambda)z\|^2 = \lambda\|y\|^2 + (1 - \lambda)\|z\|^2 -$

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Received March 6, 2020

$\lambda(1-\lambda)\|y-z\|^2$, $\forall y,z \in H, \lambda \in [0,1]$. Given a mapping T on C. The notation $\mathrm{Fix}(T)$ is used to denote the fixed-point set of T, i.e., $\mathrm{Fix}(T) = \{u \in C : Tu = u\}$. One recalls that T is said to be asymptotically nonexpansive if $\exists\{\theta_k\} \subset [0,\infty)$ with $\lim_{k\to\infty} \theta_k = 0$ such that

$$\|T^k u - T^k v\| \leq (1+\theta_k)\|u-v\|, \quad \forall u,v \in C, k \geq 1.$$

In particular, if $\theta_k = 0$, then T is said to be a nonexpansive mapping. T is said to be pseudocontractive if $\langle u-v, Tu-Tv\rangle \leq 0$, $\forall u,v \in C$. T is said to be a contraction if $\exists\delta \in [0,1)$ such that $\|f(u)-f(v)\| \leq \delta\|u-v\|$, $\forall u,v \in C$. Recently, the approximation of fixed points of (asymptotically) nonexpansive mappings and pseudocontractive mappings has extensively studied via iterative techniques by lots of investigators; see, e.g., [1,2,3,4,5,6,7]. Let $\{S_n\}_{n=0}^{\infty}$ be a sequence of continuous pseudocontractive mappings defined on C. One recalls that $\{S_n\}_{n=0}^{\infty}$ is a countable family of ℓ-uniformly Lipschitzian pseudocontractive mappings on C if $\exists\ell > 0$ such that each S_n is ℓ-Lipschitz continuous. It is easy to see that if $\{S_n\}_{n=0}^{\infty}$ is a sequence of nonexpansive mappings, then it is a countable family of ℓ-uniformly Lipschitzian pseudocontractive mappings with $\ell = 1$. From [8], one knows that if $\sum_{k=1}^{\infty} \sup_{x\in C} \|S_k x - S_{k-1}x\| < \infty$, then, for each $y \in C$, $\{S_k y\}$ converges strongly to some point of C. Moreover, let S be a mapping on C defined by $Sy = \lim_{k\to\infty} S_k y$ $\forall y \in C$. Then $\lim_{k\to\infty} \sup_{x\in C} \|Sx - S_k x\| = 0$.

Next, one gives the mapping of monotone type. Recall that a mapping $F : C \to H$ is called monotone if $\langle Fu - Fv, u-v\rangle \geq 0$, $\forall u,v \in C$. It is called α-strongly monotone if $\exists\alpha > 0$ such that $\langle Fu - Fv, u-v\rangle \geq \alpha\|u-v\|^2$, $\forall u,v \in C$. Also, it is called β-inverse-strongly monotone (or β-cocoercive) if $\exists\beta > 0$ such that $\langle Fu - Fv, u-v\rangle \geq \beta\|Fu - Fv\|^2$, $\forall u,v \in C$.

Consider the classical variational inequality problem (VIP) of finding a point $z \in C$ such that

$$\langle Az, y-z\rangle \geq 0, \quad \forall y \in C.$$

The solution set of the VIP is denoted by $\mathrm{VI}(C,A)$. Fixed point techniques are efficient and important for studying solutions of the VIP and various convergence theorems were obtained recently; see, e.g., [9,10,11,12,13,14,15]. Among them, Korpelevich [16] studied the following iterative sequence

$$\begin{cases} v_k = P_C(u_k - \tau A u_k), \\ u_{k+1} = P_C(u_k - \tau A v_k) \quad \forall k \geq 0, \end{cases}$$

where the constant τ in $(0, \frac{1}{L})$. If $\mathrm{VI}(C,A) \neq \emptyset$, $\{u_k\}$ converges weakly to a point in $\mathrm{VI}(C,A)$.

Let $B : C \to 2^H$ be a set-valued operator with $Bx \neq \emptyset$, $\forall x \in C$. B is called monotone if for each $x, y \in C$, $\langle u - v, x - y \rangle \geq 0$, $\forall u \in Bx, v \in By$. B is called maximal monotone if $(I + \lambda B)C = H$ for all $\lambda > 0$. For a maximal monotone operator B, we define the mapping $J_\lambda^B : (I + \lambda B)C \to C$ by $J_\lambda^B = (I + \lambda B)^{-1}$ for each $\lambda > 0$. Such J_λ^B is called the resolvent of B for $\lambda > 0$. One knows that $\mathrm{Fix}(J_\lambda^B) = B^{-1}0$, which is useful for solving inclusion problems. One refers the readers to [17,18,19,20,21] and the references therein.

Let $B_i : C \to H$ be ν_i-inverse-strongly monotone mappings for $i = 1, 2$, $f : C \to C$ a δ-contraction and $F : C \to H$ a κ-Lipschitzian η-strongly monotone mapping with $\delta < \tau$ and $0 < \rho < \frac{2\eta}{\kappa^2}$, where $\tau = 1 - \sqrt{1 - \rho(2\eta - \rho\kappa^2)} \in (0, 1]$. Let $T : C \to C$ be an asymptotically nonexpansive mapping with a sequence $\{\theta_k\}$, and $\{S_k\}_{k=0}^\infty$ be a countable family of ℓ-uniformly Lipschitzian pseudocontractive mappings on C. Let Ω denote the common solution set of the variational inequality problems (VIPs) for B_1 and B_2 and the common fixed point problem (CFPP) for T and $\{S_k\}_{k=0}^\infty$. Recently, Ceng and Wen [22] suggested the following hybrid extragradient-like implicit algorithm

$$\begin{cases} u_k = \beta_k x_k + (1 - \beta_k) S_k u_k, \\ v_k = P_C(u_k - \mu_2 B_2 u_k), \\ y_k = P_C(v_k - \mu_1 B_1 v_k), \\ x_{k+1} = P_C[\alpha_k f(x_k) + (I - \alpha_k \rho F) T^k y_k] \quad \forall k \geq 0, \end{cases}$$

where $\mu_i \in (0, 2\nu_i)$ for $i = 1, 2$. Suppose that $\{\alpha_k\}, \{\beta_k\} \subset (0, 1]$ are such that (i) $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=0}^\infty \alpha_k = \infty$, $\sum_{k=0}^\infty |\alpha_{k+1} - \alpha_k| < \infty$; (ii) $\lim_{k\to\infty} \frac{\theta_k}{\alpha_k} = 0$; (iii) $0 < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$, $\sum_{k=0}^\infty |\beta_{k+1} - \beta_k| < \infty$; and (iv) $\sum_{k=0}^\infty \|T^{k+1} y_k - T^k y_k\| < \infty$. They proved strong convergence of $\{x_k\}$ to $x^* \in \Omega$, which solves the hierarchical variational inequality (HVI): $\langle (f - \rho F)x^*, y - x^* \rangle \geq 0$, $\forall y \in \Omega$.

Recently, Takahashi et al. [23] introduced the following Mann-Halpern iterative scheme for finding a common solution of a fixed point problem (FPP) of a nonexpansive mapping S

and a variational inclusion (VI) for an α-inverse-strongly monotone mapping A and a maximal monotone operator B

$$x_{k+1} = \beta_k x_k + (1-\beta_k)S(\alpha_k x + (1-\alpha_k)J^B_{\lambda_k}(x_k - \lambda_k A x_k)), \quad \forall k \geq 1,$$

where $\{\lambda_k\} \subset (0,2\alpha)$ and $\{\alpha_k\}, \{\beta_k\} \subset (0,1)$ are such that (i) $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = \infty$; (ii) $0 < a \leq \lambda_k \leq b < 2\alpha$, $\lim_{k\to\infty}(\lambda_k - \lambda_{k+1}) = 0$; and (iii) $0 < c \leq \beta_k \leq d < 1$. They proved that $\{x_k\}$ converges strongly to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$.

Motivated and inspired by the above and related results, in this paper, we investigate a viscosity computation method for solving a variational inclusion (VI) for two monotone operators A, B and a common fixed point problem (CFPP) T and $\{S_n\}_{n=0}^{\infty}$. Our results are strong convergence without additional assumptions on spaces and sets. The following six lemmas play an important role in the proof of our main convergence result.

Let $A : C \to H$ be an α-inverse-strongly monotone mapping and $B : C \to 2^H$ be a maximal monotone operator. In the sequel, we will use the notation $T_\lambda := J^B_\lambda(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A)\ \forall \lambda > 0$.

Lemma 1.1. [24] Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a continuous and strongly pseudocontractive mapping. Then T has a unique fixed point in C.

Lemma 1.2. [23] Let C be a nonempty closed convex subset of a Hilbert space H. Let $B : C \to 2^H$ be a maximal monotone operator and A an inverse-strongly monotone mapping. Then (i) $\mathrm{Fix}(T_\lambda) = (A+B)^{-1}0\ \forall \lambda > 0$; (ii) $\|y - T_\lambda y\| \leq 2\|y - T_r y\|$ for $0 < \lambda \leq r$ and $y \in C$.

Lemma 1.3 [23,25] Let $B : C \to 2^H$ be a maximal monotone operator. Then the following statements hold: (i) the resolvent identity: $J^B_\lambda x = J^B_\mu(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J^B_\lambda x)\ \forall \lambda, \mu > 0,\ x \in H$; (ii) if J^B_λ is a resolvent of B for $\lambda > 0$, then J^B_λ is a firmly nonexpansive mapping with $\mathrm{Fix}(J^B_\lambda) = B^{-1}0$, where $B^{-1}0 = \{x \in C : 0 \in Bx\}$.

Lemma 1.4. [26] Let $\lambda \in (0,1]$ and the mapping $T : C \to C$ be nonexpansive. Let the mapping $T^\lambda : C \to H$ be defined as $T^\lambda x := Tx - \lambda\mu F(Tx)\ \forall x \in C$, where $F : C \to H$ is κ-Lipschitzian and η-strongly monotone. Then T^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$, i.e., $\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|\ \forall x \in C$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0,1]$.

Lemma 1.5. [27] Let X be a Banach space which admits a weakly continuous duality mapping, C be a nonempty closed convex subset of X, and $T : C \to C$ be an asymptotically nonexpansive mapping with a fixed point. Then $I - T$ is demiclosed at zero, i.e., if the sequence $\{x_n\} \subset C$ satisfies $x_n \rightharpoonup x \in C$ and $(I-T)x_n \to 0$, then $(I-T)x = 0$, where I is the identity mapping of X.

Lemma 1.6. [28] Let $\{a_k\}$ be a sequence in $[0,\infty)$ such that $a_{k+1} \le (1-s_k)a_k + s_k \nu_k \ \forall k \ge 0$, where $\{s_k\}$ and $\{\nu_k\}$ satisfy the conditions: (i) $\{s_k\} \subset [0,1]$, $\sum_{k=0}^{\infty} s_k = \infty$; (ii) $\limsup_{n\to\infty} \nu_k \le 0$ or $\sum_{k=0}^{\infty} |s_k \nu_k| < \infty$. Then $\lim_{k\to\infty} a_k = 0$.

2. MAIN RESULTS

Theorem 2.1. Let H be an infinite dimensional real Hilbert space and C its convex and closed subset. Let T be an asymptotically nonexpansive mapping on C with a sequence $\{\theta_n\}$ and $\{S_n\}_{n=0}^{\infty}$ a countable family of ℓ-uniformly Lipschitzian pseudocontractive mappings on C. Let f be a δ-contraction on C and $F : C \to H$ a κ-Lipschitzian η-strongly monotone mapping, where $\delta < \tau$ and $0 < \rho < \frac{2\eta}{\kappa^2}$ and $\tau = 1 - \sqrt{1-\rho(2\eta - \rho\kappa^2)} \in (0,1]$. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and $B : C \to 2^H$ a maximal monotone mapping. Define a vector sequence $\{x_n\}$ by

$$\begin{cases} u_n = \beta_n x_n + (1-\beta_n) S_n u_n, \\ y_n = J_{\lambda_n}^B (u_n - \lambda_n A u_n), \\ z_n = J_{\lambda_n}^B (u_n - \lambda_n A y_n + r_n(y_n - u_n)), \\ x_{n+1} = P_C[\alpha_n f(u_n) + \gamma_n u_n + ((1-\gamma_n)I - \alpha_n \rho F)T^n z_n] \quad \forall n \ge 0, \end{cases}$$

where $\{\lambda_n\} \subset (0,\infty)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{r_n\} \subset (0,1)$ are such that $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = \frac{\theta_n}{\alpha_n} = 0$, $0 < a \le \beta_n \le b < 1$; $\alpha_n + \gamma_n \le 1$, $0 < c \le \gamma_n \le d < 1$, $0 < r \le r_n$ and $0 < \lambda \le \lambda_n < \frac{\lambda_n}{r_n} \le \mu < 2\alpha$. Assume $\Omega := \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap (A+B)^{-1}0 \cap \mathrm{Fix}(T)$ is not empty, $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n-1}x\| < \infty$ for any bounded set D in set C, and $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$, where S is a mapping on C defined by $Sx = \lim_{n\to\infty} S_n x$, $\forall x \in C$. If $T^n z_n - T^{n+1} z_n \to 0$, then $x_n \to x^* \in \Omega \Leftrightarrow x_n - x_{n+1} \to 0$, where $x^* \in \Omega$.

Proof. Since $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} = 0$, we may assume, without loss of generality, that $\theta_n \le \frac{\alpha_n(\tau-\delta)}{2}$, $\forall n \ge 0$. Moreover, it can be easily seen that, for each $n \ge 0$, there is an unique vector u_n in set C with $u_n = (1-\beta_n)S_n u_n + \beta_n x_n$. In fact, one sets $F_n u = (1-\beta_n)S_n u + \beta_n x_n$, $\forall u \in C$. Since $S_n : C \to C$ is a continuous pseudocontraction, one concludes

$$\langle F_n u - F_n v, u - v\rangle = (1-\beta_n)\langle S_n u - S_n v, u - v\rangle \leq (1-\beta_n)\|u-v\|^2, \quad \forall u, v \in C.$$

It follows that F_n is a continuous strong pseudocontractive mapping. In terms of Lemma 1.1, one asserts that, for each $n \geq 0$, there exists an unique vector u_n in set C satisfying the above equality. It is now easy to see that the necessity of the theorem is valid. One next shows the sufficiency of the theorem. To the aim, we assume $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$ and divide the proof of the sufficiency into several steps. Taking $p \in \Omega$ arbitrarily, one from Lemma 1.2 that $S_n p = p$, $Tp = p$ and $J^B_{\lambda_n}(I - \lambda_n A)p = p$. Since $S_n : C \to C$ is a pseudocontraction, one sees

$$\|u_n - p\|^2 \leq \beta_n \|x_n - p\|\|u_n - p\| + (1-\beta_n)\|u_n - p\|^2,$$

which yields $\|u_n - p\| \leq \|x_n - p\|$. One also observes that

$$p = J^B_{\lambda_n}(p - \lambda_n Ap) = J^B_{\lambda_n}((1-r_n)p + r_n(p - \frac{\lambda_n}{r_n}Ap)).$$

By Lemmas 1.3, one gets

$$\begin{aligned}\|y_n - p\|^2 &= \|J^B_{\lambda_n}(u_n - \lambda_n Au_n) - J^B_{\lambda_n}(p - \lambda_n Ap)\|^2\\ &\leq \|u_n - p\|^2 - \lambda_n(2\alpha - \lambda_n)\|Au_n - Ap\|^2,\end{aligned}$$

which hence yields

$$\|y_n - p\| \leq \|u_n - p\|.$$

Further, the convexity of $\|\cdot\|^2$ yields that

$$\begin{aligned}&\|z_n - p\|^2\\ &= \|J^B_{\lambda_n}((1-r_n)u_n + r_n(y_n - \tfrac{\lambda_n}{r_n}Ay_n)) - J^B_{\lambda_n}((1-r_n)p + r_n(p - \tfrac{\lambda_n}{r_n}Ap))\|^2\\ &\leq \|((1-r_n)u_n + r_n(y_n - \tfrac{\lambda_n}{r_n}Ay_n)) - ((1-r_n)p + r_n(p - \tfrac{\lambda_n}{r_n}Ap))\|^2\\ &\leq (1-r_n)\|u_n - p\|^2 + r_n\|(y_n - \tfrac{\lambda_n}{r_n}Ay_n) - (p - \tfrac{\lambda_n}{r_n}Ap)\|^2\\ &\leq (1-r_n)\|u_n - p\|^2 + r_n[\|y_n - p\|^2 - \tfrac{\lambda_n}{r_n}(2\alpha - \tfrac{\lambda_n}{r_n})\|Ay_n - Ap\|^2]\\ &\leq (1-r_n)\|u_n - p\|^2 + r_n[\|u_n - p\|^2 - \lambda_n(2\alpha - \lambda_n)\|Au_n - Ap\|^2\\ &\quad - \tfrac{\lambda_n}{r_n}(2\alpha - \tfrac{\lambda_n}{r_n})\|Ay_n - Ap\|^2]\\ &= \|u_n - p\|^2 - r_n\lambda_n(2\alpha - \lambda_n)\|Au_n - Ap\|^2 - \lambda_n(2\alpha - \tfrac{\lambda_n}{r_n})\|Ay_n - Ap\|^2,\end{aligned}$$

which immediately leads to $\|z_n - p\| \le \|u_n - p\|$. Putting

$$v_n := \alpha_n f(u_n) + \gamma_n u_n + ((1-\gamma_n)I - \alpha_n \rho F)T^n z_n,$$

one has $x_{n+1} = P_C v_n$ and

$$\begin{aligned} v_n - p \quad &= \alpha_n(f(u_n) - f(p)) + \gamma_n(u_n - p) + (1-\gamma_n)[(I - \tfrac{\alpha_n}{1-\gamma_n}\rho F)T^n z_n - (I - \tfrac{\alpha_n}{1-\gamma_n}\rho F)p] \\ &\quad + \alpha_n(f - \rho F)p. \end{aligned}$$

Using Lemma 1.4, it follows that

$$\begin{aligned} &\|x_{n+1} - p\| \\ &\le \alpha_n \|f(u_n) - f(p)\| + \gamma_n \|u_n - p\| + (1-\gamma_n)\|(I - \tfrac{\alpha_n}{1-\gamma_n}\rho F)T^n z_n - (I - \tfrac{\alpha_n}{1-\gamma_n}\rho F)p\| \\ &\quad + \alpha_n \|(f - \rho F)p\| \\ &\le \alpha_n \delta \|u_n - p\| + \gamma_n \|u_n - p\| + (1-\gamma_n)(1 - \tfrac{\alpha_n}{1-\gamma_n}\tau)(1+\theta_n)\|z_n - p\| + \alpha_n \|(f - \rho F)p\| \\ &\le \alpha_n \delta \|u_n - p\| + \gamma_n \|u_n - p\| + (1 - \gamma_n - \alpha_n \tau)(1+\theta_n)\|u_n - p\| + \alpha_n \|(f - \rho F)p\| \\ &\le (1 - \alpha_n(\tau - \delta) + \theta_n)\|u_n - p\| + \alpha_n \|(f - \rho F)p\| \\ &\le [1 - \tfrac{\alpha_n(\tau-\delta)}{2}]\|x_n - p\| + \alpha_n \|(f - \rho F)p\| \\ &= [1 - \tfrac{\alpha_n(\tau-\delta)}{2}]\|x_n - p\| + \tfrac{\alpha_n(\tau-\delta)}{2} \cdot \tfrac{2\|(f-\rho F)p\|}{\tau-\delta}. \end{aligned}$$

By induction, one obtains $\|x_n - p\| \le \{\|x_0 - p\|, \frac{2\|(f-\rho F)p\|}{\tau-\delta}\}$. Thus, $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$. Next, one claims that

$$\Gamma_{n+1} \le [1 - \frac{\alpha_n(\tau-\delta)}{2}]\Gamma_n + \delta_n \quad \forall n \ge 0,$$

where $x^* = P_\Omega(f + I - \rho F)x^*$, $\Gamma_n = \|x_n - x^*\|^2$ and $\delta_n = 2\alpha_n\langle (f - \rho F)x^*, x_{n+1} - x^*\rangle$. From Lemma 1.4, we know that $P_\Omega(f + I - \rho F)$ is a contraction operator. The Banach contraction principle implies that there is unique fixed point of $P_\Omega(f + I - \rho F)$, i.e., $x^* = P_\Omega(f + I - \rho F)x^*$. Since each $S_n : C \to C$ is a pseudocontraction, one has

$$\begin{aligned} \|u_n - x^*\|^2 \quad &\le \beta_n\langle x_n - x^*, u_n - x^*\rangle + (1-\beta_n)\|u_n - x^*\|^2 \\ &= \tfrac{\beta_n}{2}[\|x_n - x^*\|^2 + \|u_n - x^*\|^2 - \|x_n - u_n\|^2] + (1-\beta_n)\|u_n - x^*\|^2, \end{aligned}$$

which leads to $\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - u_n\|^2$. Since $x_{n+1} = P_C v_n$, where $v_n := \alpha_n f(u_n) + \gamma_n u_n + ((1-\gamma_n)I - \alpha_n \rho F)T^n z_n$, it follows that $\langle P_C v_n - v_n, P_C v_n - x^*\rangle \le 0$, i.e., $\langle x_{n+1} - v_n, x_{n+1} -$

$x^*\rangle \le 0$. Hence,

$$
\begin{aligned}
&\|x_{n+1}-x^*\|^2 \le \langle \alpha_n f(u_n)+\gamma_n u_n+((1-\gamma_n)I-\alpha_n\rho F)T^n z_n - x^*, x_{n+1}-x^*\rangle\\
&= \alpha_n\langle f(u_n)-f(x^*), x_{n+1}-x^*\rangle + \gamma_n\langle u_n - x^*, x_{n+1}-x^*\rangle + (1\\
&\quad -\gamma_n)\langle (I-\tfrac{\alpha_n}{1-\gamma_n}\rho F)T^n z_n - (I-\tfrac{\alpha_n}{1-\gamma_n}\rho F)x^*, x_{n+1}-x^*\rangle + \alpha_n\langle (f-\rho F)x^*, x_{n+1}-x^*\rangle\\
&\le (\alpha_n\delta+\gamma_n)\|u_n-x^*\|\|x_{n+1}-x^*\| + (1-\gamma_n-\alpha_n\tau)(1+\theta_n)\|z_n-x^*\|\|x_{n+1}-x^*\|\\
&\ +\alpha_n\langle (f-\rho F)x^*, x_{n+1}-x^*\rangle\\
&\le \tfrac{\alpha_n\delta+\gamma_n}{2}(\|u_n-x^*\|^2+\|x_{n+1}-x^*\|^2) + \tfrac{1-\gamma_n-\alpha_n\tau+\theta_n}{2}(\|z_n-x^*\|^2+\|x_{n+1}-x^*\|^2)\\
&\ +\alpha_n\langle (f-\rho F)x^*, x_{n+1}-x^*\rangle\\
&\le \tfrac{\alpha_n\delta+\gamma_n}{2}(\|u_n-x^*\|^2+\|x_{n+1}-x^*\|^2) + \tfrac{1-\gamma_n-\alpha_n\tau+\theta_n}{2}\{\|u_n-x^*\|^2\\
&\ -r_n\lambda_n(2\alpha-\lambda_n)\|Au_n-Ax^*\|^2 - \lambda_n(2\alpha-\tfrac{\lambda_n}{r_n})\|Ay_n-Ax^*\|^2\\
&\ +\|x_{n+1}-x^*\|^2\} + \alpha_n\langle (f-\rho F)x^*, x_{n+1}-x^*\rangle\\
&\le \tfrac{1}{2}\|x_{n+1}-x^*\|^2 + \tfrac{1-\alpha_n(\tau-\delta)+\theta_n}{2}\|u_n-x^*\|^2 - \tfrac{1-\gamma_n-\alpha_n\tau+\theta_n}{2}[r_n\lambda_n(2\alpha-\lambda_n)\|Au_n-Ax^*\|^2\\
&\ +\lambda_n(2\alpha-\tfrac{\lambda_n}{r_n})\|Ay_n-Ax^*\|^2] + \alpha_n\langle (f-\rho F)x^*, x_{n+1}-x^*\rangle\\
&\le \tfrac{1}{2}\|x_{n+1}-x^*\|^2 + \tfrac{1-\alpha_n(\tau-\delta)+\theta_n}{2}(\|x_n-x^*\|^2-\|x_n-u_n\|^2) - \tfrac{1-\gamma_n-\alpha_n\tau+\theta_n}{2}[r_n\lambda_n(2\alpha\\
&\quad -\lambda_n)\|Au_n-Ax^*\|^2 + \lambda_n(2\alpha-\tfrac{\lambda_n}{r_n})\|Ay_n-Ax^*\|^2] + \alpha_n\langle (f-\rho F)x^*, x_{n+1}-x^*\rangle\\
&\le \tfrac{1}{2}\|x_{n+1}-x^*\|^2 + \tfrac{1-\alpha_n(\tau-\delta)+\theta_n}{2}\|x_n-x^*\|^2 - \tfrac{1-\gamma_n-\alpha_n\tau+\theta_n}{2}[r_n\lambda_n(2\alpha-\lambda_n)\|Au_n-Ax^*\|^2\\
&\ +\lambda_n(2\alpha-\tfrac{\lambda_n}{r_n})\|Ay_n-Ax^*\|^2 + \|x_n-u_n\|^2] + \alpha_n\langle (f-\rho F)x^*, x_{n+1}-x^*\rangle\\
&\le \tfrac{1}{2}\|x_{n+1}-x^*\|^2 + \tfrac{1-\frac{\alpha_n(\tau-\delta)}{2}}{2}\|x_n-x^*\|^2 - \tfrac{1-\gamma_n-\alpha_n\tau+\theta_n}{2}[r_n\lambda_n(2\alpha-\lambda_n)\|Au_n-Ax^*\|^2\\
&\ +\lambda_n(2\alpha-\tfrac{\lambda_n}{r_n})\|Ay_n-Ax^*\|^2 + \|x_n-u_n\|^2] + \alpha_n\langle (f-\rho F)x^*, x_{n+1}-x^*\rangle.
\end{aligned}
$$

For each $n \ge 0$, one next puts $\varepsilon_n = \frac{\alpha_n(\tau-\delta)}{2}$, $\Gamma_n = \|x_n - x^*\|^2$, $\delta_n = 2\alpha_n\langle (f-\rho F)x^*, x_{n+1}-x^*\rangle$ and

$$
\begin{aligned}
\eta_n = (1-\gamma_n-\alpha_n\tau+\theta_n)[&r_n\lambda_n(2\alpha-\lambda_n)\|Au_n-Ax^*\|^2\\
&+\lambda_n(2\alpha-\tfrac{\lambda_n}{r_n})\|Ay_n-Ax^*\|^2+\|x_n-u_n\|^2].
\end{aligned}
$$

So it follows from (3.8) that $\Gamma_{n+1} \le (1-\varepsilon_n)\Gamma_n + \delta_n$. On the other hand, from the assumption $x_n - x_{n+1} \to 0$, one obtains $\Gamma_n - \Gamma_{n+1} \le (\|x_n - x^*\| + \|x_{n+1}-x^*\|)\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$). That is, $\Gamma_n - \Gamma_{n+1} \to 0$. From $\Gamma_{n+1} \le (1-\varepsilon_n)\Gamma_n - \eta_n + \delta_n$, one gets

$$0 \le \eta_n \le \Gamma_n - \Gamma_{n+1} + \delta_n - \varepsilon_n\Gamma_n.$$

Since $\varepsilon_n \to 0$, $\delta_n \to 0$ and $\Gamma_n - \Gamma_{n+1} \to 0$, we have $\eta_n \to 0$. This immediately implies that

$$\lim_{n\to\infty} \|Au_n - Ax^*\| = \lim_{n\to\infty} \|Ay_n - Ax^*\| = 0 = \lim_{n\to\infty} \|x_n - u_n\| = 0.$$

Since $J^B_{\lambda_n}$ is firmly nonexpansive, one sees

$$\begin{aligned}
&\|y_n - x^*\|^2 = \|J^B_{\lambda_n}(u_n - \lambda_n Au_n) - J^B_{\lambda_n}(x^* - \lambda_n Ax^*)\|^2 \\
&\leq \langle (u_n - \lambda_n Au_n) - (x^* - \lambda_n Ax^*), y_n - x^* \rangle \\
&= \tfrac{1}{2}[\|(u_n - \lambda_n Au_n) - (x^* - \lambda_n Ax^*)\|^2 + \|y_n - x^*\|^2 - \|u_n - \lambda_n(Au_n - Ax^*) - y_n\|^2],
\end{aligned}$$

which yields

$$\begin{aligned}
\|y_n - x^*\|^2 &\leq \|(u_n - \lambda_n Au_n) - (x^* - \lambda_n Ax^*)\|^2 - \|u_n - \lambda_n(Au_n - Ax^*) - y_n\|^2 \\
&\leq \|u_n - x^*\|^2 - \|u_n - \lambda_n(Au_n - Ax^*) - y_n\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\leq \tfrac{\alpha_n\delta+\gamma_n}{2}(\|u_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \tfrac{1-\gamma_n-\alpha_n\tau+\theta_n}{2}(\|z_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
&\quad +\alpha_n\langle (f-\rho F)x^*, x_{n+1} - x^* \rangle \\
&\leq \tfrac{\alpha_n\delta+\gamma_n}{2}(\|u_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \tfrac{1-\gamma_n-\alpha_n\tau+\theta_n}{2}[(1-r_n)\|u_n - x^*\|^2 + r_n\|y_n - x^*\|^2 \\
&\quad +\|x_{n+1} - x^*\|^2] + \alpha_n\langle (f-\rho F)x^*, x_{n+1} - x^* \rangle \\
&\leq \tfrac{\alpha_n\delta+\gamma_n}{2}(\|u_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \tfrac{1-\gamma_n-\alpha_n\tau+\theta_n}{2}[(1-r_n)\|u_n - x^*\|^2 + r_n(\|u_n - x^*\|^2 \\
&\quad -\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|^2) + \|x_{n+1} - x^*\|^2] + \alpha_n\langle (f-\rho F)x^*, x_{n+1} - x^* \rangle \\
&\leq \tfrac{1}{2}\|x_{n+1} - x^*\|^2 + \tfrac{1}{2}\|u_n - x^*\|^2 - \tfrac{(1-\gamma_n-\alpha_n\tau+\theta_n)r_n}{2}\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|^2 \\
&\quad +\alpha_n\langle (f-\rho F)x^*, J(x_{n+1} - x^*) \rangle \\
&\leq \tfrac{1}{2}\|x_{n+1} - x^*\|^2 + \tfrac{1}{2}\|x_n - x^*\|^2 - \tfrac{(1-\gamma_n-\alpha_n\tau+\theta_n)r_n}{2}\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|^2 \\
&\quad +\alpha_n\|(f-\rho F)x^*\|M_0,
\end{aligned}$$

where $\sup_{n\geq 0}\|x_n - x^*\| \leq M_0$ for some $M_0 > 0$. This immediately implies that

$$\frac{(1-\gamma_n-\alpha_n\tau+\theta_n)r_n}{2}\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|^2 \leq \frac{1}{2}(\Gamma_n - \Gamma_{n+1}) + \alpha_n\|(f-\rho F)x^*\|M_0.$$

Since $\alpha_n \to 0,\ \theta_n \to 0$ and $\Gamma_n - \Gamma_{n+1} \to 0$, one has $\lim_{n\to\infty} \|u_n - y_n\| = 0$. In a similar way, one gets

$$\begin{aligned}\|z_n - x^*\|^2 &= \|J^B_{\lambda_n}(u_n - \lambda_n A y_n + r_n(y_n - u_n)) - J^B_{\lambda_n}(x^* - \lambda_n A x^*)\|^2\\ &\le \langle (u_n - \lambda_n A y_n + r_n(y_n - u_n)) - (x^* - \lambda_n A x^*), z_n - x^*\rangle\\ &= \tfrac{1}{2}[\|(u_n - \lambda_n A y_n + r_n(y_n - u_n)) - (x^* - \lambda_n A x^*)\|^2 + \|z_n - x^*\|^2\\ &\quad -\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|^2].\end{aligned}$$

So,

$$\begin{aligned}\|z_n - x^*\|^2 &\le \|(u_n - \lambda_n A y_n + r_n(y_n - u_n)) - (x^* - \lambda_n A x^*)\|^2\\ &\quad -\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|^2\\ &\le \|u_n - x^*\|^2 - \|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|^2\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \tfrac{\alpha_n\delta+\gamma_n}{2}(\|u_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \tfrac{1-\gamma_n-\alpha_n\tau+\theta_n}{2}(\|z_n - x^*\|^2\\ &\quad +\|x_{n+1} - x^*\|^2) + \alpha_n\langle (f-\rho F)x^*, x_{n+1} - x^*\rangle\\ &\le \tfrac{\alpha_n\delta+\gamma_n}{2}(\|u_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) + \tfrac{1-\gamma_n-\alpha_n\tau+\theta_n}{2}[\|u_n - x^*\|^2\\ &\quad -\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|^2 + \|x_{n+1} - x^*\|^2]\\ &\quad +\alpha_n\langle (f-\rho F)x^*, x_{n+1} - x^*\rangle\\ &\le \tfrac{1}{2}\|x_{n+1} - x^*\|^2 + \tfrac{1}{2}\|u_n - x^*\|^2 + \alpha_n\langle (f-\rho F)x^*, x_{n+1} - x^*\rangle\\ &\quad -\tfrac{1-\gamma_n-\alpha_n\tau+\theta_n}{2}\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|^2\\ &\le \tfrac{1}{2}\|x_{n+1} - x^*\|^2 + \tfrac{1}{2}\|x_n - x^*\|^2 + \alpha_n\langle (f-\rho F)x^*, x_{n+1} - x^*\rangle\\ &\quad -\tfrac{1-\gamma_n-\alpha_n\tau+\theta_n}{2}\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|^2,\end{aligned}$$

which immediately attains

$$\frac{1-\gamma_n-\alpha_n\tau+\theta_n}{2}\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|^2 \le \frac{1}{2}(\Gamma_n - \Gamma_{n+1}) + \alpha_n\|(f-\rho F)x^*\|M_0.$$

Since $\alpha_n \to 0,\ \theta_n \to 0$ and $\Gamma_n - \Gamma_{n+1} \to 0$, one arrives at $\lim_{n\to\infty}\|u_n - z_n\| = 0$. Next, one shows that $x_n - S_n x_n \to 0,\ x_n - Tx_n \to 0$ and $x_n - T_\lambda x_n \to 0$, where $T_\lambda = J^B_\lambda(I - \lambda A)$. Observe $\|S_n u_n - u_n\| \le \frac{b}{1-b}\|x_n - u_n\| \to 0$ as $n \to \infty$. Since $\{S_n\}_{n=0}^\infty$ is ℓ-uniformly Lipschitzian on C, one obtains $\|S_n x_n - x_n\| \le (\ell+1)\|x_n - u_n\| + \|S_n u_n - u_n\| \to 0$ as $n \to \infty$. Note that

$$\begin{aligned}\|x_{n+1} - T^n z_n\| &\le \|\alpha_n f(u_n) + \gamma_n u_n + ((1-\gamma_n)I - \alpha_n\rho F)T^n z_n - T^n z_n\|\\ &\le \alpha_n\|f(u_n) - \rho F T^n z_n\| + \gamma_n(\|u_n - x_{n+1}\| + \|x_{n+1} - T^n z_n\|),\end{aligned}$$

which yields

$$\begin{aligned}\|x_{n+1}-T^n z_n\| &\leq \tfrac{\alpha_n}{1-\gamma_n}\|f(u_n)-\rho F T^n z_n\|+\tfrac{\gamma_n}{1-\gamma_n}\|u_n-x_{n+1}\| \\ &\leq \tfrac{\alpha_n}{1-d}\|f(u_n)-\rho F T^n z_n\|+\tfrac{d}{1-d}(\|u_n-x_n\|+\|x_n-x_{n+1}\|).\end{aligned}$$

So it follows that

$$\|z_n-T^n z_n\| \leq \|z_n-u_n\|+\frac{d}{1-d}(\|u_n-x_n\|+\|x_n-x_{n+1}\|)+\frac{\alpha_n}{1-d}\|f(u_n)-\rho F T^n z_n\|.$$

Thanks to $\alpha_n \to 0$ and $x_n - x_{n+1} \to 0$, one has $\lim_{n\to\infty}\|z_n - T^n z_n\| = 0$. Also, note that

$$\|z_n-Tz_n\| \leq \|z_n-T^n z_n\|+\|T^n z_n-T^{n+1}z_n\|+\|T^{n+1}z_n-Tz_n\|.$$

By the assumption $T^n z_n - T^{n+1} z_n \to 0$, the uniform continuity of T sends us to $\lim_{n\to\infty}\|z_n - Tz_n\| = 0$. By virtu of

$$\|x_n-Tx_n\| \leq \|x_n-u_n\|+\|u_n-z_n\|+\|z_n-Tz_n\|+\|Tz_n-Tu_n\|+\|Tu_n-Tx_n\|,$$

one has $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. In addition, for each $n \geq 0$, we put $T_{\lambda_n} := J^B_{\lambda_n}(I-\lambda_n A)$. Then $\lim_{n\to\infty}\|u_n - T_{\lambda_n}u_n\| = 0$. Since $0 < \lambda \leq \lambda_n$, we have

$$\begin{aligned}\|x_n-T_\lambda x_n\| &\leq 2\|x_n-T_{\lambda_n}x_n\| \\ &\leq 2(\|x_n-u_n\|+\|u_n-T_{\lambda_n}u_n\|+\|T_{\lambda_n}u_n-T_{\lambda_n}x_n\|) \\ &\leq 2(2\|x_n-u_n\|+\|u_n-T_{\lambda_n}u_n\|) \to 0 \quad (n\to\infty).\end{aligned}$$

That is, $\lim_{n\to\infty}\|x_n - T_\lambda x_n\| = 0$. One next proves that $x_m - \widetilde{S}x_m \to 0$, where $\widetilde{S} := (2I-S)^{-1}$. Indeed, one first shows that $S : C \to C$ is pseudocontractive ℓ-Lipschitzian with $x_m - Sx_m \to 0$, where $Sx = \lim_{m\to\infty} S_m x$, $\forall x \in C$. It is clear that $S_m u \to Su$ and $S_m v \to Sv$ for each $u, v \in C$. So, $\langle Su - Sv, u - v\rangle = \lim_{m\to\infty}\langle S_m u - S_m v, u - v\rangle \leq \|u - v\|^2$. Hence, S is a pseudocontraction. Since $\{S_m\}_{m=0}^{\infty}$ is ℓ-uniformly Lipschitzian on C, one gets $\|Su - Sv\| = \lim_{m\to\infty}\|S_m u - S_m v\| \leq \ell\|u - v\|$. Thus, S is ℓ-Lipschitzian. Set $D = \overline{\text{conv}}\{x_m : m \geq 0\}$. Because of the boundedness of the set $\{x_m : m \geq 0\}$, we know from the assumption that $\sum_{m=1}^{\infty}\sup_{y\in D}\|S_m y - S_{m-1}y\| < \infty$. Consequently,

$$\lim_{m\to\infty}\|S_m x_m - Sx_m\| = 0. \tag{3.22}$$

So it follows from (3.16) and (3.22) that $\|x_m - Sx_m\| \leq \|x_m - S_m x_m\| + \|S_m x_m - Sx_m\| \to 0$ as $m \to \infty$. That is, $\lim_{m\to\infty}\|x_m - Sx_m\| = 0$. Putting $\widetilde{S} := (2I-S)^{-1}$, we have that $\widetilde{S} : C \to C$

is nonexpansive, $\mathrm{Fix}(\widetilde{S}) = \mathrm{Fix}(S) = \bigcap_{m=0}^{\infty}\mathrm{Fix}(S_m)$ and $\lim_{m\to\infty}\|x_m - \widetilde{S}x_m\| = 0$. In fact, set $\widetilde{S} := (2I - S)^{-1}$, where I is the identity operator of H. Then we know that $\widetilde{S}$ is nonexpansive and $\mathrm{Fix}(\widetilde{S}) = \mathrm{Fix}(S) = \bigcap_{m=0}^{\infty}\mathrm{Fix}(S_m)$ as a consequence of [**?**, Theorem 6]. It follows that

$$\|x_m - \widetilde{S}x_m\| = \|\widetilde{S}\widetilde{S}^{-1}x_m - \widetilde{S}x_m\| \leq \|\widetilde{S}^{-1}x_m - x_m\| = \|x_m - Sx_m\| \to 0 \quad (m\to\infty).$$

Now, one proves that $x_m \to x^*$, where $x^* = P_\Omega(f + I - \rho F)x^*$. Indeed, $\limsup_{m\to\infty}\langle (f - \rho F)x^*, x_{m+1} - x^*\rangle \leq 0$, where $x^* = P_\Omega(f + I - \rho F)x^*$. As a matter of fact, it has a subsequence $\{x_{m_l}\} \subset \{x_m\}$ with

$$\limsup_{m\to\infty}\langle (f-\rho F)x^*, x_m - x^*\rangle = \lim_{l\to\infty}\langle (f-\rho F)x^*, x_{m_l} - x^*\rangle.$$

Thanks to the framework of the space and the boundedness of the sequence, one supposes that $x_{m_l} \rightharpoonup \hat{x} \in C$. Note that T_λ and $\widetilde{S}$ both are nonexpansive and that T is asymptotically nonexpansive. Since $(I - T_\lambda)x_m \to 0$ and $(I - \widetilde{S})x_m \to 0$, using Lemma 1.5, one concludes that $\hat{x} \in \mathrm{Fix}(T_\lambda) = (A+B)^{-1}0$ and $\hat{x} \in \mathrm{Fix}(\widetilde{S}) = \mathrm{Fix}(S) = \bigcap_{m=0}^{\infty}\mathrm{Fix}(S_m)$ and $x_{m_l} - Tx_{m_l} \to 0$ for $\{x_{m_l}\} \subset \{x_m\}$. Lemma 1.5 yields that $\hat{x} \in \mathrm{Fix}(T)$. Consequently, $\hat{x} \in \Omega$. So, $x_m - x_{m+1} \to 0$ and $x_{m_l} \rightharpoonup \hat{x}$ and

$$\begin{aligned}&\limsup_{m\to\infty}\langle (f-\rho F)x^*, x_{m+1} - x^*\rangle = \limsup_{m\to\infty}\langle (f-\rho F)x^*, x_m - x^*\rangle \\ &= \langle (f-\rho F)x^*, \tilde{x} - x^*\rangle \leq 0.\end{aligned}$$

Observe that

$$\begin{aligned}&\|x_{m+1} - x^*\|^2 \\ &\leq \tfrac{1}{2}\|x_{m+1} - x^*\|^2 + \tfrac{1-\alpha_m(\tau-\delta)+\theta_m}{2}\|x_m - x^*\|^2 + \alpha_m\langle (f-\rho F)x^*, x_{m+1} - x^*\rangle \\ &\leq \tfrac{1-\alpha_m(\tau-\delta)}{2}\|x_m - x^*\|^2 + \tfrac{1}{2}\|x_{m+1} - x^*\|^2 + \tfrac{\theta_m}{2}M_0^2 + \alpha_m\langle (f-\rho F)x^*, x_{m+1} - x^*\rangle,\end{aligned}$$

which immediately leads to

$$\begin{aligned}\|x_{m+1} - x^*\|^2 &\leq (1 - \alpha_m(\tau-\delta))\|x_m - x^*\|^2 + \alpha_m(\tau-\delta)\tfrac{2\langle (f-\rho F)x^*, x_{m+1} - x^*\rangle}{\tau-\delta} \\ &\quad + \alpha_m(\tau-\delta)\cdot\tfrac{\theta_m M_0^2}{\alpha_m(\tau-\delta)}.\end{aligned}$$

By using Lemma 1.6, one obtains $x_m \to x^*$. This complete the proof.

Remark 2.1. Compared with the corresponding results in Ceng and Wen [22] and Takahashi, Takahashi and Toyoda [23], our results improve and extend them in the following aspects. The problem of solving the VIPs for two monotone operators A, B with the CFPP constraint of an asymptotically nonexpansive mapping T and countably many uniformly Lipschitzian pseudocontractions $\{S_n\}_{n=0}^{\infty}$ in [22, Theorem 3.1] is extended to develop our problem of solving the VI for two monotone operators A, B with the CFPP constraint of an asymptotically nonexpansive mapping T and countably many uniformly Lipschitzian pseudocontractions $\{S_n\}_{n=0}^{\infty}$. The hybrid extragradient-like implicit rule in [22, Theorem 3.1] is extended to develop our hybrid implicit extragradient method. The problem of solving the VI for two monotone operators A, B with the FPP constraint of a nonexpansive mapping S in [23, Theorem 3.1] is extended to develop our problem of solving the VI for two monotone operators A, B with the CFPP constraint of an asymptotically nonexpansive mapping T and countably many uniformly Lipschitzian pseudocontractions $\{S_n\}_{n=0}^{\infty}$. The Mann-type Halpern iterative scheme in [23, Theorem 3.1] are extended to develop our hybrid implicit extragradient method.

ACKNOWLEDGMENTS

This work was supported by Gyeongnam National University of Science and Technology Grant in 2020.3.1–2022.2.28.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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