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# COMMON FIXED POINT THEOREM FOR TWO SELFMAPS OF A G-METRIC SPACE

K. RAJANI DEVI[1], V. KIRAN[2,*], J. NIRANJAN GOUD[2]

[1]Department of Mathematics, K.V.R (W) Government Degree College, Kurnool, India

[2]Department of Mathematics, Osmania University, Hyderabad, 500007, India



**Abstract.** In this paper, we prove a common fixed point theorem for two compatible self maps of a G-metric space.

**Keywords:** G-metric space; compatible mappings; fixed point; associated sequence of a point relative to two self maps; contractive modulus.

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## 1. Introduction

commuting mappings were generalizied as weakly commuting maps by Sessa[8]. Later G.Jungck[4, 5] introduced compatibility as a further generalization of weakly commuting maps. Among all generalizations[1,2,3,9] of metric spaces, $G$- metric spaces initiated by Zead Mustafa and Brailey Sims[6, 7] evinced interest in many researchers.

in the present paper we prove a common fixed point theorem for two compatible self maps of a $G$ -metric space.

*Corresponding author

E-mail address: kiranmathou@gmail.com

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## 2. Preliminaries

Before proving the main result we begin with,

**Definition 2.1:** Let $X$ be a non empty set and

$G : X^3 \to [0,\infty)$ be a function satisfying

$(G1)$ $G(x,y,z) = 0$ if $x = y = z$

$(G2)$ $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$

$(G3)$ $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$

$(G4)$ $G(x,y,z) = G(\sigma(x,y,z))$ for all $x,y,z \in X$ where $\sigma(x,y,z)$ is a permutation of the set $\{x,y,z\}$ and

$(G5)$ $G(x,y,z) \leq G(x,w,w) + G(w,y,z)$ for all $x,y,z,w \in X$

Then $G$ is called a $G$-metric on $X$ and the pair $(X,G)$ is called a $G$- metric space.

**Definition 2.2:** Let $(X,G)$ be a $G$-metric Space. A sequence $\{x_n\}$ in $X$ is said to be $G$-convergent if there is a $x_0 \in X$ such that to each $\varepsilon > 0$ there is a natural number $N$ for which $G(x_n,x_n,x_0) < \varepsilon$ for all $n \geq N$.

**Definition 2.3:** Let $(X,G)$ be a $G$-metric Space. A sequence $\{x_n\}$ in $X$ is said to be $G$-Cauchy if for each $\varepsilon > 0$ there exists is a natural number $N$ such that $G(x_n,x_m,x_l) < \varepsilon$ for all $n,m,l \geq N$. Note that every $G$-convergent sequence in a $G$-metric space $(X,G)$ is $G$-Cauchy.

**Definition 2.4:** Let $f$ and $g$ be two self maps of a $G$-metric space $(X,G)$ such that $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$ for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$, then the functions $f$ and $g$ are said to be compatible.

**Definition 2.5:** A function $\psi : [0,\infty) \to [0,\infty)$ is said to be a contractive modulus if $\psi(0) = 0$ and $\psi(t) < t$ for $t > 0$

**Definition 2.6:** Let $f$ and $g$ be self maps of a non-empty set $X$ and let $x_0 \in X$,we can find a sequence $\{x_n\}$ in $X$ satisfying that $fx_n = gx_{n-1}$ for $n \geq 0$ then $\{x_n\}$ is called an associated sequence of $x_0$ relative to the self maps $f$ and $g$.

## 3. MAIN RESULT

**Theorem 3.1:** Suppose $f$ is continuous selfmap of a $G$-metric space $(X,G)$, then $f$ has a fixed point in $X$ if and only if there is a contractive modulus $\psi$ and a continuous selfmap $g$ of $X$ such that:

(i) $f$ and $g$ are compatible

(ii) $G(gx,gy,gy) \leq \psi(G(fx,fy,fy))$ for all $x,y \in X$

and

(iii) there is a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of $x_0$ relative to the selfmaps $f$ and $g$ such that the sequence $\{fx_n\}$ converges to some point $t$ of $X$.

Further $gt$ is the unique common fixed point of $f$ and $g$.

**Proof:** To prove the necessary part, suppose that $f$ has a fixed point, say 'a', $a \in X$, then $fa = a$. Define $g : X \to X$ by $gx = a$ for all $x \in X$. Now for any $x \in X$, we have $(gf)x = g(fx) = a$ and $(fg)x = fgx = fa = a$ for any $x \in X$, giving that $fg = gf$, so that f and g are compatible.
Now let $\psi$ be a contractive modulus, then $\psi(0) = 0$ and $\psi(t) < t$ for $t > 0$ and for any $x,y \in X$ $G(gx,gy,gy) = G(a,a,a) = 0 \leq \psi(G(fx,fy,fy)$.
Further an associated sequence of $x_0 = a$ relative to the selfmaps $f$ and $g$ is given by $x_n = a$ for $n = 0,1,2,3\cdots$ , and since the sequence $\{fx_n\}$ is a constant sequence converging to $a$, which is a point in $X$.
Thus the conditions (i) (ii) and (iii) of the theorem are satisfied.
Conversely, suppose that there is a contractive modulus $\psi$ and a selfmap $g$ of $X$ satisfying (i) (ii) and (iii) of the theorem hold.
From the condition (iii) of the theorem there is an associated sequence $\{x_n\}$ of $x_0$ relative to the selfmaps $f$ and $g$ such that $fx_n = gx_{n-1}$ for $n = 1,2,3\cdots$ and $fx_n \to t$ as $n \to \infty$ for some $t \in X$. Then since $gx_n = fx_{n+1}$, it follows that $gx_n \to t$ as $n \to \infty$.
Now we show that $g$ is continuous on $X$. To see this, suppose that $\{y_n\}$ is a sequence in $X$ with $y_n \to y$ as $n \to \infty$, $y \in X$. Since $f$ is continuous $fy_n \to fy$ as $n \to \infty$, this together with inequality (ii) of the theorem, we get $G(gy_n,gy,gy) \leq \psi(G(fy_n,fy,fy)) \to 0$ as $n \to \infty$, which implies that $gy_n \to gy$ as $n \to \infty$, showing that $g$ is continuous.
Using the continuity of $f$ and $g$,we get $gfx_n \to gt, fgx_n \to ft$ as $n \to \infty$. Since $fx_n \to t, gx_n \to t$

as $n \to \infty$ and $f$ and $g$ are compatible, we have $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$ which implies that $G(ft, gt, gt) = 0$ gives $ft = gt$. To show that $fgt = gft$, take $z_n = t$ for $n = 1, 2, 3\cdots$ so that $fz_n \to ft$ and $gz_n \to gt$ as $n \to \infty$. Since $ft = gt$, $f$ and $g$ are compatible , we get $\lim_{n\to\infty} G(fgz_n, gfz_n, gfz_n) = 0$.

Using the continuity of G, $f$ and $g$, we get $gfz_n \to gft$ and $fgz_n \to fgt$ as $n \to \infty$. It follows that $G(fgt, gft, gft) = 0$ and hence $fgt = gft$

Consequently

$$fft = fgt = gft = ggt \tag{1}$$

If possible suppose that $gt \neq ggt$, then $G(gt, ggt, ggt) > 0$ and hence

$$\psi(G(gt, ggt, ggt)) < G(gt, ggt, ggt) \tag{2}$$

But from (ii)of the theorem and 1 we get

$$G(gt, ggt, ggt) \leq \psi(G(ft, fgt, fgt)) = \psi(G(gt, ggt, ggt))$$

which is contradicts to 2, hence $gt = ggt$.

Using this in 1 we get $ggt = gt = fgt$, showing that$gt$ is a common fixed point of $f$ and $g$.

**Uniqueness:** Suppose that $u = fu = gu$ and $v = fv = gv$ for some $u, v \in X$.

if possible suppose that $u \neq v$,then $G(u, v, v) \neq 0$ so that

$$\psi(G(u, v, v)) < G(u, v, v) \tag{3}$$

from (ii) of the theorem we have

$$G(u, v, v) = G(gu, gv, gv) \leq \psi(G(fu, fv, fv)) = \psi(G(u, v, v))$$

which is contradiction to 3, hence $u = v$, proving the theorem.

**Corollary 3.2:** Suppose $f$ is continuous selfmap of a $G$-metric space $(X, G)$, then $f$ has a fixed point in $X$ if and only if there is a contractive modulus $\psi$ and a selfmap $g$ of $X$ such that

(i) $fg = gf$

(ii) $G(gx, gy, gy) \leq \psi(G(fx, fy, fy))$ for all $x, y \in X$

and

(iii) there is a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of $x_0$ relative to the selfmaps $f$ and $g$ such that the sequence $\{fx_n\}$ converges to some point $t$ of $X$. Further $gt$ is the unique common fixed point of $f$ and $g$.

**Proof:** From the fact that the commutativity implies the compatibility of a pair of selfmaps proof of the corollary follows from the Theorem 3.

**Corollary 3.3:** Suppose $f$ and $g$ are selfmaps of a $G$-metric space $(X,G)$. Let $f$ is continuous and if there is a contractive modulus $\psi$ and a positive integer k such that:

(i) $fg = gf$

(ii) $G(g^k x, g^k y, g^k y) \leq \psi(G(fx, fy, fy))$ for all $x,y \in X$

and

(iii) there is a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of $x_0$ relative to the selfmaps $f$ and $g^k$ such that the sequence $\{fx_n\}$ converges to some point $t$ of $X$. Further $gt$ is the unique common fixed point of $f$ and $g$.

**Proof:** From the condition (i) of the corollary 3 we get $fg^k = g^k f$. Thus $f$ and $g^k$ are commuting and hence satisfying the hypothesis of 3, and therefore $f$ and $g^k$ have a unique common fixed point say $b$, then $g^k b = b = fb$.

Now $g^k gb = g^{k+1} b = gg^k b = gb$ and $fgb = gfb = gb$.

This shows that $gb$ is a common fixed point of $f$ and $g^k$. The uniqueness of $b$ implies that $gb = b$ since $fb = b$ , $b$ is a common fixed point of $f$ and $g$.

To prove that $f$ and $g$ have unique common fixed point, suppose that $u = fu = gu$ and $v = fv = gv$ for some $u,v \in X$, so that $g^k u = u$ and $g^k v = v$, this shows that $u,v$ are common fixed points of $f$ and $g^k$. The uniqueness of common fixed point of $f$ and $g^k$ implies $u = v$.

**Corollary 3.4:** Let $p$ be a positive integer. If $g$ is continuous selfmap of a $G$-metric space $(X,G)$, such that:

(i) $G(fx, fy, fy) \leq \psi(G(g^P x, g^p y, g^p y))$ for all $x,y \in X$

and

(ii) there is a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of $x_0$ relative to the selfmaps $g^p$ and $I$( where $I$ is the identity map on$X$) such that the sequence $\{g^p x_n\}$ converges to some point $t$ of $X$. Then $g$ has a unique common fixed point in $X$.

**Proof:** We know that $g^p I = I g^p$. From (ii) of the corollary 3, we have
$G(x,y,y) = G(Ix,Iy,Iy) \leq \psi G(g^p x, g^p y, g^p y)$ for all $x,y \in X$.
Since $g$ is continuous, $g^p$ is continuous. Applying corollary 5.3.1 to the function $g^p$ and $I$, we have unique common fixed point, showing that $g$ has unique fixed point as every point of $X$ is a fixed point of $I$.

## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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