Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 5, 1522-1531
ISSN: 1927-5307

SOME INTERESTING APPLICATIONS OF GRAPH LABELLINGS

V.YEGNANARAYANAN[1,*] AND P.VAIDHYANATHAN[2]

[1]Department of Mathematics, Velammal Engineering College, Chennai-600066, India

[2]Department of Mathematics, Bharathiar University Extension Centre, Bharathiar University, Coimbatore, India

Abstract. A graph $G(p, q)$ is said to be (1,1) edge-magic with the common edge count k_0 if there exists a bijection $f : V(G) \cup E(G) \to \{1, ..., p + q\}$ such that $f(u) + f(v) + f(e) = k_0$ for all $e = (u, v) \in E(G)$. A graph $G(p, q)$ is said to be (1,1) vertex-magic with the common vertex number count k_1 if there exists a bijection $f : V(G) \cup E(G) \to \{1, ..., p + q\}$ such that for each $u \in V(G), f(u) + \sum_e f(e) = k_1$ for all $e = (u, v) \in E(G)$ with $v \in V(G)$. We call f a nice (1,1) edge-magic labeling of G if $f(V(G)) = \{1, ..., p\}$. A graph $G = (V, E)$ is said to be (0,1) edge-magic (modulo $p =| V |$) if there exists a bijection $f : E(G) \to \{1, ..., q =| E |\}$ such that the induced mapping $g : V(G) \to N$ defined by $g(u) = \sum_{(u,v)\in E(G), v\in V(G)} f(u, v) (\bmod p)$ is a constant map. In this paper we discuss briefly the applications of these labelings.

Keywords: (1,1) edge magic graphs, (1,1) vertex magic graphs, (o,1), Sidon sets .

2000 AMS Subject Classification: Primary 05C78, Secondary: 05B10, 05C69, 05C75

1. Introduction

Labeling of Graphs are very extended area of research. It is an active area of research in graph theory with hundreds of references in the literature. The main problem is to

*Corresponding author

Received June 2, 2012

assign a set of integers or the elements of a group to elements of the graph (vertices, edges or both) such that some arithmetic properties hold. One of the motivations is to address long-standing conjectures on decompositions of graphs, like the celebrated Ringel conjecture on the decomposition of the complete graph by isomorphic copies of a given tree. However the area of graph labelings has many applications both within mathematics and to several areas of computer science and communication networks.

A labeling of a graph can be described by a function that maps pieces of a graph (vertices - vertex labeling, edges - edge labeling, or both - total labeling) into some set of numbers (possibly N, Z or even R) usually called labels, such that some property is satisfied. In this paper we focus on vertex labeling on finite, simple undirected graphs. Labeling is an assignment of numbers in a graph such that some numerical property holds. Other names that have been used in the past instead of labeling are valuations and numberings. Despite the fact that these names seem more descriptive according to the above observation the name labeling has prevailed. It is obvious from the abstraction of the definition that it is possible to express a variety of concepts as labeling of graphs. The most obvious might be for example weight functions. However it is also possible to express many graph properties.

For instance,1) let $G(V, E)$ be a simple unweighted undirected graph and $s \in V$ a vertex. Consider a vertex labeling $\Phi : V \rightarrow \{1, ..., diam(G)\}$, where $\Phi(s) = 0$ and for every other $v \in V$, $\Phi(v) = \min\{\Phi(u) \mid u \in N(v)\} + 1$ (diam(G) stands for the diameter of G and $N(v)$ stands for the neighborhood of v). A labeling like this describes the length of a shortest path between s and any other vertex v. Then let $G(V, E)$ be a simple undirected graph. Consider a bijective vertex labeling $\Phi' : V \rightarrow \{1, ..., |V|\}$ such that for any two vertices $u, v \in V$ if $|\Phi'(u) - \Phi'(v)| = 1$ then $(u, v) \in E$. Finding such a labeling in G would actually produce a Hamiltonian path. Being able to describe a wide variety of graph properties makes graph labeling a very powerful tool with a plethora of applications in many different areas.

Let us imagine that in a two party parliament of a democratic country, the legislators from two parties are approximately the same. In order to form fair committees which focusses on different affairs in the parliament, lereaat, what are the possible arrangements of the legislators of parties such that a balanced situation for committees is reached? That is how to make the number of committees dominated by either party are approximately the same? The situtaion can be modeled using edge balanced graph labeling [5].

2. Some Interesting Labelings

Optimal Linear Arrangements. The optimal linear arrangement problem arose in solving wiring problems or some placement problems. For a labeling π of a graph G, we define: $f_\pi(G) = \sum_{\{u,v\}\in E(G)} \mid \pi(u) - \pi(v) \mid$. The cost of G, denoted by $f(G)$, is the minimum value of $f_\pi(G)$ for π ranging over all labelings of G. A labeling π with $f_\pi(G) = f(G)$ is called an optimal linear arrangement. It can easily checked that for a path P_n with n vertices, a star $S_n = K_{1,n-1}$, a cycle C_n and the complete graph K_n, we have respectively, $f(P_n) = n-1$, $f(C_n) = 2(n-1)$, $f(S_n) = [\frac{n^2}{4}]$, and $f(K_n) = n(n^2-1)/6$.

Folding Number. For a labeling π of a graph G we define: $t_\pi(G) = \max_i \mid \{(u,v) \in E : \pi(u) \le i < \pi(v)\} \mid$. The folding number of G, denoted by $t(G)$, is the minimum value of $t_\pi(G)$ as π ranges over all labelings of G. A labeling π with $t_\pi(G) = t(G)$ is called a folding labeling. A folding labeling is also occasionally called a minimum cut linear arrangement. It can be easily checked that $t(P_n) = 1$, $t(C_n) = 2$, $t(K_n) = \lfloor \frac{n}{2} \rfloor . \lceil \frac{n}{2} \rceil$, and $t(S_n) = \lfloor \frac{n}{2} \rfloor$. Stockmeyer.et al[9] proved that the problem of determining the folding number of a graph is NP-complete. The folding number problem arose in VLSI (Very large scale integration) design problems.

Harmonious Labelings. We call a connected graph with v vertices and $e > v$ edges harmonious if it is possible to label the vertices v with distinct elements $\pi(v)$ of $\mathbb{Z}_e$ (the integers modulo e) in such a way that, when each edge $\{u,v\}$ is labeled with $\pi(u) + \pi(v)(\text{mod } e)$, the resulting edge labels are distinct. If the graph is a tree we allow exactly one vertex label to be repeated. Such a labeling of the vertices and edges is called a harmonious labeling of the graph. Graham and Sloane[3] first investigated harmonious labelings for various graphs. The harmonious labellings are closely related to problems in error correcting codes.

3. Edge Magic Labelings

A graph $G = (V, E)$ is said to be (0,1) edge-magic (modulo $p =\mid V \mid$) if there exists a bijection $f : E(G) \to \{1, ..., q =\mid E \mid\}$ such that the induced mapping $g : V(G) \to N$ defined by $g(u) = \sum_{(u,v)\in E(G), v\in V(G)} f(u, v)(\text{mod}\, p)$ is a constant map. For example, consider the simplest cubic graph, the prism, $C_3 \times K_2$. It has several modulo p (0,1) edge-magic labeling. Figure 1 shows some of them.

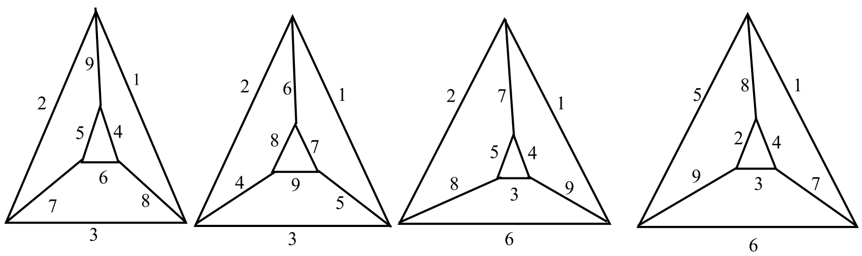

FIGURE 1

By carefully looking at the way the labels are arranged in the first two diagrams of Figure 1 we infer that a procedure can be drawn in general to label graphs of this type. Suppose that $G = C_n \times K_2$ is a prism with $V(G) = p = 2n, p \equiv 2 \pmod 4$ and $\mid E(G) \mid= q = \frac{3p}{2}$. We proceed as follows to produce a modulo p (0,1) edge-magic labeling: Start at any edge of one of the cycles. Assign successive integers from 1,2,...,$p/2$ to every alternative edge on the cycle in clockwise direction. When we arrive at a previously labeled edge then, continue the labeling in the same direction and same way beginning with the edge of the other cycle that is parallel to the first labeled edge on the first cycle. Now to label the linking edges between two cycles in G, start with the label $p+1$. [Observe that we have used the labels 1,2,...p successively to label the edges of the outer and inner cycles of G]. Assign the label $(p + 1)$ to that linking edge which joins the vertex common to the edges of the inner cycle with labels $p - \lfloor n/2 \rfloor$ and p. Then assign the rest of the labels $p+2, p+3, ...q$ to other linking edges in the anti-clockwise direction. For example, look at Figure 2. It gives a modulo p, (0,1) edge-magic labeling of the prism $G = C_9 \times K_2$.

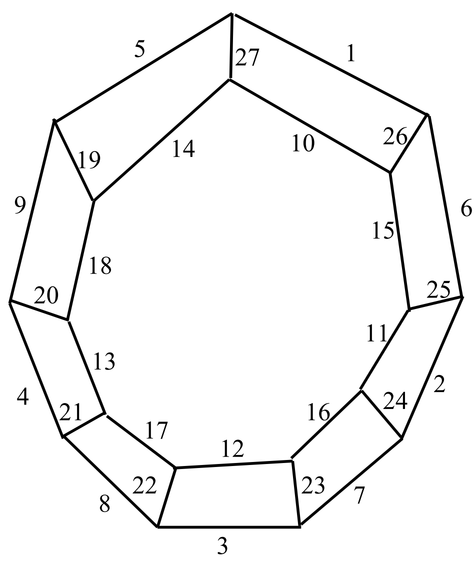

FIGURE 2

The above method took the advantage of the property of a cubic graph viz., A cubic graph on p vertices will have $3p/2$ edges and the set of integers from 1 to $3p/2$ can be partitioned into the sets $\{1, 2, ..., p/2\}$, $\{(p/2)+1, (p/2)+2, ..., p\}$ and $\{p+1, p+2, ..., 3p/2\}$.

Next let us consider the Mobius ladder, M_n, which is a cubic circulant graph with an even number of vertices, formed from an n-cycle by adding edges called "rungs" connecting opposite pairs of vertices in the cycle. It is so named because with the exception of M_6, which is the $K_{3,3}$, M_n has exactly $(n/2)$ 4-cycles which link together by their shared edges to form a topological Mobius strip. Every Mobius ladder is a non planar apex graph with crossing number one, and can be embedded without crossings on a torus or projective plane. They are vertex transitive but not edge transitive. When $n \equiv 2 \pmod 4$, M_n is bipartite. When $n \equiv 0 \pmod 4$ it has chromatic number 3. They are uniquely determined by their chromatic polynomials. It plays an important role in the theory of graph minors. Walba et.al [10] first synthesized molecular structures in the form of Mobius ladder, and since then this structure has been of interest in chemistry and chemical stereography, especially in view of the ladder-like form of DNA molecules. Mobius ladders have also been used as the shape of a superconducting ring in experiments to study the effects of conductor topology on electron interactions[7]. They have also been used in computer science as part of integer programming approaches to problems of set packing and linear ordering.

The prism graph $G_p = C_p \times K_2$ are both planar and polyhedral. It has $2p$ vertices and $3p$ edges and is equivalent to the generalized peterson graph. For odd p, it is isomorphic to a circulant graph as can be seen by rotating the inner cycle by $180°$ and increasing its radius to equal that of the outer cycle in top embeddings above. It is equivalent to the cayley graph of the dihedral graph D_{2n} with respect to the generating set $\{x, x^{-1}, y\}$, G_3 is the line graph of $K_{2,3}$ and G_4 is isomorphic with the cubical graph. Figure 3 shows the modulo p (0,1) edge-magic labeling of M_6 and M_{10}.

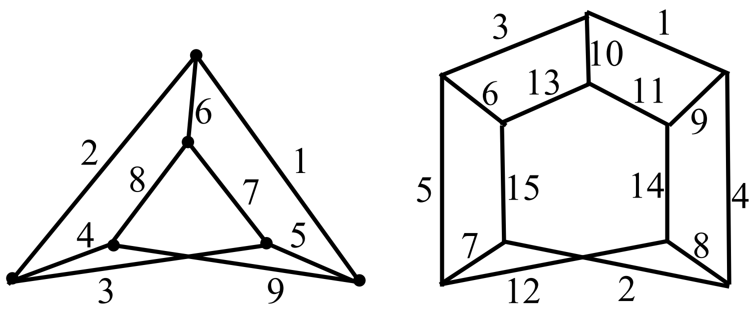

FIGURE 3

4. A Method to obtain (0,1) edge magic labels

To produce a modulo p (0,1) edge-magic labeling for M_p, where $p \equiv 2(\text{mod } 4)$, start at any edge on one of the cycles. Assign a label from the set $\{1, 2, ...p/2\}$ to each alternative edge on this cycle in a clockwise direction. Upon reaching the edge this cycle with label $p/2$, label the edge that joins the vertex common to the edge with label $p/2$ and the edge with label $p/2 - \lfloor n/2 \rfloor$ to the other cycle as $(p/2) + 1$. Assign labels successively to the other connecting edges in the reverse direction (anti clockwise). Then label the remaining edges $E_j (j = p+1, p+2, ..., q)$ with the label $p+i$, where i is the label of the edge parallel to E_j.

5. (1,1) edge-magic graphs and Sidon Sets

A graph $G(p,q)$ is said to be (1,1) edge-magic with the common edge count k_0 if there exists a bijection $f : V(G) \cup E(G) \to \{1, ..., p+q\}$ such that $f(u) + f(v) + f(e) = k_0$ for all $e = (u,v) \in E(G)$. A graph $G(p,q)$ is said to be (1,1) vertex-magic with the common vertex number count k_1 if there exists a bijection $f : V(G) \cup E(G) \to \{1, ..., p+q\}$ such that for each $u \in V(G)$, $f(u) + \sum_e f(e) = k_1$ for all $e = (u,v) \in E(G)$ with $v \in V(G)$. We call f a nice (1,1) edge-magic labeling of G if $f(V(G)) = \{1, ..., p\}$ and G is said to be nice (1,1) edge-magic. These labelings are illustrated in Figure 4 and Figure 5. For more see [11],[12],[13] and [14].

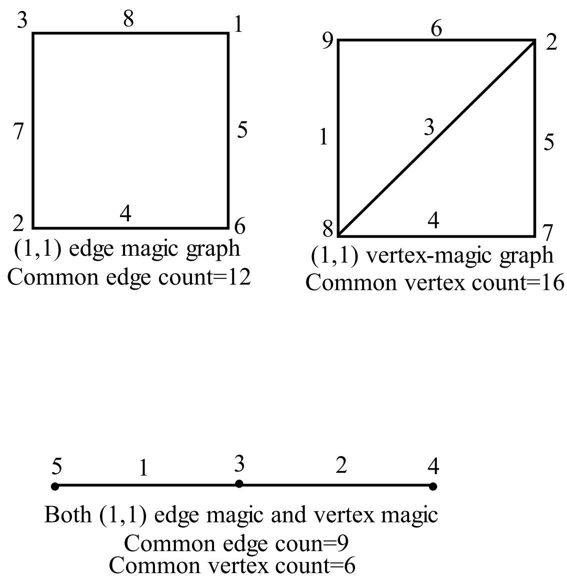

FIGURE 4

Take a set of integers, say $A = \{1, 2, 3, 4\}$, and calculate all the possible sums of two elements of the set: $A + A = \{2, 3, 4, 5, 6, 7, 8\}$. If you give me the sum 5 then I can not deduce if you have picked 1 and 4 or 2 and 3. Now take a different set, say $S = \{1, 2, 4, 8\}$, and again calculate all the possible sums of two of its elements: $S + S = \{2, 3, 4, 5, 6, 8, 9, 10, 12, 16\}$. If you give me the sum 8 I know that you picked 4 twice, if you give me 10 I know that you picked 2 and 8, and the same thing happens with every possible sum. S is a Sidon set.

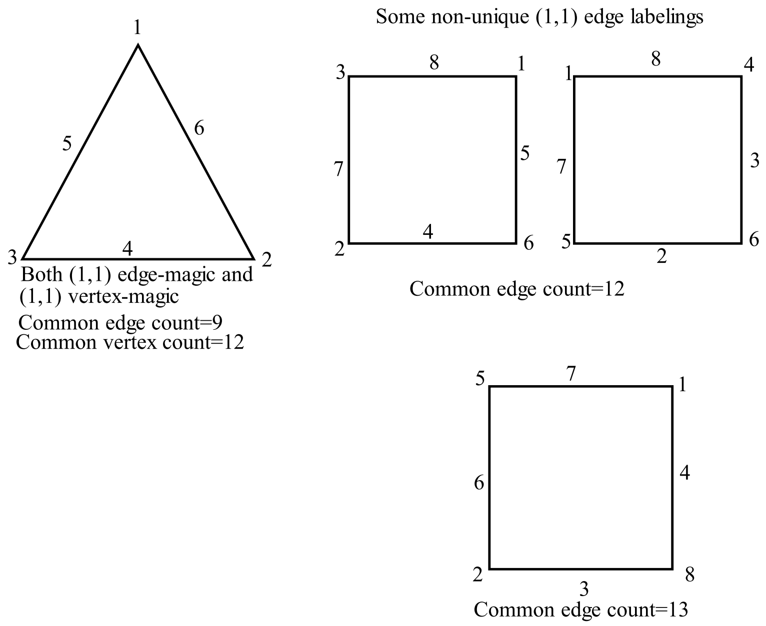

FIGURE 5

In other words, A is a Sidon set if all the sums $a_1 + a_2$, $a_i \in A$, are different (except when they coincide because of conmutativity, $a_1 + a_2 = a_2 + a_1$. If we pick 5 numbers in arithmetic progression, there will be a lot of repeated sums and they will not form a Sidon set. On the other hand, if we pick 5 numbers at random, between say 1 and 100, very probably they will form a Sidon set. So constructing Sidon sets is not difficult. The interesting thing is constructing Sidon sets with as many elements as we can. We can consider infinite sets instead of finite ones (we will call them sequences instead of sets). We can allow 5 repetitions of each sum instead of only one. We can add up seven elements instead of two. In any case, the goal will be to give bounds for the number of elements such a generalized Sidon set can have.

A set A of integers is said to be a Sidon set if all sums of pairs of elements (not necessarily different) of A, are pairwise distinct. Sidon introduced the concept in his investigations of Fourier series.The main problem, posed by Sidon is how many elements can A have up to some number x. Despite a large body of research the question remained unsolved for almost 80 years. Recently, it was finally settled by J. Cilleruelo, I. Ruzsa and C. Vinuesa.[1]

In 1941 Erdos and Turan [2] proved that a Sidon set $A \subset [1,n]$ always satisfies, $\mid A \mid \leq n^{1/2}+n^{1/4}+1$. Kotzig [6] calls a set $A \subset Z$ a well spread sequence if all sums of distinct elements in A are pairwise different. He showed that, if $A \subset [1,n]$ with $n \geq 8$, then $n \geq 4+\begin{pmatrix} \mid A \mid -1 \\ 2 \end{pmatrix}$...(1) Ruzsa [8] calls such a set a weak sidon set. He proved that a weak sidon set in $[1,n]$ satisfies $\mid A \mid \leq n^{1/2}+n^{1/4}+1$....(2). If $A \subset V$ induces a clique in a (1,1) edge magic graph $G=(V,E)$ with (1,1) edge magic labeling f then $f(A)$ is a weak sidon set. That is, for each pair of vertices $x,y \in A$, we have $f(x)+f(y)=k-f(xy)$, so that the sums of labels of pairs of vertices in A are pairwise distinct. Therefore $\mid A \mid$ is bounded by (2) with $n=\mid V \cup E \mid$, or $n=\mid V \mid$ if f is (1,1) nice edge magic. These are explicit constructions of (weak) sidon sets whose cardinality is close to the upper bound in (2). For instance, for any prime p, Singer gives a construction of a sidon set of cardinality $p+1$ in $[1,n]$ with $n=p^2+p+1$ and Bose gives one of cardinality p with $n=p^2-1$. See for instance [4]. Ruzsa [8] gives also such a construction of a sidon set with $(p-1)$ elements in $[1,p^2-p]$. Since for each positive integer n_0 there is a prime p such that $p \leq n+0(n_0)$, these construction provide a sidon set of order n_0 in $[1,n]$ with $n \leq n_0^2+0(n_0^2)$. The existence of dense sidon sets provide the means to obtain lower bounds for the largest possible clique in a connected (1,1) edge magic graph.

References

[1] Cilleruelo, J.; Ruzsa, I.; Vinuesa, C., "Generalized Sidon sets", Advances in Mathematics 225: 2786-2807,2010.

[2] Erdos.P and Turan.P, On a problem of Sidon in additive number theory and some related problems, Journal of London Math. Soc. 16,212-215, 1941.

[3] Graham .R.L, and Sloane. N.J.A, On additive bases and harmonious graphs, Siam J. Alg. Disc. Meth, Vol-I (4), 382-404, Dec 1980.

[4] Halberstan.H and Roth.K.F, Sequences, Springer Verlag, New York, 1983.

[5] M.C.Kong and S.M.Lee., On Edge Balancved Graphs, Graph Theory, Combinatorics and Algorithms, 1, 711-722, 1995.

[6] Kotzig.A, On well spread sets of integers, Publications du CRM-161, 1972.

[7] Mila, Frederic, Stafford, C.A, Capponi, Sylvain., Persistent currents in a Mobius ladder, A Test of Inter chain Coherence of interacting electrons, Physical Review, B57(3), 1457-1460, 1998.

[8] Ruzsa. I.Z, Solving a linear equation in a set of integers, Acta Arithmetica, LXV.3, 259-282,1993.

[9] Stockmeyer.L.J, M.R.Garey, D.S. Johnson, Some Simplified NP-complete graph Problems, Theoretical Computer Science, 1, 237-267, 1976.

[10] Walba.D, Richards.R, Haliwanger.R.C, Total Synthesis of the first molecular mobius strip, J. of the American Chemical Society, 104(11), 3219-3221, 1982.

[11] Yegnanarayanan.V and Vaidhyanathan.P, Magic Graphs II, Accepted for publication on June 29, 2010, Utilitas Mathematica, To appear.

[12] Yegnanarayanan.V, On Magic Graphs, Utilitas Mathematica 59,181-201, 2001.

[13] Yegnanarayanan.V and Vaidhyanathan.P, On Nice (1,1) Edge-Magic Graphs, Electronic Notes in Discrete Mathematics 33,115-122, 2009.

[14] Yegnanarayanan.V, Vaidhyanathan.P and Manoharan.P, On Magic Labeling of Graphs, appered in the Proc. of the Int. National Conference on Maths. in Engineering & Business Management March 9-10, 2012, pp 110-114, 2012, ISBN:978-81-8286-015-5.