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SOFT VALUATION ON A GENERALIZED SOFT LATTICE

D. SUSHA, MANJU JOHN*

Department of Mathematics, Catholicate College, Pathanamthitta-689645, Kerala, India

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Abstract. Lattice theory play an important role in mathematics as well as in other disciplines such as computer science, engineering, cryptography, etc. In this paper, we introduce the concept of generalized soft lattice (gs lattice) and investigate some of its fundamental properties. Further we define soft valuation on a generalized soft lattice (gs lattice) and study its major properties. In the last section we discuss the notion of soft distance function and express it in terms of soft valuation. Here we discuss the notions of soft pseudo metric lattice and soft metric lattice.

Keywords: gs-lattice; soft valuation; soft pseudo metric lattice and soft metric lattice.

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1. INTRODUCTION

D Molodstov in [10] introduced the concept of soft sets and is defined as a parameterized family of subsets of an initial universal set. He defined some basic notions and showed that it can be applied to more fields of mathematics as well as various fields which contain uncertain data.

*Corresponding author

E-mail address: manjujohn75@gmail.com

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The theory of lattices were developed by R. Dedekind in early 1980's. In every lattice, meet and join are commutative and associative binary operations. So it can be considered as commutative semigroup with the binary algebraic operations meet and join. In [2], Birkhoff present deeper ideas of lattice theory. Presently lattice theory is developing rapidly. Applications of soft sets to lattices is described in [11]. The concept of soft elements of a soft set was introduced by S. Das and S.K. Samanta in [6] and they introduced soft metric space in terms of soft elements in [5]. Also, S. Das, P. Majumdar and S.K. Samanta introduced the notion of soft linear space in terms of soft elements in [4], M. Chiney and S.K. Samanta introduced a new version of soft topology in [3]. Based on this, we would like to express certain concepts in lattice theory to soft sets in terms of soft elements.

This paper is organized as follows. In section 2 we recall some preliminaries. In section 3 we introduce generalized soft lattices (in short gs lattices) and investigate its properties. Here we observe that any gs lattice satisfy the distributive inequality and soft modular identity. Section 4 deals with the notion of soft valuation on a gs lattice. Here we prove that any gs lattice with positive soft valuation is soft modular. In the last section we define soft distance function and express in terms of soft valuation and verify that the soft distance function satisfies the axioms of soft pseudo metric space. We conclude this section by defining soft pseudometric lattice and soft metric lattice with some properties.

2. PRELIMINARIES

Definition 2.1. [2] A poset is a set P in which a binary relation $x \leq y$ is defined which satisfies the following conditions:

- (i) $x \leq x, \forall x \in P$ (Reflexive)
- (ii) If $x \leq y$ and $y \leq x$, then $x = y$ (Antisymmetry) and
- (iii) If $x \leq y$ and $y \leq z$, then $x \leq z$. (Transitivity)

Definition 2.2. [2] A lattice is a poset P with the property that every pair of elements $x, y \in P$ have a g.l.b. (meet) and a l.u.b. (join) in P and are denoted by $x \wedge y$ and $x \vee y$ respectively.

Definition 2.3. [10] Let X be an initial universe and E be a set of parameters. Let $P(X)$ denotes the power set of X and $A \subseteq E$. Then a soft set over X is a pair (F, A) , where F is a mapping from A to $P(X)$.

Definition 2.4. [8] Let E be the set of parameters and $A \subseteq E$. Then for each soft set (F, A) over an initial universal set X , a soft set (H, E) is constructed over X , where for all $\lambda \in E$,

$$H(\lambda) = \begin{cases} F(\lambda) & \text{if } \lambda \in A \\ \phi & \text{if } \lambda \in E \setminus A \end{cases}$$

Thus the soft sets (F, A) and (H, E) are equivalent to each other and the usual set operations on both are same. So we have to consider soft sets over the same parameter set A .

Definition 2.5. [7] Let (F, A) and (G, B) be two soft sets over a non empty common universal set X .

Then (a) (F, A) is a soft subset of (G, B) if (i) $A \subseteq B$ and

(ii) $F(a) \subseteq G(a), \forall a \in A$.

We write $(F, A) \tilde{\subset} (G, B)$.

(b) (F, A) and (G, B) are soft equal if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A) .

Definition 2.6. [9] Let X be a nonempty universal set and A be a nonempty parameter set. Then a soft set (F, A) over X is said to be a null soft set if $F(\lambda) = \phi, \forall \lambda \in A$ and absolute soft set if $F(\lambda) = X, \forall \lambda \in A$.

The absolute soft set over X with parameter set A is denoted by \tilde{X} or (\tilde{X}, A) and the null soft set is denoted by $\tilde{\Phi}$ or $(\tilde{\Phi}, A)$.

Definition 2.7. [1] Let (F, A) and (G, B) be two soft sets over X , then the Cartesian product of (F, A) and (G, B) is defined as $(F, A) \times (G, B) = (H, A \times B)$, where $H : A \times B \rightarrow P(X \times X)$ and $H(a, b) = F(a) \times G(b)$.

Definition 2.8. [11] Let (F, A) be a soft set over a lattice L . Then (F, A) is said to be a soft lattice over L if $F(\lambda)$ is a sublattice of $L, \forall \lambda \in A$.

Definition 2.9. [6] Let X be a nonempty universal set and A be a nonempty parameter set. Then a function $\tilde{x} : A \rightarrow X$ is called a soft element of \tilde{X} and we write $\tilde{x} \in \tilde{X}$.

A soft element \tilde{x} is said to belong to a soft set (F, A) over X if $\tilde{x}(\lambda) \in F(\lambda), \forall \lambda \in A$ and we write $\tilde{x} \in (F, A)$.

Thus for a soft set (F, A) over X with the parameter set A with $F(\lambda) \neq \phi, \forall \lambda \in A$, we have $F(\lambda) = \{\tilde{x}(\lambda) : \tilde{x} \in (F, A)\}$. Soft elements are usually denoted by $\tilde{x}, \tilde{y}, \tilde{z}$, etc.

Definition 2.10. [5] Let X be a nonempty set. The collection of all soft sets (F, A) over X for which $F(\lambda) \neq \phi, \forall \lambda \in A$ together with the null soft set $(\tilde{\Phi}, A)$ is denoted by $S(\tilde{X})$.

For any non null soft set in $S(\tilde{X})$, the collection of all soft elements of (F, A) is denoted by $SE((F, A))$.

Let \mathcal{B} be a collection of soft elements of (F, A) . Then the soft set generated by \mathcal{B} is given by $SS(\mathcal{B}) = (G, A)$, where $G(\lambda) = \{\tilde{x}(\lambda) : \tilde{x} \in \mathcal{B}\}$.

Proposition 2.11. [5] For any soft subsets $(F, A), (G, A) \in S(\tilde{X})$, $(F, A) \tilde{\subset} (G, A)$ iff every soft element of (F, A) is also a soft element of (G, A) .

Definition 2.12. [5] Let R be the set of all real numbers and $B(R)$ denotes the set all bounded subsets of R and A be a nonempty set of parameters. Then a mapping $F : A \rightarrow R$ is called a soft real set and is denoted by (F, A) . In particular, if (F, A) is a singleton set then after identifying (F, A) with the corresponding soft element, it will be called a soft real number and are usually denoted by $\tilde{r}, \tilde{s}, \tilde{t}$, etc. Also the soft real numbers \bar{r}, \bar{s}, \dots are particular types such that $\bar{r}(\lambda) = r$ and $\bar{s}(\lambda) = s, \forall \lambda \in A$

For two soft real numbers \tilde{r} and \tilde{s} , we define

$$(1) \tilde{r} \leq \tilde{s} \text{ if } \tilde{r}(\lambda) \leq \tilde{s}(\lambda), \forall \lambda \in A.$$

$$(2) \tilde{r} \geq \tilde{s} \text{ if } \tilde{r}(\lambda) \geq \tilde{s}(\lambda), \forall \lambda \in A.$$

$$(3) \tilde{r} < \tilde{s} \text{ if } \tilde{r}(\lambda) < \tilde{s}(\lambda), \forall \lambda \in A.$$

$$(4) \tilde{r} > \tilde{s} \text{ if } \tilde{r}(\lambda) > \tilde{s}(\lambda), \forall \lambda \in A.$$

Definition 2.13. [2] A valuation on a lattice (L, \leq, \vee, \wedge) is a real valued function v on L such that $v[x] + v[y] = v[x \vee y] + v[x \wedge y]$.

Result. [2] In any modular lattice of finite length, the height function is a positive valuation.

Definition 2.14. [3] Let X and Y be two nonempty sets and $\{f_\lambda : \lambda \in A\}$ be a collection of functions from X to Y . Then a function $f : SE(\tilde{X}) \rightarrow SE(\tilde{Y})$ defined by $f[\tilde{x}](\lambda) = f_\lambda(\tilde{x}(\lambda)) \quad \forall \lambda \in A$ is called a soft function.

Definition 2.15. [5] Let \tilde{X} be the absolute soft set over the nonempty set X with the nonempty set of parameters A and $R(A)^*$ denote the set of all non negative soft real numbers. A mapping $d : SE(\tilde{X}) \times SE(\tilde{X}) \rightarrow R(A)^*$ is said to be a soft metric on the soft set \tilde{X} if d satisfies the following axioms.

$$M_1 : d(\tilde{x}, \tilde{y}) \geq \tilde{0} \quad \forall \tilde{x}, \tilde{y} \in \tilde{X}$$

$$M_2 : d(\tilde{x}, \tilde{y}) = \tilde{0} \text{ iff } \tilde{x} = \tilde{y}$$

$$M_3 : d(\tilde{x}, \tilde{y}) = d(\tilde{y}, \tilde{x}) \quad \forall \tilde{x}, \tilde{y} \in \tilde{X}$$

$$M_4 : d(\tilde{x}, \tilde{z}) \leq d(\tilde{x}, \tilde{y}) + d(\tilde{y}, \tilde{z}) \quad \forall \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{X}$$

Definition 2.16. [5] A mapping $d : SE(\tilde{X}) \times SE(\tilde{X}) \rightarrow R(A)^*$ is said to be a soft pseudo metric on the soft set \tilde{X} if d satisfies the following axioms.

$$M_1 : d(\tilde{x}, \tilde{y}) \geq \tilde{0} \quad \forall \tilde{x}, \tilde{y} \in \tilde{X}$$

$$M_2 : d(\tilde{x}, \tilde{y}) = \tilde{0} \text{ if } \tilde{x} = \tilde{y}$$

$$M_3 : d(\tilde{x}, \tilde{y}) = d(\tilde{y}, \tilde{x}) \quad \forall \tilde{x}, \tilde{y} \in \tilde{X}$$

$$M_4 : d(\tilde{x}, \tilde{z}) \leq d(\tilde{x}, \tilde{y}) + d(\tilde{y}, \tilde{z}) \quad \forall \tilde{x}, \tilde{y}, \tilde{z} \in \tilde{X}$$

3. GENERALIZED SOFT LATTICES(GS LATTICES)

In this section we introduce generalized soft lattice (gs lattice) and study its properties. We first define gs poset in terms of soft elements.

Definition 3.1. Let (F, A) be a soft set in $S(\tilde{X})$ with a non empty parameter set A . Then a soft relation $\tilde{\leq}$ on (F, A) is a binary relation on $SE(F, A)$.

Definition 3.2. Let $\tilde{\leq}$ be a soft relation on a soft set (F, A) in $S(\tilde{X})$. Then it is said to be

(i)soft reflexive if $\tilde{x} \tilde{\leq} \tilde{x} \quad \forall \tilde{x} \in (F, A)$

(ii)soft symmetric if $\tilde{x} \tilde{\leq} \tilde{y} \Rightarrow \tilde{y} \leq \tilde{x} \quad \forall \tilde{x}, \tilde{y} \in (F, A)$

(iii)soft antisymmetric if $\tilde{x} \leq \tilde{y}$ and $\tilde{y} \leq \tilde{x} \Rightarrow \tilde{x} = \tilde{y} \quad \forall \tilde{x}, \tilde{y} \in (F, A)$

(iv)soft transitive if $\tilde{x} \leq \tilde{y}$ and $\tilde{y} \leq \tilde{z} \Rightarrow \tilde{x} \leq \tilde{z} \quad \forall \tilde{x}, \tilde{y}, \tilde{z} \in (F, A)$

Definition 3.3. A soft partial ordering on a soft set (F, A) in $S(\tilde{X})$ is a partial ordering on $SE(F, A)$. i.e, a soft relation $\tilde{\leq}$ on (F, A) is a soft partial ordering if it is soft reflexive, soft antisymmetric and soft transitive.

Definition 3.4. A generalized soft poset (gs poset) is a soft set (F, A) in $S(\tilde{X})$ together with a soft partial ordering on it and is denoted by $((F, A), \tilde{\leq})$.

Example 3.5. Consider the absolute soft set \tilde{R} with a non empty parameter set A , where R is the set of all real numbers. Define the soft relation $\tilde{\leq}$ on $SE(\tilde{R})$ by $\tilde{r} \tilde{\leq} \tilde{s}$ if and only if $\tilde{r}(\lambda) \leq \tilde{s}(\lambda), \forall \lambda \in A$.

Then $\tilde{\leq}$ is soft reflexive, soft antisymmetric and soft transitive on \tilde{R} . Hence \tilde{R} is a gs poset with the soft partial ordering $\tilde{\leq}$.

Definition 3.6. Let (F, A) and (G, B) be two gs posets over the universal sets X and Y respectively. Their soft partial orderings are respectively \leq_1 and \leq_2 . Then the direct product of (F, A) and (G, B) is the Cartesian product of (F, A) and (G, B) and is denoted by $(F, A)(G, B)$.

A soft element of $(F, A)(G, B)$ is a function $(\tilde{x}, \tilde{y}) : A \times B \rightarrow F(a) \times G(b)$ given by $(\tilde{x}, \tilde{y})(a, b) = (\tilde{x}(a), \tilde{y}(b))$. The soft elements of $(F, A)(G, B)$ are given by the set $SE((F, A)(G, B)) = \{(\tilde{x}, \tilde{y}) : \tilde{x} \in (F, A), \tilde{y} \in (G, B)\}$

Theorem 3.7. *The direct product of two gs posets is again a gs poset.*

Proof. (F, A) and (G, B) be two gs posets with the soft partial orderings \leq_1 and \leq_2 respectively. Define the soft relation \leq on the direct product $(F, A)(G, B)$ by $(\tilde{x}_1, \tilde{y}_1) \leq (\tilde{x}_2, \tilde{y}_2)$ iff $\tilde{x}_1 \leq_1 \tilde{x}_2$ in (F, A) and $\tilde{y}_1 \leq_2 \tilde{y}_2$ in (G, B) .

(i) Since $\tilde{x}_1 \leq_1 \tilde{x}_1$ and $\tilde{y}_1 \leq_2 \tilde{y}_1$, $(\tilde{x}_1, \tilde{y}_1) \leq (\tilde{x}_1, \tilde{y}_1)$. Hence the soft relation \leq is soft reflexive.

(ii) Let $(\tilde{x}_1, \tilde{y}_1) \leq (\tilde{x}_2, \tilde{y}_2)$ and $(\tilde{x}_2, \tilde{y}_2) \leq (\tilde{x}_1, \tilde{y}_1)$.

$\Rightarrow \tilde{x}_1 \leq_1 \tilde{x}_2$ and $\tilde{x}_2 \leq_1 \tilde{x}_1$ in (F, A) and $\tilde{y}_1 \leq_2 \tilde{y}_2$ and $\tilde{y}_2 \leq_2 \tilde{y}_1$ in (G, B) .

$\Rightarrow \tilde{x}_1 = \tilde{x}_2$ and $\tilde{y}_1 = \tilde{y}_2$.

$\Rightarrow (\tilde{x}_1, \tilde{y}_1) = (\tilde{x}_2, \tilde{y}_2)$.

Hence the soft relation \leq is soft symmetric.

(iii) Let $(\tilde{x}_1, \tilde{y}_1) \leq (\tilde{x}_2, \tilde{y}_2)$ and $(\tilde{x}_2, \tilde{y}_2) \leq (\tilde{x}_3, \tilde{y}_3)$.

$\Rightarrow \tilde{x}_1 \leq_1 \tilde{x}_2$ and $\tilde{x}_2 \leq_1 \tilde{x}_3$ in (F, A) and $\tilde{y}_1 \leq_2 \tilde{y}_2$ and $\tilde{y}_2 \leq_2 \tilde{y}_3$ in (G, B) .

$\Rightarrow \tilde{x}_1 \leq_1 \tilde{x}_3$ and in (F, A) and $\tilde{y}_1 \leq_2 \tilde{y}_3$ in (G, B) .

$\Rightarrow (\tilde{x}_1, \tilde{y}_1) \leq (\tilde{x}_3, \tilde{y}_3)$.

Hence the soft relation \leq is soft transitive. Consequently it is a soft partial ordering on $(F, A)(G, B)$ and the direct product $(F, A)(G, B)$ is a gs poset.

Definition 3.8. soft set (F, A) over a non empty universal set X in $S(\tilde{X})$ with a non empty parameter set A is called a generalized soft lattice (gs-lattice) if its soft elements form a lattice. i.e, if (F, A) is a soft poset and every pair of soft elements have a a greatest soft element(g.l.b. is called the soft meet) and a least soft element(l.u.b called the soft join) in $SE(F, A)$ then it is called a generalized soft lattice(gs-lattice).

Definition 3.9. If the absolute soft set (\tilde{P}, A) is a generalized soft lattice (gs-lattice) then it is called the absolute gs-lattice.

Theorem 3.10. Any lattice $(L, \leq_L, \vee_L, \wedge_L)$ generate a gs-lattice.

Proof. Let $(L, \leq_L, \vee_L, \wedge_L)$ be any lattice and (\tilde{L}, A) be the absolute soft poset generated by \leq_L . i.e, the soft partial ordering on \tilde{L} is given by

for $\tilde{x}, \tilde{y} \in \tilde{L}$, we have $\tilde{x} \leq \tilde{y}$ if and only if $\tilde{x}(\lambda) \leq_\lambda \tilde{y}(\lambda), \forall \lambda \in A$.

Define \vee on \tilde{L} by $(\tilde{x} \vee \tilde{y})(\lambda) = \tilde{x}(\lambda) \vee_L \tilde{y}(\lambda), \forall \lambda \in A$.

Also $(\tilde{x} \vee \tilde{y})(\lambda) = \tilde{x}(\lambda) \vee_L \tilde{y}(\lambda) \geq_L \tilde{x}(\lambda)$ and $\tilde{x}(\lambda) \vee_L \tilde{y}(\lambda) \geq_L \tilde{y}(\lambda), \forall \lambda \in A$.

$\therefore \tilde{x} \vee \tilde{y}$ exists and it is the least upper bound of \tilde{x} and \tilde{y} .

So it is the soft join of \tilde{x} and \tilde{y} .

Similarly, we get the soft meet of \tilde{x} and \tilde{y} given by $(\tilde{x} \wedge \tilde{y})(\lambda) = \tilde{x}(\lambda) \wedge_L \tilde{y}(\lambda), \forall \lambda \in A$.

Hence $(\tilde{L}, \leq, \vee, \wedge)$ form a gs-lattice.

Note. The soft meet and soft join defined in the above proof are respectively called the soft meet and the soft join generated by meet(\wedge_L) and the join(\vee_L) on L . Also, \tilde{L} is called the absolute gs-lattice generated by the lattice $(\tilde{L}, \leq, \vee, \wedge)$.

Theorem 3.11. Any soft lattice is a gs-lattice.

Proof. By definition of soft lattice, any soft lattice is a soft set over a lattice. Let (F, A) be a soft lattice over a lattice $(L, \leq_L, \vee_L, \wedge_L)$. Consider the absolute gs-lattice $(\tilde{L}, \leq, \vee, \wedge)$ generated by the lattice L .

Then since (F, A) is a soft subset of (\tilde{L}, A) , it is a gs poset.

Since (F, A) is a soft lattice, $F(\lambda)$ is a sublattice of $L, \forall \lambda \in A$.

Let $\tilde{x}, \tilde{y} \in (F, A)$.

Since (F, A) is a soft subset of (\tilde{L}, A) , $\tilde{x}, \tilde{y} \in \tilde{L}$. [By proposition 2.11]

Also, since $(\tilde{x} \vee \tilde{y})(\lambda) = \tilde{x}(\lambda) \vee_L \tilde{y}(\lambda)$ and $F(\lambda)$ is a sublattice of $L, (\tilde{x} \vee \tilde{y})(\lambda) \in F(\lambda), \forall \lambda \in A$.

Hence $\tilde{x} \vee \tilde{y} \in (F, A)$.

Similarly, we get $\tilde{x} \wedge \tilde{y} \in (F, A)$.

Hence (F, A) is a gs-lattice.

Remark. Any non null soft subset of an absolute gs-lattice need not be a gs-lattice.

Proof. Consider Z^+ which is a lattice with the partial ordering ' $'$ '. Then \tilde{Z}^+ is a gs lattice with the parameter set $A = \{e_1, e_2\}$. Define the soft set (F, A) by $F(e_1) = \{1, 2, 3\}$ and $F(e_2) = \{1, 2, 3\}$. Its soft elements $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5$ and \tilde{x}_6 are given by

$$\begin{aligned} \tilde{x}_1(e_1) = 1, \tilde{x}_1(e_2) = 1, \tilde{x}_2(e_1) = 1, \tilde{x}_2(e_2) = 2, \tilde{x}_3(e_1) = 1, \tilde{x}_3(e_2) = 3, \tilde{x}_4(e_1) = 2, \tilde{x}_4(e_2) = \\ 1, \tilde{x}_5(e_1) = 21, \tilde{x}_5(e_2) = 2, \tilde{x}_6(e_1) = 2, \tilde{x}_6(e_2) = 3, \tilde{x}_7(e_1) = 3, \tilde{x}_7(e_2) = 1, \tilde{x}_8(e_1) = 3, \tilde{x}_8(e_2) = \\ 2 \text{ and } \tilde{x}_9(e_1) = 3, \tilde{x}_9(e_2) = 3. \end{aligned}$$

\tilde{x}_2 and \tilde{x}_6 has no soft meet (l.u.b).

Theorem 3.12. The direct product of any two gs lattices is a gs lattice.

Proof. Let $(F, A), \vee_F, \wedge_F$ and $(G, B), \vee_G, \wedge_G$ be two gs lattices. Then by theorem 3.7 their direct product $(F, A)(G, B)$ is a gs poset.

Now we have to prove that gs poset possess the soft join and soft meet.

Let $(\tilde{x}_1, \tilde{y}_1), (\tilde{x}_2, \tilde{y}_2) \in (F, A)(G, B)$.

Then $(\tilde{x}_i, \tilde{y}_i) \leq (\tilde{x}_1, \tilde{y}_1) \vee (\tilde{x}_2, \tilde{y}_2) = (\tilde{x}_1 \vee \tilde{x}_2, \tilde{y}_1 \vee \tilde{y}_2)$ for $i = 1, 2$ and

$(\tilde{x}_1 \vee \tilde{x}_2, \tilde{y}_1 \vee \tilde{y}_2) \in (F, A)(G, B)$. ($\because (F, A)$ and (G, B) are gs lattices, $\tilde{x}_1, \tilde{x}_2 \in (F, A)$ and $\tilde{y}_1, \tilde{y}_2 \in (G, B)$).

Also any other upper bound (\tilde{p}, \tilde{q}) of $(\tilde{x}_1, \tilde{y}_1)$ and $(\tilde{x}_2, \tilde{y}_2)$ is such that $\tilde{p} \geq \tilde{x}_i$ and $\tilde{q} \geq \tilde{y}_i$ for

$i = 1, 2$.

Hence $\tilde{p} \geq \tilde{x}_1 \vee \tilde{x}_2$ and $\tilde{q} \geq \tilde{y}_1 \vee \tilde{y}_2$ so that $(\tilde{p}, \tilde{q}) \geq (\tilde{x}_1 \vee \tilde{x}_2, \tilde{y}_1 \vee \tilde{y}_2)$.

Hence $(\tilde{x}_1, \tilde{y}_1) \vee (\tilde{x}_2, \tilde{y}_2) = (\tilde{x}_1 \vee \tilde{x}_2, \tilde{y}_1 \vee \tilde{y}_2)$.

Similarly (or by duality), we get $(\tilde{x}_1, \tilde{y}_1) \wedge (\tilde{x}_2, \tilde{y}_2) = (\tilde{x}_1 \wedge \tilde{x}_2, \tilde{y}_1 \wedge \tilde{y}_2)$.

Hence the direct product of gs lattices $(F, A)(G, B)$ is also a gs lattice.

Theorem 3.13. *In any gs poset \tilde{P} , the soft operations, soft meet and soft join satisfy the following axioms if they exist.*

$$(1) \tilde{x} \vee \tilde{x} = \tilde{x}, \tilde{x} \wedge \tilde{x} = \tilde{x} (\text{idempotent})$$

$$(2) \tilde{x} \vee \tilde{y} = \tilde{y} \vee \tilde{x}, \tilde{x} \wedge \tilde{y} = \tilde{y} \wedge \tilde{x} (\text{Commutativity})$$

$$(3) \tilde{x} \vee (\tilde{y} \vee \tilde{z}) = (\tilde{x} \vee \tilde{y}) \vee \tilde{z}, \tilde{x} \wedge (\tilde{y} \wedge \tilde{z}) = (\tilde{x} \wedge \tilde{y}) \wedge \tilde{z} (\text{Associativity})$$

$$(4) \tilde{x} \wedge (\tilde{x} \vee \tilde{y}) = \tilde{x} \vee (\tilde{x} \wedge \tilde{y}) = \tilde{x} (\text{Absorption})$$

$$(5) \tilde{x} \leq \tilde{y} \Leftrightarrow \tilde{x} \wedge \tilde{y} = \tilde{x}$$

$$\Leftrightarrow \tilde{x} \vee \tilde{y} = \tilde{y} (\text{Consistency})$$

Proof. *If the above relations exist, we have*

$$(1) \tilde{x} \vee \tilde{x} = l.u.b.(\tilde{x}, \tilde{x}) = \tilde{x} \text{ and } \tilde{x} \wedge \tilde{x} = g.l.b.(\tilde{x}, \tilde{x}) = \tilde{x}.$$

$$(2) \tilde{x} \vee \tilde{y} = \tilde{y} \vee \tilde{x} = l.u.b.(\tilde{x}, \tilde{y}) \text{ and } \tilde{x} \wedge \tilde{y} = \tilde{y} \wedge \tilde{x} = g.l.b.(\tilde{x}, \tilde{y})$$

$$(3) \tilde{x} \vee (\tilde{y} \vee \tilde{z}) = (\tilde{x} \vee \tilde{y}) \vee \tilde{z} = l.u.b.(\tilde{x}, \tilde{y}, \tilde{z}) \text{ and } \tilde{x} \wedge (\tilde{y} \wedge \tilde{z}) = (\tilde{x} \wedge \tilde{y}) \wedge \tilde{z} = g.l.b.(\tilde{x}, \tilde{y}, \tilde{z})$$

$$(4) \tilde{x} \wedge (\tilde{x} \vee \tilde{y}) = g.l.b.(\tilde{x}, \tilde{x} \vee \tilde{y}) = \tilde{x} (\cdot : \tilde{x} \leq \tilde{x} \vee \tilde{y}) \text{ and } \tilde{x} \vee (\tilde{x} \wedge \tilde{y}) = l.u.b.(\tilde{x}, \tilde{x} \wedge \tilde{y}) = \tilde{x} (\cdot : \tilde{x} \wedge \tilde{y} \leq \tilde{x}).$$

$$(5) \tilde{x} \leq \tilde{y} \Leftrightarrow \tilde{x} \wedge \tilde{y} = g.l.b.(\tilde{x}, \tilde{y}) = \tilde{x} \text{ and } \tilde{x} \leq \tilde{y} \Leftrightarrow \tilde{x} \vee \tilde{y} = l.u.b.(\tilde{x}, \tilde{y}) = \tilde{y}.$$

Lemma 3.14. *If a soft poset \tilde{P} have universal bounds $\tilde{\theta}$ (least soft element) and \tilde{i} (greatest soft element), then $\forall \tilde{x} \in \tilde{X}$, we have*

$$(1) \tilde{\theta} \wedge \tilde{x} = \tilde{\theta} \text{ and } \tilde{\theta} \vee \tilde{x} = \tilde{x}$$

$$(2) \tilde{i} \wedge \tilde{x} = \tilde{x} \text{ and } \tilde{i} \vee \tilde{x} = \tilde{i}$$

Proof. *Let $\tilde{x} \in \tilde{X}$. Then*

$$(1) \tilde{\theta} \wedge \tilde{x} = g.l.b.(\tilde{\theta}, \tilde{x}) = \tilde{\theta} (\cdot : \tilde{\theta} \leq \tilde{x} \quad \forall \tilde{x} \in \tilde{X}) \text{ and } \tilde{\theta} \vee \tilde{x} = l.u.b.(\tilde{\theta}, \tilde{x}) = \tilde{x}$$

$$(2) \tilde{i} \wedge \tilde{x} = g.l.b.(\tilde{i}, \tilde{x}) = \tilde{x} \text{ and } \tilde{i} \vee \tilde{x} = l.u.b.(\tilde{i}, \tilde{x}) = \tilde{i}$$

Lemma 3.15. *In any gs-lattice, the operations of soft meet and soft join are isotone (order preserving): If $\tilde{y} \leq \tilde{z}$ then $\tilde{x} \wedge \tilde{y} \leq \tilde{x} \wedge \tilde{z}$ and $\tilde{x} \vee \tilde{y} \leq \tilde{x} \vee \tilde{z}$.*

Proof. Let $\tilde{y} \leq \tilde{z}$.

$$\Rightarrow \tilde{y} \wedge \tilde{z} = \tilde{y} \dots \dots \dots (a) \text{ [By consistency]}$$

$$\text{Also } \tilde{x} \wedge \tilde{x} = \tilde{x} \dots \dots \dots (b)$$

$$\begin{aligned} \text{From (a) and (b), we get } \tilde{x} \wedge \tilde{y} &= (\tilde{x} \wedge \tilde{x}) \wedge (\tilde{y} \wedge \tilde{z}) \\ &= (\tilde{x} \wedge \tilde{y}) \wedge (\tilde{x} \wedge \tilde{z}) \text{ (By associative law)} \\ &\Rightarrow \tilde{x} \wedge \tilde{y} \leq \tilde{x} \wedge \tilde{z} (\because \text{by consistency, } \tilde{x} \leq \tilde{y} \Leftrightarrow \tilde{x} \wedge \tilde{y} = \tilde{x}.) \end{aligned}$$

By duality principle, we get $\tilde{x} \vee \tilde{y} \leq \tilde{x} \vee \tilde{z}$.

Theorem 3.16. In any gs-lattice, the distributive inequalities hold.

$$(1) \tilde{x} \wedge (\tilde{y} \vee \tilde{z}) \geq (\tilde{x} \wedge \tilde{y}) \vee (\tilde{x} \wedge \tilde{z})$$

$$(2) \tilde{x} \vee (\tilde{y} \wedge \tilde{z}) \leq (\tilde{x} \vee \tilde{y}) \wedge (\tilde{x} \vee \tilde{z})$$

Proof. We have $\tilde{x} \wedge \tilde{y} \leq \tilde{x}$ and $\tilde{x} \wedge \tilde{y} \leq \tilde{y} \leq \tilde{y} \vee \tilde{z}$

$$\Rightarrow \tilde{x} \wedge \tilde{y} \leq \tilde{x} \wedge (\tilde{y} \vee \tilde{z}) \text{ --- (a)}$$

Also, we have $\tilde{x} \wedge \tilde{z} \leq \tilde{x}$ and $\tilde{x} \wedge \tilde{z} \leq \tilde{z} \leq \tilde{y} \vee \tilde{z}$

$$\Rightarrow \tilde{x} \wedge \tilde{z} \leq \tilde{x} \wedge (\tilde{y} \vee \tilde{z}) \text{ --- (b)}$$

From (a) and (b), we get $\tilde{x} \wedge (\tilde{y} \vee \tilde{z})$ is an upper bound of $\tilde{x} \wedge \tilde{y}$ and $\tilde{x} \wedge \tilde{z}$.

Hence $\tilde{x} \wedge (\tilde{y} \vee \tilde{z}) \geq (\tilde{x} \wedge \tilde{y}) \vee (\tilde{x} \wedge \tilde{z})$.

(2) follows by duality principle.

Theorem 3.17. The soft elements of any gs-lattice (F, A) over L with the non empty parameter set A satisfies the soft modular identity: $\tilde{x} \leq \tilde{z} \Rightarrow \tilde{x} \vee (\tilde{y} \wedge \tilde{z}) \leq (\tilde{x} \vee \tilde{y}) \wedge \tilde{z}$.

Proof. We have $\tilde{x} \leq \tilde{x} \vee \tilde{y}$ and let $\tilde{x} \leq \tilde{z}$.

$$\text{Hence } \tilde{x} \leq (\tilde{x} \vee \tilde{y}) \wedge \tilde{z} \text{ --- (c)}$$

Also $\tilde{y} \wedge \tilde{z} \leq \tilde{y} \leq \tilde{x} \vee \tilde{y}$ and $\tilde{y} \wedge \tilde{z} \leq \tilde{z}$.

$$\therefore \tilde{y} \wedge \tilde{z} \leq (\tilde{x} \vee \tilde{y}) \wedge \tilde{z} \text{ --- (d)}$$

Hence from (c) and (d), we get $\tilde{x} \vee (\tilde{y} \wedge \tilde{z}) \leq (\tilde{x} \vee \tilde{y}) \wedge \tilde{z}$.

Definition 3.18. A gs-lattice is said to be soft modular if it satisfy the soft modular identity.

4. SOFT VALUATION ON A GS LATTICE

Here we define soft valuation on a gs lattice. We verify that any gs lattice with positive soft valuation is a modular gs lattice.

Definition 4.1. [3] Let (F, A) and (G, B) be two non null soft sets over two non empty universal sets X and Y in $S(\tilde{X})$ and $S(\tilde{Y})$ respectively.

Then a soft function from (F, A) to (G, B) is a function $f : SE(F, A) \rightarrow SE(G, B)$.

Definition 4.2. A soft function $f : (F, A)$ to (G, B) is called a soft real valued function if $(G, B) \subseteq \tilde{R}$. i.e, for each soft element \tilde{x} of (F, A) , there exists a soft real number \tilde{r} such that $f(\tilde{x}) = \tilde{r}$, where $f(\tilde{x}) : B \rightarrow R$ is a soft element of (\tilde{R}, B) .

Example 4.3. Let (\tilde{X}, A) be any absolute soft set over a non empty universal set X with a nonempty parameter set A and \tilde{R} be the absolute soft set over the set of real numbers R with the nonempty parameter set B . Then define

$f : SE(\tilde{X}) \rightarrow \tilde{R}$ by $f(\tilde{x}) = \tilde{0}$, where $\tilde{0}$ is a soft element of (\tilde{R}, B) such that $\tilde{0}(b) = 0, \forall b \in B$.

Then f is a soft real valued function.

Definition 4.4. Let (F, A) be any gs-lattice. A soft valuation on (F, A) is a soft real valued function $V : SE(F, A) \rightarrow R(A)$ on $SE(F, A)$, which satisfies $V[\tilde{x}] + V[\tilde{y}] = V[\tilde{x} \vee \tilde{y}] + V[\tilde{x} \wedge \tilde{y}]$.

Definition 4.5. A soft valuation V on a gs lattice is isotone iff $\tilde{x} \succeq \tilde{y} \Rightarrow V[\tilde{x}] \succeq V[\tilde{y}]$ and positive iff $\tilde{x} \succ \tilde{y} \Rightarrow V[\tilde{x}] \succ V[\tilde{y}]$.

Theorem 4.6. Any valuation on a lattice L generate a soft valuation on \tilde{L} .

Proof. Let (L, \vee_L, \wedge_L) be a lattice and V_L be a valuation on L .

Define $V : SE(\tilde{L}) \rightarrow SE(\tilde{R})$ by

$$(V[\tilde{x}])(\lambda) = V_L(\tilde{x}(\lambda)), \forall \lambda \in A. \dots\dots(i)$$

Let $\tilde{x}, \tilde{y} \in \tilde{L}$ and $\lambda \in A$.

Let \vee and \wedge be the soft join and soft meet on \tilde{L} generated by \vee_L and \wedge_L on L respectively.

Then

$$(\tilde{x} \vee \tilde{y})(\lambda) = \tilde{x}(\lambda) \vee_L \tilde{y}(\lambda) \text{ and } (\tilde{x} \wedge \tilde{y})(\lambda) = \tilde{x}(\lambda) \wedge_L \tilde{y}(\lambda), \forall \lambda \in A. \dots\dots\dots(ii)$$

Then

$$\begin{aligned} (V[\tilde{x}] + V[\tilde{y}])(\lambda) &= V[\tilde{x}](\lambda) + V[\tilde{y}](\lambda) \\ &= V_L(\tilde{x}(\lambda)) + V_L(\tilde{y}(\lambda)) \text{ [By (i)]} \\ &= V_L[\tilde{x}(\lambda) \vee \tilde{y}(\lambda)] + V_L[\tilde{x}(\lambda) \wedge \tilde{y}(\lambda)]. \text{ [By definition 2.13]} \\ &= V_L[(\tilde{x} \vee \tilde{y})(\lambda)] + V_L[(\tilde{x} \wedge \tilde{y})(\lambda)]. \text{ [By (ii)]} \\ &= V[(\tilde{x} \vee \tilde{y})(\lambda)] + V[\tilde{x} \wedge \tilde{y}](\lambda). \text{ [By (i)]} \\ &= (V[\tilde{x} \vee \tilde{y}]) + (V[\tilde{x} \wedge \tilde{y}])(\lambda). \end{aligned}$$

i.e. $V[\tilde{x}] + V[\tilde{y}] = V[\tilde{x} \vee \tilde{y}] + (V[\tilde{x} \wedge \tilde{y}]).$

Hence V is a soft valuation on \tilde{L} .

Example 4.7. Let L be a modular lattice of finite length and \tilde{L} is the absolute gs lattice with a nonempty parameter set A . Let h_L be the height function on L and h be the soft valuation generated by h_L on \tilde{L} and is given by

$$h[\tilde{x}](\lambda) = h_L(\tilde{x}(\lambda)) \quad \forall \lambda \in A. \text{ This soft function is called soft height function and satisfy the relation } h[\tilde{x}] + h[\tilde{y}] = h[\tilde{x} \vee \tilde{y}] + h[\tilde{x} \wedge \tilde{y}].$$

Theorem 4.8. *If V_F and V_G be two positive soft valuations on gs lattices (F,A) and (G,B) respectively. Then the sum $V_F + V_G$ defines a positive soft valuation on the direct product $(F,A)(G,B)$.*

Proof. *Let V_F and V_G be two soft valuations on gs lattices (F,A) and (G,B) respectively.*

Define the sum $V_F + V_G$ by $(V_F + V_G)[(\tilde{x}_1, \tilde{y}_1)] = V_F[\tilde{x}_1] + V_G[\tilde{y}_1], \forall \tilde{x}_1 \in (F,A)$ and $\tilde{y}_1 \in (G,B)$.

Since V_F and V_G are two positive soft valuations, we have

$$V_F[\tilde{x}_1] + V_F[\tilde{x}_2] = V_F[\tilde{x}_1 \vee \tilde{x}_2] + V_F[\tilde{x}_1 \wedge \tilde{x}_2] \text{ and } V_G[\tilde{y}_1] + V_G[\tilde{y}_2] = V_G[\tilde{y}_1 \vee \tilde{y}_2] + V_G[\tilde{y}_1 \wedge \tilde{y}_2].$$

$$\begin{aligned} \therefore (V_F + V_G)[(\tilde{x}_1, \tilde{y}_1)] + (V_F + V_G)[(\tilde{x}_2, \tilde{y}_2)] &= V_F[\tilde{x}_1] + V_G[\tilde{y}_1] + V_F[\tilde{x}_2] + V_G[\tilde{y}_2] \\ &= V_F[\tilde{x}_1] + V_F[\tilde{x}_2] + V_G[\tilde{y}_1] + V_G[\tilde{y}_2] \\ &= V_F[\tilde{x}_1 \vee \tilde{x}_2] + V_F[\tilde{x}_1 \wedge \tilde{x}_2] + V_G[\tilde{y}_1 \vee \tilde{y}_2] + V_G[\tilde{y}_1 \wedge \tilde{y}_2] \\ &= V_F[\tilde{x}_1 \vee \tilde{x}_2] + V_G[\tilde{y}_1 \vee \tilde{y}_2] + V_F[\tilde{x}_1 \wedge \tilde{x}_2] + V_G[\tilde{y}_1 \wedge \tilde{y}_2] \\ &= (V_F + V_G)[(\tilde{x}_1 \vee \tilde{x}_2, \tilde{y}_1 \vee \tilde{y}_2)] + (V_F + V_G)[(\tilde{x}_1 \wedge \tilde{x}_2, \tilde{y}_1 \wedge \tilde{y}_2)] \\ &= (V_F + V_G)[(\tilde{x}_1, \tilde{y}_1) \vee (\tilde{x}_2, \tilde{y}_2)] + (V_F + V_G)[(\tilde{x}_1, \tilde{y}_1) \wedge (\tilde{x}_2, \tilde{y}_2)] \end{aligned}$$

Hence $V_F + V_G$ is a soft valuation on $(F, A)(G, B)$.

$$\begin{aligned} \text{Also } (\tilde{x}_1, \tilde{y}_1) > (\tilde{x}_2, \tilde{y}_2) &\Rightarrow \tilde{x}_1 > \tilde{x}_2 \text{ and } \tilde{y}_1 > \tilde{y}_2 \\ &\Rightarrow V_F[\tilde{x}_1] > V_F[\tilde{x}_2] \text{ and } V_G[\tilde{y}_1] > V_G[\tilde{y}_2] \\ &\Rightarrow V_F[\tilde{x}_1] + V_G[\tilde{y}_1] > V_F[\tilde{x}_2] + V_G[\tilde{y}_2] \\ &\Rightarrow (V_F + V_G)[(\tilde{x}_1, \tilde{y}_1)] > (V_F + V_G)[(\tilde{x}_2, \tilde{y}_2)] \end{aligned}$$

Hence $V_F + V_G$ is positive.

Thus $V_F + V_G$ defines a positive soft valuation on the direct product.

Theorem 4.9. Any gs lattice with a positive soft valuation is soft modular.

Proof. Let (F, A) be a gs lattice and $\tilde{x}, \tilde{z} \in (F, A)$ with $\tilde{x} \leq \tilde{z}$.

We have $V[\tilde{x}] + V[\tilde{y}] = V[\tilde{x} \vee \tilde{y}] + V[\tilde{x} \wedge \tilde{y}]$.

$$\begin{aligned} \therefore V[\tilde{x} \vee (\tilde{y} \wedge \tilde{z})] - V[(\tilde{x} \vee \tilde{y}) \wedge \tilde{z}] &= (V[\tilde{x}] + V[\tilde{y} \wedge \tilde{z}] - V[\tilde{x} \wedge \tilde{y} \wedge \tilde{z}]) - (V[\tilde{x} \vee \tilde{y}] + V[\tilde{z}] - V[\tilde{x} \vee \tilde{y} \vee \tilde{z}]) \\ &= (V[\tilde{x}] + V[\tilde{y} \wedge \tilde{z}] - V[\tilde{x} \wedge \tilde{y}]) - (V[\tilde{x} \vee \tilde{y}] + V[\tilde{z}] - V[\tilde{y} \vee \tilde{z}]) \quad (\because \tilde{x} \leq \tilde{z}) \\ &= V[\tilde{x}] - V[\tilde{x} \wedge \tilde{y}] - V[\tilde{x} \vee \tilde{y}] + V[\tilde{y} \wedge \tilde{z}] + V[\tilde{y} \vee \tilde{z}] - V[\tilde{z}] \\ &= -V[\tilde{y}] + V[\tilde{y}] \\ &= \tilde{0}. \end{aligned}$$

Also we have $\tilde{x} \vee (\tilde{y} \wedge \tilde{z}) \leq (\tilde{x} \vee \tilde{y}) \wedge \tilde{z}$. [By theorem 3.17]

Hence $\tilde{x} \vee (\tilde{y} \wedge \tilde{z}) = (\tilde{x} \vee \tilde{y}) \wedge \tilde{z}$. [Since V is positive $\tilde{x} \vee (\tilde{y} \wedge \tilde{z}) < (\tilde{x} \vee \tilde{y}) \wedge \tilde{z} \Rightarrow V[\tilde{x} \vee (\tilde{y} \wedge \tilde{z})] < V[(\tilde{x} \vee \tilde{y}) \wedge \tilde{z}]$.]

Remark. In the theorem 4.8, $V_F + V_G$ defines a positive soft valuation on the direct product $(F,A)(G,B)$. Hence by theorem 4.9, $(F,A)(G,B)$ is soft modular.

5. SOFT METRIC LATTICE

In this section we define soft distance function. We express soft distance function in terms of soft valuation and prove some of its properties. Here we have the main result that any gs lattice with isotone soft valuation is a soft pseudo metric space and any gs lattice with positive soft valuation is a soft metric space. Also, we introduce soft pseudo metric lattice and soft metric lattice.

Definition 5.1. Let (F,A) be any non null soft set over a nonempty universal set X with a non empty parameter set A . Then a soft distance function on (F,A) is a soft function $d : SE(F,A) \times SE(F,A) \rightarrow R(A)^*$, where for each pair of soft elements $\tilde{x}, \tilde{y} \in (F,A)$, $d(\tilde{x}, \tilde{y})$ is a function from A to R^* .

Note. Here R^* and $R(A)^*$ are respectively the set of all non negative real numbers and the set of all soft non negative real numbers with parameter set A . Hence $d(\tilde{x}, \tilde{y})(\lambda)$ is a non negative real number and $d(\tilde{x}, \tilde{y})$ is a soft non negative real number.

Example 5.2. Let (\tilde{R}, A) be the absolute soft set over \mathbb{R} , the set of all real numbers. Then the function $d : SE(\tilde{R}) \times SE(\tilde{R}) \rightarrow R(A)^*$, where $d(\tilde{x}, \tilde{y}) : A \rightarrow R^*$ defined by $d(\tilde{x}, \tilde{y})(\lambda) = |\tilde{x}(\lambda) - \tilde{y}(\lambda)| \forall \lambda \in A, \forall \tilde{x}, \tilde{y} \in SE(\tilde{R})$. Here $d(\tilde{x}, \tilde{y})$ is a soft distance function.

Theorem 5.3. Let (F,A) be a gs-lattice with an isotone soft valuation V , then soft distance function $d : SE(F,A) \times SE(F,A) \rightarrow R(A)^*$ defined by $d(\tilde{x}, \tilde{y}) = V[\tilde{x} \vee \tilde{y}] - V[\tilde{x} \wedge \tilde{y}]$ satisfy the following axioms.

$$M_1 : d(\tilde{x}, \tilde{x}) = \tilde{0} \quad \forall \tilde{x} \in (F,A)$$

$$M_2 : d(\tilde{x}, \tilde{y}) = d(\tilde{y}, \tilde{x}) \quad \forall \tilde{x}, \tilde{y} \in (F,A)$$

$$M_3 : d(\tilde{x}, \tilde{y}) \geq \tilde{0} \quad \forall \tilde{x}, \tilde{y} \in (F, A)$$

$$M_4 : d(\tilde{a} \vee \tilde{x}, \tilde{a} \vee \tilde{y}) + d(\tilde{a} \wedge \tilde{x}, \tilde{a} \wedge \tilde{y}) \leq d(\tilde{x}, \tilde{y}) \quad \forall \tilde{x}, \tilde{y}, \tilde{a} \in (F, A)$$

$$M_5 : d(\tilde{x}, \tilde{y}) + d(\tilde{y}, \tilde{z}) \geq d(\tilde{x}, \tilde{z}) \quad \forall \tilde{x}, \tilde{y}, \tilde{z} \in (F, A).$$

Proof. Let (F, A) be a gs-lattice with an isotone soft valuation V .

Define $d(\tilde{x}, \tilde{y}) = V[\tilde{x} \vee \tilde{y}] - V[\tilde{x} \wedge \tilde{y}] \quad \forall \tilde{x}, \tilde{y} \in (F, A)$

M_1 :By definition, we have

$$\begin{aligned} d(\tilde{x}, \tilde{x}) &= V[\tilde{x} \vee \tilde{x}] - V[\tilde{x} \wedge \tilde{x}] \\ &= V[\tilde{x}] - V[\tilde{x}] \\ &= \tilde{0} \end{aligned}$$

M_2 :By definition, we have

$$\begin{aligned} d(\tilde{x}, \tilde{y}) &= V[\tilde{x} \vee \tilde{y}] - V[\tilde{x} \wedge \tilde{y}] \\ &= V[\tilde{y} \vee \tilde{x}] - V[\tilde{y} \wedge \tilde{x}] \\ &= d(\tilde{y}, \tilde{x}) \end{aligned}$$

M_3 :Since V is isotone and $\tilde{x} \vee \tilde{y} \geq \tilde{x} \wedge \tilde{y}$,

$$\begin{aligned} \text{we have } V[\tilde{x} \vee \tilde{y}] &\geq V[\tilde{x} \wedge \tilde{y}] \\ \text{i.e., } V[\tilde{x} \vee \tilde{y}] - V[\tilde{x} \wedge \tilde{y}] &\geq \tilde{0} \\ \text{i.e., } d(\tilde{x}, \tilde{y}) &\geq \tilde{0} \end{aligned}$$

M_4 :By definition, we have

$$\begin{aligned}
& d(\tilde{a} \vee \tilde{x}, \tilde{a} \vee \tilde{y}) + d(\tilde{a} \wedge \tilde{x}, \tilde{a} \wedge \tilde{y}) \\
&= V[(\tilde{a} \vee \tilde{x}) \vee (\tilde{a} \vee \tilde{y}) - V[(\tilde{a} \vee \tilde{x}) \wedge (\tilde{a} \vee \tilde{y})] + V[(\tilde{a} \wedge \tilde{x}) \vee (\tilde{a} \wedge \tilde{y})] - V[(\tilde{a} \wedge \tilde{x}) \wedge (\tilde{a} \wedge \tilde{y})] \\
&\stackrel{\geq}{=} V[\tilde{a} \vee \tilde{x} \vee \tilde{y}] - V[\tilde{a} \vee (\tilde{x} \wedge \tilde{y})] + V[(\tilde{a} \wedge (\tilde{x} \vee \tilde{y}))] - V[\tilde{a} \wedge \tilde{x} \wedge \tilde{y}] \\
&\text{(Since } V \text{ is isotone and } (\tilde{x} \wedge \tilde{y}) \vee (\tilde{x} \wedge \tilde{z}) \stackrel{\geq}{=} \tilde{x} \wedge (\tilde{y} \vee \tilde{z}) \text{ and } (\tilde{x} \vee \tilde{y}) \wedge (\tilde{x} \vee \tilde{z}) \stackrel{\leq}{=} \tilde{x} \vee (\tilde{y} \wedge \tilde{z})) \\
&\stackrel{\geq}{=} V[\tilde{a} \vee \tilde{x} \vee \tilde{y}] + V[(\tilde{a} \wedge (\tilde{x} \vee \tilde{y}))] - V[\tilde{a} \vee (\tilde{x} \wedge \tilde{y})] - V[\tilde{a} \wedge \tilde{x} \wedge \tilde{y}] \\
&= V[\tilde{a}] + V[\tilde{x} \vee \tilde{y}] - (V[\tilde{a}] + V[\tilde{x} \wedge \tilde{y}]) \text{ [since } V \text{ is a soft valuation]} \\
&= V[\tilde{x} \vee \tilde{y}] - V[\tilde{x} \wedge \tilde{y}] \\
&= d(\tilde{x}, \tilde{y})
\end{aligned}$$

$$\begin{aligned}
M_5 : \text{ We have } d(\tilde{x} \vee \tilde{y}, \tilde{y}) + d(\tilde{y}, \tilde{x} \wedge \tilde{y}) &= V[(\tilde{x} \vee \tilde{y}) \vee \tilde{y}] - V[(\tilde{x} \vee \tilde{y}) \wedge \tilde{y}] + V[\tilde{y} \vee (\tilde{x} \wedge \tilde{y})] - V[\tilde{y} \wedge (\tilde{x} \wedge \tilde{y})] \\
&= V[\tilde{x} \vee \tilde{y}] - V[(\tilde{x} \vee \tilde{y}) \wedge \tilde{y}] + V[\tilde{y} \vee (\tilde{x} \wedge \tilde{y})] - V[\tilde{x} \wedge \tilde{y}] \\
&= V[\tilde{x} \vee \tilde{y}] - V[\tilde{x} \wedge \tilde{y}] + V[\tilde{y} \vee (\tilde{x} \wedge \tilde{y})] - V[(\tilde{x} \vee \tilde{y}) \wedge \tilde{y}] \\
&= V[\tilde{x} \vee \tilde{y}] - V[\tilde{x} \wedge \tilde{y}] + V[\tilde{y}] - V[\tilde{y}] \text{ [By absorption]} \\
&= V[\tilde{x} \vee \tilde{y}] - V[\tilde{x} \wedge \tilde{y}] \\
&= d(\tilde{x}, \tilde{y}).
\end{aligned}$$

Similarly, we get $d(\tilde{y}, \tilde{z}) = d(\tilde{y} \vee \tilde{z}, \tilde{y}) + d(\tilde{y}, \tilde{y} \wedge \tilde{z})$

$$\begin{aligned}
\therefore d(\tilde{x}, \tilde{y}) + d(\tilde{y}, \tilde{z}) &= d(\tilde{x} \vee \tilde{y}, \tilde{y}) + d(\tilde{y}, \tilde{x} \wedge \tilde{y}) + d(\tilde{y} \vee \tilde{z}, \tilde{y}) + d(\tilde{y}, \tilde{y} \wedge \tilde{z}) \\
&\stackrel{\geq}{=} d(\tilde{x} \vee \tilde{y} \vee \tilde{z}, \tilde{y} \vee \tilde{z}) + d(\tilde{y} \vee \tilde{z}, \tilde{y}) + d(\tilde{y}, \tilde{x} \wedge \tilde{y}) + d(\tilde{x} \wedge \tilde{y}, \tilde{x} \wedge \tilde{y} \wedge \tilde{z})
\end{aligned}$$

(\because by M_4 , we have $d(\tilde{x} \vee \tilde{y}, \tilde{y}) \succeq d(\tilde{x} \vee \tilde{y} \vee \tilde{z}, \tilde{y} \vee \tilde{z})$ and $d(\tilde{y}, \tilde{y} \wedge \tilde{z}) \succeq d(\tilde{x} \wedge \tilde{y}, \tilde{x} \wedge \tilde{y} \wedge \tilde{z})$)

$$\begin{aligned}
 \therefore d(\tilde{x}, \tilde{y}) + d(\tilde{y}, \tilde{z}) &= V[\tilde{x} \vee \tilde{y} \vee \tilde{z}] - V[(\tilde{x} \vee \tilde{y} \vee \tilde{z}) \wedge (\tilde{y} \vee \tilde{z})] + V[\tilde{y} \vee \tilde{z}] - V[(\tilde{y} \vee \tilde{z}) \wedge \tilde{y}] \\
 &\quad + V[\tilde{y} \vee (\tilde{x} \wedge \tilde{y})] - V[\tilde{y} \wedge \tilde{x}] + V[(\tilde{x} \wedge \tilde{y}) \vee (\tilde{x} \wedge \tilde{y} \wedge \tilde{z})] - V[\tilde{x} \wedge \tilde{y} \wedge \tilde{z}] \\
 &= V[\tilde{x} \vee \tilde{y} \vee \tilde{z}] - V[\tilde{y} \vee \tilde{z}] + V[\tilde{y} \vee \tilde{z}] - V[\tilde{y}] + V[\tilde{y}] - V[\tilde{y} \wedge \tilde{x}] + V[\tilde{x} \wedge \tilde{y}] \\
 &= V[\tilde{x} \vee \tilde{y} \vee \tilde{z}] - V[\tilde{x} \wedge \tilde{y} \wedge \tilde{z}] \\
 &= d(\tilde{x} \vee \tilde{y} \vee \tilde{z}, \tilde{x} \wedge \tilde{y} \wedge \tilde{z}) \\
 &\succeq d(\tilde{x} \vee \tilde{y}, \tilde{x} \wedge \tilde{y}) \\
 &= V[(\tilde{x} \vee \tilde{y}) \vee (\tilde{x} \wedge \tilde{y})] - V[(\tilde{x} \vee \tilde{y}) \wedge (\tilde{x} \wedge \tilde{y})] \\
 &= V[(\tilde{x} \vee \tilde{y})] - V[\tilde{x} \wedge \tilde{y}] \\
 &= d(\tilde{x}, \tilde{y})
 \end{aligned}$$

Corollary 5.4. A gs-lattice (F, A) with an isotone soft valuation is a soft pseudo metric (soft quasi metric) space.

Proof. From M_1, M_2, M_3 and M_5 in the above theorem we get the result.

Definition 5.5. A gs-lattice with an isotone soft valuation is said to be a soft pseudo metric lattice.

Theorem 5.6. Let (F, A) be a gs-lattice with the mapping $d : SE(F, A) \times SE(F, A) \rightarrow R(A)^*$ defined by $d(\tilde{x}, \tilde{y}) = V[\tilde{x} \vee \tilde{y}] - V[\tilde{x} \wedge \tilde{y}] \quad \forall \tilde{x}, \tilde{y} \in (F, A)$, where V is an isotone soft valuation on (F, A) . Then in (F, A) , the following are equivalent.

(i) $d(\tilde{x}, \tilde{y}) = \tilde{0} \Rightarrow \tilde{x} = \tilde{y}$.

(ii) $\tilde{x} \vee \tilde{y} > \tilde{x} \wedge \tilde{y} \Rightarrow V[\tilde{x} \vee \tilde{y}] > V[\tilde{x} \wedge \tilde{y}]$.

(iii) V is positive.

Proof. We prove the theorem by proving (i) \Rightarrow (ii), (ii) \Rightarrow (i), (ii) \Rightarrow (iii) and (iii) \Rightarrow (ii).

First we prove that (i) \Rightarrow (ii)

Assume (i). i.e, $d(\tilde{x}, \tilde{y}) = \tilde{0} \Rightarrow \tilde{x} = \tilde{y}$.

Let $\tilde{x} \vee \tilde{y} > \tilde{x} \wedge \tilde{y}$.

By theorem 5.3, we have $d(\tilde{x}, \tilde{y}) \geq \tilde{0}$.

$$\begin{aligned} \text{Then } d(\tilde{x}, \tilde{y}) \geq \tilde{0} &\Rightarrow d(\tilde{x}, \tilde{y}) = V[\tilde{x} \vee \tilde{y}] - V[\tilde{x} \wedge \tilde{y}] \geq \tilde{0} \\ &\Rightarrow V[\tilde{x} \vee \tilde{y}] \geq V[\tilde{x} \wedge \tilde{y}] \\ &\Rightarrow V[\tilde{x} \vee \tilde{y}] > V[\tilde{x} \wedge \tilde{y}] \text{ or } V[\tilde{x} \vee \tilde{y}] = V[\tilde{x} \wedge \tilde{y}] \end{aligned}$$

If $V[\tilde{x} \vee \tilde{y}] = V[\tilde{x} \wedge \tilde{y}]$ then $d(\tilde{x}, \tilde{y}) = V[\tilde{x} \vee \tilde{y}] - V[\tilde{x} \wedge \tilde{y}] = \tilde{0}$.

Then by assumption $\tilde{x} = \tilde{y}$ and so $\tilde{x} \vee \tilde{y} = \tilde{x} \wedge \tilde{y}$, a contradiction.

Hence $V[\tilde{x} \vee \tilde{y}] > V[\tilde{x} \wedge \tilde{y}]$.

Next we prove that (ii) \Rightarrow (i)

Assume (ii). i.e., $\tilde{x} \vee \tilde{y} > \tilde{x} \wedge \tilde{y} \Rightarrow V[\tilde{x} \vee \tilde{y}] > V[\tilde{x} \wedge \tilde{y}]$.

Let $d(\tilde{x}, \tilde{y}) = \tilde{0}$.

i.e., $d(\tilde{x}, \tilde{y}) = V[\tilde{x} \vee \tilde{y}] - V[\tilde{x} \wedge \tilde{y}] = \tilde{0}$.

i.e., $V[\tilde{x} \vee \tilde{y}] = V[\tilde{x} \wedge \tilde{y}]$.

Then to prove that $\tilde{x} = \tilde{y}$.

If possible let $\tilde{x} \neq \tilde{y}$.

Then since $\tilde{x} \vee \tilde{y} > \tilde{x} \wedge \tilde{y}$, $V[\tilde{x} \vee \tilde{y}] > V[\tilde{x} \wedge \tilde{y}]$ and so $d(\tilde{x}, \tilde{y}) > \tilde{0}$, a contradiction.

Hence $\tilde{x} = \tilde{y}$.

Now we prove that (ii) \Rightarrow (iii)

Assume (ii). i.e., $\tilde{x} \vee \tilde{y} > \tilde{x} \wedge \tilde{y} \Rightarrow V[\tilde{x} \vee \tilde{y}] > V[\tilde{x} \wedge \tilde{y}]$.

To prove that V is positive.

Let $\tilde{x} > \tilde{y}$.

To prove that $V[\tilde{x}] > V[\tilde{y}]$.

Since $\tilde{x} > \tilde{y}$, $\tilde{x} \vee \tilde{y} = \tilde{x}$ and $\tilde{x} \wedge \tilde{y} = \tilde{y}$.

Then from (ii), we get $\tilde{x} > \tilde{y} \Rightarrow V[\tilde{x}] > V[\tilde{y}]$.

Next we prove that (iii) \Rightarrow (ii)

Assume (iii). i.e, let V be a positive soft valuation.

Then $\tilde{x} > \tilde{y} \Rightarrow V[\tilde{x}] > V[\tilde{y}], \forall \tilde{x}, \tilde{y} \in (F, A)$.

Hence in (F, A) , we have $\tilde{x} \vee \tilde{y} > \tilde{x} \wedge \tilde{y} \Rightarrow V[\tilde{x} \vee \tilde{y}] > V[\tilde{x} \wedge \tilde{y}]$.

Theorem 5.7. A gs-lattice with a positive soft valuation is a soft metric space.

Proof. Let (F, A) be a gs-lattice with a positive soft valuation V .

Then by definition 5.5, it is a soft pseudo metric lattice if V is isotone. Also, a soft pseudo metric lattice is a soft metric lattice if $d(\tilde{x}, \tilde{y}) = \tilde{0} \Rightarrow \tilde{x} = \tilde{y}$.

Hence we have enough to prove that (i) V is isotone and (ii) $d(\tilde{x}, \tilde{y}) = \tilde{0} \Rightarrow \tilde{x} = \tilde{y}$.

(i) let $\tilde{x} \geq \tilde{y} \Rightarrow \tilde{x} < \tilde{y}$ or $\tilde{x} = \tilde{y}$

Since V is positive, $\tilde{x} < \tilde{y} \Rightarrow V[\tilde{x}] < V[\tilde{y}]$.

Also if $\tilde{x} = \tilde{y}$, then $d(\tilde{x}, \tilde{y}) = \tilde{0}$. [By definition of V]

Hence $\tilde{x} \geq \tilde{y} \Rightarrow V[\tilde{x}] \geq V[\tilde{y}]$ and so V is isotone.

(ii) Since V is positive, $d(\tilde{x}, \tilde{y}) = \tilde{0} \Rightarrow \tilde{x} = \tilde{y}$. [from theorem 5.6]

Definition 5.8. A gs-lattice with a positive soft valuation is called a soft metric lattice.

Example 5.9. Let (F, A) be a gs-lattice over R^* , the set of all nonnegative real numbers with the soft partial ordering defined by $\tilde{r} \leq \tilde{s}$ if and only if $\tilde{r}(\lambda) \leq \tilde{s}(\lambda), \forall \lambda \in A$. Here soft meet and soft join on (F, A) are generated by meet and join on R^* given by $a \wedge b = \max(a, b)$ and $a \vee b = \min(a, b)$ respectively. Let v be the valuation on R^* defined by $v[a] = |a|, \forall a \in R^*$. Define the function $V : (F, A) \rightarrow R^*$ by $V[\tilde{x}](\lambda) = v[\tilde{x}(\lambda)], \forall \tilde{x} \in (F, A)$ and $\forall \lambda \in A$. Then V is a positive soft valuation on (F, A) and hence a soft metric lattice by theorem 5.7. Here the soft metric on (F, A) is given by $d(\tilde{x}, \tilde{y}) = V[\tilde{x} \vee \tilde{y}] - V[\tilde{x} \wedge \tilde{y}] \quad \forall \tilde{x}, \tilde{y} \in (F, A)$.

Example 5.10. The direct product in theorem 4.8 is a soft metric lattice. [By theorem 4.8, it is a positive soft valuation. Hence by theorem 5.8, it is a soft metric lattice.]

Theorem 5.11. A soft metric lattice is soft modular.

Proof. Let (F, A) be any soft metric lattice.

\therefore by definition 5.8, it is a gs-lattice with positive soft valuation.

Hence by theorem 4.9, it is soft modular.

6. CONCLUSION

In this paper, we introduce the concept of gs-lattice in terms of soft elements and define a soft valuation on it. Then we introduce a soft pseudo metric on the gs-lattice by means of soft valuation. Also we discuss the circumstances under which it becomes a soft metric space.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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