Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 2, 403-411
https://doi.org/10.28919/jmcs/4425
ISSN: 1927-5307

SOME RESULTS ON SET COLORINGS OF DIRECTED TREES

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Abstract. A set coloring of the digraph D is an assignment (function) of distinct subsets of a finite set X of colors to the vertices of the digraph, where the color of an arc, say (u,v) is obtained by applying the set difference from the set assigned to the vertex v to the set assigned to the vertex u which arc also distinct. A set coloring is called a strong set coloring if sets on the vertices and arcs are distinct and together form the set of all non empty subsets of X. A set coloring is called a proper set coloring if all the non empty subsets of X are obtained on the arcs of D. A digraph is called a strongly set colorable (properly set colorable) if it admits a strong set coloring (proper set coloring).

In this paper we find some classes of directed trees which admit a strong set coloring and construction of strongly set colorable directed tree $\overrightarrow{T_n'}$.

Keywords: set coloring; strong (proper)set coloring; digraphs.

2010 AMS Subject Classification: 05C20, 05C78.

1. Introduction

In this paper, we consider only finite simple digraphs. For all notations we follow Harary [1]. The notion of set coloring of a graph has been introduced by Hegde [2] in 2009. Further Hegde and Sumana [4] determined the set coloring number of certain graphs.

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Received December 10, 2019

The concept of set colorings of graph was then extended to digraphs by Hegde and Castelino [3].

Definition 1.1. Given a digraph $D=(V,E)$ with a non empty set X of n colors and m arcs, a function $f:V\to 2^X$ can be defined as the assignment of the colors $f(v)$ to each of the vertices $v\in V$ and given such a function f on the vertex set V,we define $f^*:E\to 2^X$ which assigns colors to the arcs $e=uv\in E$,$f^*(e)=f(v)-f(u)$.

A digraph D is said to be a set colorable if both f and f^* are injective functions. A digraph D is said to be properly set colorable if it is set colorable with $f^*(E)=2^X\setminus\emptyset$ and D is strongly set colorable if $f(V)\cup f^*(E)=2^X\setminus\emptyset$ and $f(V)\bigcap f^*(E)=\emptyset$. They also determined the necessary condition for strong(proper) set colorings of digraphs.

Definition 1.2. Set coloring number [3],$\sigma(D)$ of a digraph D is the least cardinality of a set X with respect to which D has a set coloring. Further, if $f:V\to 2^X$ is a set coloring of D with $|X|=\sigma(D)$ we call f an optimal set coloring of D.

Theorem 1.3. [3] For any digraph D,$\lceil log_2(q+1)\rceil\leq\sigma(D)\leq p-1$,where $\lceil x\rceil$ denotes the least integer not less than the real x, and bounds are best possible.

In this paper, we find the set coloring number of unipath, some classes of digraphs which admit a strong(proper) set coloring and construction of a strongly set colorable directed tree.

2. Set Coloring Number of a Digraph

In this section we find set coloring number of unipath.

Definition 2.1. A oriented path is called unipath if $id(v)=ed(v)=1$ for every vertex v except the first and last of the oriented path.

Theorem 2.2. Given any positive integer $n\geq 2$,$\sigma(\overrightarrow{P_{2^n}})>n$.

Proof. Let the vertices of $\overrightarrow{P_{2^n}}$ be denoted by $v_1,v_2,...,v_{2^n}$ such that $f^*(v_i,v_{i+1})=f(v_i)-f(v_{i+1})$, $\forall(v_i,v_{i+1})\in E(\overrightarrow{P_{2^n}})$. Let us assume that there exist a set coloring (f,f^*) of $\overrightarrow{P_{2^n}}$ with respect to a set X of $|X|=n$ and both f and f^* are injective functions. That is sum of the number vertices and the number edges greater than 2^n which contradicts the fact that $|X|=n$. Therefore $\sigma(\overrightarrow{P_{2^n}})>n$. □

3. Strongly (Properly) Set Colorable Directed Trees

In this section we present some results on strong(proper) set colorings of some classes of directed trees.

Definition 3.1. A n-centipede $\overrightarrow{Cn}$ is a directed tree obtained by joining each vertex of the unipath to a pendent vertex whose in degree is zero.

Theorem 3.2. A directed centipede tree $\overrightarrow{Cn}$ is strongly set colorable if and only if $n = 2^{k-1}$, where $k = 2,3,4$.

Proof. A directed centipede $\overrightarrow{Cn}$ has n vertices and $n-1$ arcs. Let $\overrightarrow{Cn}$ be strongly set colorable directed tree with respect to a set X having k colors. Then$|V(\overrightarrow{Cn})| + |E(\overrightarrow{Cn})| = 2^k - 1 \Rightarrow n + (n-1) = 2^k - 1 \Rightarrow n = 2^{k-1}$.

Conversely, let $\overrightarrow{Cn}$ be a directed tree such that $n = 2^k - 1$. Let $X = \{1,2,3,...,k\}$. Also let $X_1 = \{1,2,3,...,k\}$,the full set of X and X_2 is a subset containing $k-1$ elements of X which doesn't contain the element $a, a \in X$. Then assign the set X_1 to the sink of $\overrightarrow{Cn}$, that is vertex v of $\overrightarrow{Cn}$, where $od(v) = 0$. Also assign the set X_2 to the vertex say u adjacent to v and $id(u) = 0$ and the remaining subsets of X to the $n-2$ vertices of $\overrightarrow{Cn}$. Then one can observe that the elements on the arcs are also subsets of X and together form the set of all nonempty subsets of X. Hence $\overrightarrow{Cn}$ is strongly set colorable. □

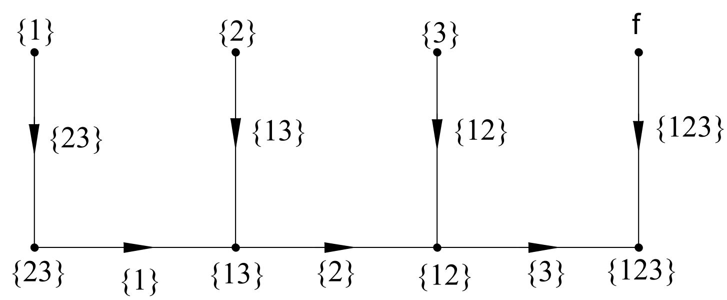
Remark 3.3. A directed centipede tree $\overrightarrow{Cn}$ is not strongly set colorable if and only if $n = 2^{k-1}$, where $k > 4$.

Theorem 3.4. A directed centipede tree $\overrightarrow{Cn}$ is properly set colorable if and only if $n = 2^k$, where $k = 2,3,4$.

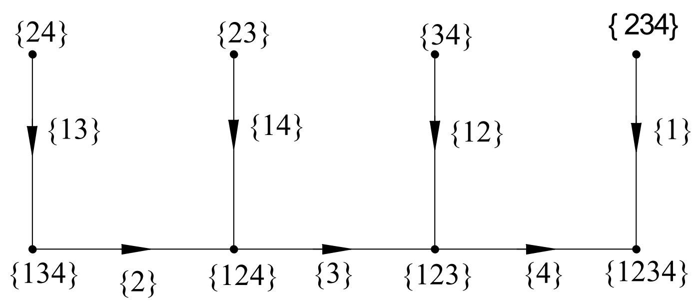
Proof. A directed centipede $\overrightarrow{Cn}$ has n vertices and $n-1$ arcs. Let $\overrightarrow{Cn}$ be properly set colorable directed tree with respect to a set X having k colors. Then$|E(\overrightarrow{Cn})| = 2^k - 1 \Rightarrow (n-1) = 2^k - 1$ $\Rightarrow n = 2^k$.

Conversely, let $\overrightarrow{Cn}$ be a directed tree such that $n = 2^k$. Let $X = \{1,2,3,...,k\}$. Also let $X_1 = \{1,2,3,...,k\}$, the full set of X, assigned to the sink i.e., vertex v of $\overrightarrow{Cn}$ where $od(v) = 0$ and assign empty set to the source i.e., vertex u of $\overrightarrow{Cn}$ where $id(u)$. Let X_2 is a subset containing $k-1$ elements of X which doesn't contain the element $a, a \in X$. Also assign the set X_2 to the vertex say v_1 adjacent to v and the remaining subsets of X to the $n-2$ vertices of $\overrightarrow{Cn}$. Then one can observe that the elements on the arcs are also subsets of X and form the set of all nonempty subsets of X. Hence $\overrightarrow{Cn}$ is properly set colorable. □

Remark 3.5. A directed centipede tree $\overrightarrow{Cn}$ is not properly set colorable if and only if $n = 2^k$, where $k > 4$.



Properly set coloring of a centipede.



Strongly set coloring of a centipede.

FIGURE 1. Properly and strongly set colorable directed centipede $\overrightarrow{Cn}$

Definition 3.6. Let $\overrightarrow{S_k}$ be a directed star with k vertices such that $od(w) = 0$. The directed banana tree $\overrightarrow{B}(n,k)$ is a directed tree obtained by joining one leaf of each n copies of a $k-star$ $\overrightarrow{S_k}$ to a single vertex w_0 where $od(w_0) = 0$.

Theorem 3.7. For any positive integer $r \geq 1$ a directed banana tree $\overrightarrow{B}(n,k)$ is strongly set colorable if and only if $n = 2^{r\backslash 2} - 1$ and $k = 2^{r\backslash 2} + 1$.

Proof. A directed banana tree $\overrightarrow{B}(n,k)$ has n vertices and $n-1$ arcs. Let $\overrightarrow{B}(n,k)$ be strongly set colorable with respect to a set X having k colors. Then $|V(\overrightarrow{B}(n,k))| + |E(\overrightarrow{B}(n,k))| = 2^k - 1 \Rightarrow n + (n-1) = 2^k - 1 \Rightarrow n = 2^{k-1}$.

Conversely, let w_0 be the root vertex of $\overrightarrow{B}(n,k)$ and $w_1, w_2, w_3, ..., w_k$ be the central vertices of the $k-stars$ joining the central vertex. Let $u_{i,1}, u_{i,2}, u_{i,3}, ..., u_{i,n}$ denote the pendent vertices joining $w_i, 1 \leq i \leq k$. Let X be a non empty set with $|X| = k$. Let X_1 be the full set of X and X_2 be the $(k-1)$-element set of X. Then we define a mapping $f : V(\overrightarrow{B}(n,k)) \bigcup E(\overrightarrow{B}(n,k)) \rightarrow 2^X \setminus \emptyset$ as follows $f(w_0) = X_1$, $f(w_i) = A$, where $A \subseteq X_2$ and $f(u_{i,j}) = B$, where B is the remaining $(k-1)$-element sets and $(k-2)$-element sets. Since A and B are disjoint, vertices are assigned by the distinct subsets. Therefore the mapping f is injective. Hence the $\overrightarrow{B}(n,k)$ is strongly set colorable. □

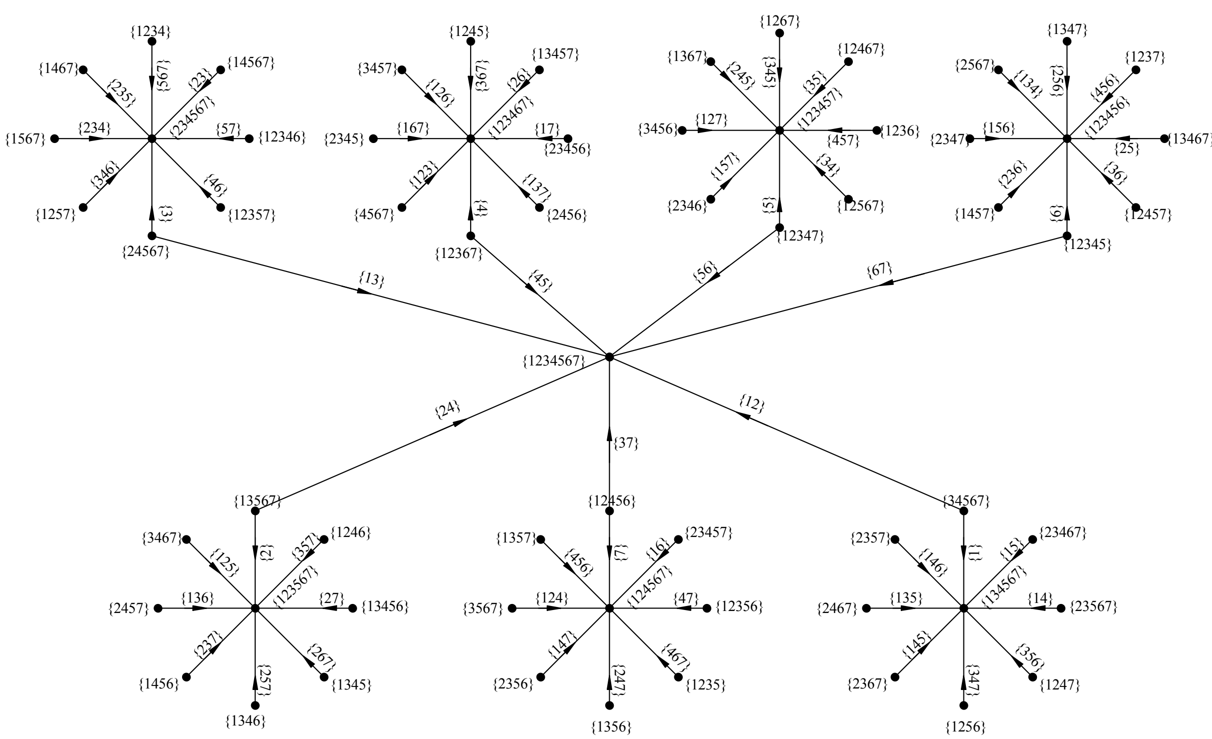


FIGURE 2. Strong set colorable directed banana tree $\overrightarrow{B}(7,9)$.

Definition 3.8. A lobster Lb is a tree in which all the vertices are within the distance 2 of a central path. A directed lobster $\overrightarrow{Lb}$ is a oriented tree which gives a directed caterpillar when all its pendant vertices are deleted.

Theorem 3.9. A directed lobster $\overrightarrow{Lb}$ is strongly set colorable if and only if $n = 2^{k-1}$, $k \geq 5$.

Proof. A directed Lobster $\overrightarrow{Lb}$ has n vertices and $n-1$ arcs. Let $\overrightarrow{Lb}$ be strongly set colorable with respect to a set X having k colors. Then $|V(\overrightarrow{Lb})|+|E(\overrightarrow{Lb})| = 2^k-1 \Rightarrow n+(n-1) = 2^k-1 \Rightarrow n = 2^{k-1}$.

Conversely, let $\overrightarrow{Lb}$ be a directed lobster such that $n = 2^{k-1}$. Let v be the central vertex of $\overrightarrow{Lb}$ and od(v)=0. Let $X_1 = \{1,2,...,k\}$, the full set of X. Let P be the longest path from v. Then assign $(k-1)$-elements subsets of X say, A to the vertices of P. Let N_1 be the set of all vertices which are at a distance one from P. Then assign remaining $(k-1)$-elements subsets of X together with $(k-2)$-elements subsets of X other than A say, B to the vertices of N_1. Let N_2 be the set of all pendant vertices of $\overrightarrow{Lb}$. Then assign $(k-2)$-elements subsets of X together with the remaining subsets of X other than A and B say, C to the vertices of N_2. Assign all the subsets of X which contains the element a, except the singleton set a, to the remaining vertices of $\overrightarrow{Lb}$. Hence $\overrightarrow{Lb}$ is strongly set colorable. □

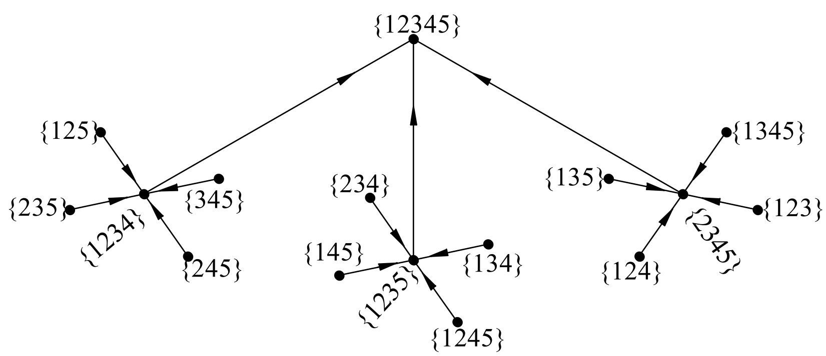


FIGURE 3. Strong set colorable directed Lobster $\overrightarrow{Lb}$.

Theorem 3.10. Every properly set colorable directed tree is strongly set colorable.

Proof. Let $\overrightarrow{T_n}$ be a properly set colorable tree with proper set coloring with proper coloring f with respect to a set X of cardinality m. Let $X' = X\bigcup\{x\}$. Since $\overrightarrow{T_n}$ is properly set colorable, all the subsets of X are assigned to the vertices and $f(\overrightarrow{T_n}) = \{f(v) : v \in V(\overrightarrow{T_n})\} = 2^X$ and $f^*(\overrightarrow{T_n}) = \{f(e) : e \in E(\overrightarrow{T_n}\} = 2^X \setminus \emptyset$.

Define a function $F : V(\overrightarrow{T_n}) \to 2^{X'}$ by $F(v) = f(v)\bigcup\{x\}$ for all $v \in V(\overrightarrow{T_n})$. Since f and f^* are injective,F and F^* are also injective. Also $F(V(\overrightarrow{T_n}))\bigcap F^*(E(V(\overrightarrow{T_n})) = \emptyset$.

Since $f(\overrightarrow{T_n}) = 2^X$ and $f^*(\overrightarrow{T_n}) = 2^X \setminus \emptyset$, we get $F(\overrightarrow{T_n}) = 2^{X'} - 2^X$ and $F^*(\overrightarrow{T_n}) = 2^X \setminus \emptyset$. That is, $f^*(\overrightarrow{T_n}) = F^*(\overrightarrow{T_n})$. Further, $|F(\overrightarrow{T_n})| = 2^{|X'|} - 2^{|X|} = 2^{m+1} - 2^m = 2^m(2-1) = 2^m$ and $|F^*(\overrightarrow{T_n})| = 2^m - 1$.

Therefore $|F(\overrightarrow{T_n})| + |F^*(\overrightarrow{T_n})| = 2^m + 2^m - 1 = 2^{m+1} - 1 = 2^{|X'|} - 1$. This implies that F is a strong set coloring of $\overrightarrow{T_n}$. □

4. Construction of Strongly Set Colorable Directed Tree by Adding an Arc to the Root Vertex of a Complete Binary Tree

Given below is a construction of a strongly set colored directed tree.

Definition 3.11. A directed tree $\overrightarrow{T_n'}$ with n vertices is said to be multi-scale if an arc is added to the root vertex of a binary tree, where the outdegree of the root vertex and indegree of all the pendant vertices of $\overrightarrow{T_n'}$ is 0.

Next, we give the construction of an infinite family of strongly set colorable directed tree by adding an arc to the root vertex of a binary tree.

Let X_1 be a non empty set with $|X_1| = m_1$, where $m_1 = 3$ is a positive integer. Consider $K_{1,2^{m_1-1}-1} = \overrightarrow{T_0'}(m_1)$ with $od(v) = 0$, where v is the central vertex. Let $v_1, v_2, \cdots, v_{2^{m_1-1}-1}$ be the pendant vertices of $T_0'(m_1)$. We define a mapping $f_1 : V(\overrightarrow{T_0'}(m_1)) \to 2^{X_1}$ as follows:

$f_1(v) = \{X_1\}$

$f_1(v_i) = A_r$, where A_r is a $(m_1 - 1)$ element subset of X_1 for $i = 1, 2, ...2^{m_1-1} - 1$.

$f_1(v_{2^{m_1-1}-1}) = X_1$.

Clearly, f_1 and f_1^* are injective functions. Let X_2 be a set of cardinality m_2, where $m_2 > m_1$. Introduce new vertices $u_{1,1}, u_{1,2}, \cdots, u_{1,k_1}$, where $k_1 = 2^{m_2-1} - 2^{m_1-1}$. Join each pair of these vertices to $v_2, v_3, ..., v_{2^{m_1-1}-1}$ such that $id((u_{1,k_{i'}}) = 0)$. Let the resulting directed tree be denoted by $\overrightarrow{T_1'}(m_2)$ and define the mapping $f_2 : V(\overrightarrow{T_1'}(m_2)) \to 2^{X_2}$ as follows:

$f_2(v) = \{X_1\} \cup \{m_2\} = A$.

$f_2(v_i) = A_r \cup \{m_2\} = A_r'$ for $i = 1, 2, ..., 2^{m_1-1} - 1$.

$f_2(v_{2^{m_1-1}-1}) = X_1 \cup \{m_2\} = X_2$.

$f_2(u_{1,i'}) = B_r$, $B_r \subset X_2$, $B_r \neq A_r'$ for $i' = 1, 2, ..., k_1$.

$f_2(u_{1,k_1}) = X_2 - \{x_0\} = B$.

Let $f_2^* : E(\overrightarrow{T_1'}(m_2)) \to 2^{X_2}$ denote the induced edge function defined by $f_2^*(u,v) = f_2(v) - f_2(u)$ where $(u,v) \in E(\overrightarrow{T_1'}(m_2))$. Then one can easily verify that both f_2 and f_2^* are injective functions and hence $\overrightarrow{T_1'}(m_2)$ is strongly set colorable.

Let X_3 be a set of cardinality m_3, where $m_3 > m_2 > m_1$.

Introduce $u_{2,1}, u_{2,2}, \cdots, u_{1,k_2}$, where $k_2 = 2^{m_3-1} - 2^{m_2-1}$, join two of them to $u_{1,1}, u_{1,2}, \cdots, u_{1,k_1}$ where $id(u_{1,k_{i''}}) = 0$. Let the resulting directed tree be $\overrightarrow{T_2'}(m_3)$. Define the mapping $f_3 : V(\overrightarrow{T_2'}(m_3)) \to 2^{X_3}$ as follows:

$f_3(v) = X \cup \{m_3\} = X_3$.

$f_3(v_i) = A_r' \cup \{m_3\} = A_r''$, $i < 2^{m_1-1} - 1$.

$f_3(v_{2^{m_1-1}-1}) = X_2$.

$f_3(u_{1,i'}) = B_r \cup \{m_3\} = B_r'$, $i' = 1, 2, ..., k_1$.

$f_3(u_{1,k_1}) = X_3$.

$f_3(u_{2,i''}) = C_r$, $C_r \subset X_3$, $C_r \neq A_r''$, $C_r \neq B_r'$ and $m_3 \in C_r$ for $i'' = 1, 2, ..., k_2$.

$f_3(u_{2,k_2}) = X_3 - \{x_0\} = C$.

Let $f_3^* : E(\overrightarrow{T_2'}(m_3)) \to 2^{X_3}$ denote the induced edge function defined by $f_3^*(u,v) = f_3(v) - f_3(u)$, where $(u,v) \in E(\overrightarrow{T_1'}(m_3))$. Then one can easily verify that both f_3 and f_3^* are injective functions and hence $\overrightarrow{T_2'}(m_3)$ is strongly set colorable.

We can continue this procedure indefinitely to obtain the strongly set colorable directed tree at the n^{th} step where $m_n > m_{n-1} > ... > m_3 > m_2 > m_1$ are chosen arbitrarily.

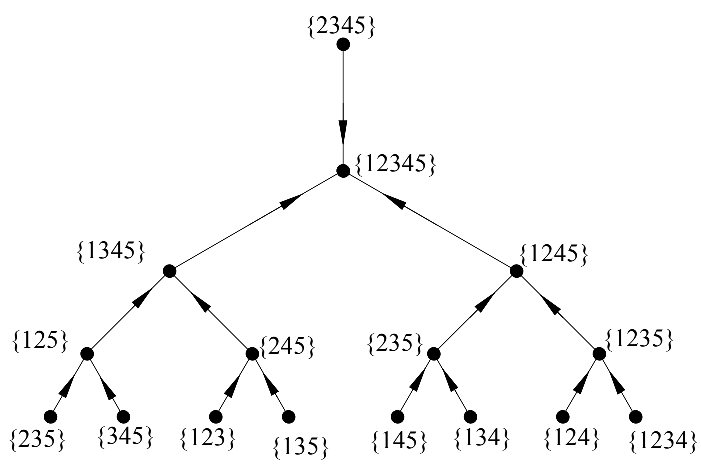


FIGURE 4. Strong set colorable digraph of complete oriented binary tree $\overrightarrow{T_2'}(m_3)$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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