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## QUASI HEMI-SLANT SUBMANIFOLDS OF SASAKIAN MANIFOLDS

RAJENDRA PRASAD, SANDEEP KUMAR VERMA, SUMEET KUMAR\*

Department of Mathematics and Astronomy, University of Lucknow, Lucknow (U.P.) 226007, India

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**Abstract.** In this paper, we define and study quasi hemi-slant submanifolds as a generalization of slant submanifolds, semi-slant submanifolds and hemi-slant submanifolds of contact metric manifolds [ In particular, for a Sasakian manifold]. Further, we obtain necessary and sufficient conditions for integrability of distributions which are involved in the definition of quasi hemi-slant submanifolds of Sasakian manifolds. After it, we investigate the necessary and sufficient condition for quasi hemi-slant submanifolds of Sasakian manifolds to be totally geodesic. Finally, we obtain the necessary and sufficient condition for a quasi hemi-slant submanifold to be local product Riemannian manifold and also give an example of such submanifolds.

**Keywords:** quasi hemi-slant submanifolds; Sasakian manifolds; totally geodesic.

**2010 AMS Subject Classification:** 53C12, 53C25, 53C40.

### 1. INTRODUCTION

Presently, the theory of submanifolds has gained prominence in computer design, image processing, economic modeling as well as in mathematical physics and in mechanics. These extensive applications of this topic makes it an active and interesting field of research for geometers. The notion of geometry of submanifolds begin with the idea of the extrinsic geometry of surface and it is developed for ambient space in the course of time.

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\*Corresponding author

E-mail address: [itssumeetkumar@gmail.com](mailto:itssumeetkumar@gmail.com)

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Firstly, the notion of slant submanifolds of almost Hermitian manifolds was studied by B. Y. Chen [10] as a natural generalization of holomorphic immersions and totally real immersions. Many consequent results on slant submanifolds are collected in his book [7]. Several examples of slant submanifolds in  $C^2$  and  $C^4$  are given by B. Y. Chen and Y. Tazawa in [11]. On the other hand, using this notion A. Lotta [14] studied slant immersions of a Riemannian manifold into almost contact metric manifold in 1996. Bejancu and Papaghiuc studied the semi-invariant submanifolds of Sasakian manifold [1]. In the course of time this interesting subject have been studied broadly by several geometers during last two decades [5], [10], [16], [17], [23]. J. L. Cabrerizo and his co-authors [6] studied slant submanifolds in Sasakian manifold. Further slant submanifold was generalized as semi-slant submanifold, pseudo-slant submanifold, bi-slant submanifold and hemi-slant submanifold etc., in different kinds of differentiable manifolds [15], [21], [23].

Motivated from above studies, we study quasi hemi-slant submanifolds of Sasakian manifolds as a generalization of semi-slant submanifolds and hemi-slant submanifolds. The present paper is organized as follows : In section 2, we mention basic definitions and some properties of Sasakian manifolds. In section 3, we define quasi-hemi-slant submanifolds and some basic properties of the submanifolds. Section 4 deals with necessary and sufficient conditions for integrability of distributions which are involved in the definition. We also find necessary and sufficient condition for the submanifolds to be totally geodesic. In the last section, we provide an example of such submanifolds.

**2. PRELIMINARIES**

We consider  $\widehat{M}$  is a  $(2n + 1)$ -dimensional (i.e. odd dimensional ) almost contact manifold [12] which carries a tensor field  $\phi$  of type  $(1, 1)$ , 1-form  $\eta$  and characteristic vector field  $\xi$  satisfying

$$(2.1) \quad \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

where  $I : T\widehat{M} \longrightarrow T\widehat{M}$  is the identity map. We have from definition  $\phi\xi = 0, \eta \circ \phi = 0$  and  $rank(\phi) = 2n$ .

Since any almost contact manifold  $(\widehat{M}, \phi, \xi, \eta)$  admits a Riemannian metric  $g$  such that

$$(2.2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields  $X, Y \in \Gamma(T\widehat{M})$ , where  $\Gamma(T\widehat{M})$  represents the Lie algebra of vector fields on  $\widehat{M}$ . The manifold  $\widehat{M}$  together with the structure  $(\phi, \xi, \eta, g)$  is called an almost contact metric manifold.

The immediate consequence of (2.2), we have

$$(2.3) \quad \eta(X) = g(X, \xi) \quad \text{and} \quad g(\phi X, Y) + g(X, \phi Y) = 0,$$

for all vector fields  $X, Y \in \Gamma(T\widehat{M})$ .

An almost contact structure  $(\phi, \xi, \eta)$  is said to be normal [3] if the almost complex structure  $J$  on the product manifold  $\widehat{M} \times R$  is given by

$$(2.4) \quad J(U, f \frac{d}{dt}) = (\phi U - f\xi, \eta(U) \frac{d}{dt}),$$

where  $J^2 = -I$  and  $f$  is the differentiable function on  $\widehat{M} \times R$ .  $J$  has no torsion i.e.,  $J$  is integrable. The condition for normality in terms of  $\phi$ ,  $\xi$  and  $\eta$  is  $[\phi, \phi] + 2d\eta \otimes \xi = 0$  on  $\widehat{M}$ , where  $[\phi, \phi]$  is the Nijenhuis tensor of  $\phi$ . A Sasakian manifold [6] is normal contact metric manifold and every Sasakian manifold is  $K$ -contact manifold. It is easy to show that an almost contact metric manifold is a Sasakian manifold if and only if

$$(2.5) \quad (\widehat{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

$$\widehat{\nabla}_X \xi = -\phi X$$

for all vector fields  $X, Y \in \Gamma(T\widehat{M})$ .

Let  $M$  be a Riemannian manifold isometrically immersed in  $\widehat{M}$  and the induced Riemannian metric on  $M$  is denoted by the same symbol  $g$  throughout this paper. Let  $A$  and  $h$  denote the shape operator and second fundamental form, respectively, of immersion of  $M$  into  $\widehat{M}$ . The Gauss and Weingarten formulas of  $M$  into  $\widehat{M}$  are given by [8]

$$(2.6) \quad \widehat{\nabla}_X Y = \nabla_X Y + h(X, Y)$$

and

$$(2.7) \quad \widehat{\nabla}_X V = -A_V X + \nabla_X^\perp V,$$

for any vector fields  $X, Y \in \Gamma(TM)$  and  $V \in \Gamma(T^\perp M)$ , where  $\nabla$  is the induced connection on  $M$  and  $\nabla^\perp$  represents the connection on the normal bundle  $T^\perp M$  of  $M$  and  $A_V$  is the shape operator of  $M$  with respect to normal vector  $V \in \Gamma(T^\perp M)$ . Moreover,  $A_V$  and the second fundamental form  $h : TM \otimes TM \rightarrow T^\perp M$  of  $M$  into  $\widehat{M}$  are related by

$$(2.8) \quad g(h(X, Y), V) = g(A_V X, Y),$$

for any vector fields  $X, Y \in \Gamma(TM)$  and  $V \in \Gamma(T^\perp M)$ .

The mean curvature vector is defined by

$$(2.9) \quad H = \frac{1}{n} \text{trace}(h) = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i),$$

where  $n$  denotes the dimension of submanifold  $M$  and  $\{e_1, e_2, \dots, e_n\}$  is the orthonormal basis of tangent space of  $M$ .

For any  $X \in \Gamma(TM)$ , we can write

$$(2.10) \quad \phi X = TX + NX,$$

where  $TX$  and  $NX$  are the tangential and normal components of  $\phi X$  on  $M$  respectively. Similarly for any  $V \in T^\perp M$ , we have

$$(2.11) \quad \phi V = tV + nV,$$

where  $tV$  and  $nV$  are the tangential and normal components of  $\phi V$  on  $M$  respectively.

A submanifold  $M$  of Sasakian manifold  $\widehat{M}$  is said to be totally umbilical if

$$(2.12) \quad h(X, Y) = g(X, Y)H,$$

where  $H$  is the mean curvature vector. If  $h(X, Y) = 0$  for all  $X, Y \in \Gamma(TM)$ , then  $M$  is said to be totally geodesic [9] and if  $H = 0$ , then  $M$  is said to be a minimal submanifold.

The covariant derivative of tangential and normal components of (2.10) and (2.11) are given as

$$(\widehat{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y,$$

$$(\widehat{\nabla}_X N)Y = \nabla_X^\perp NY - N\nabla_X Y,$$

$$(\widehat{\nabla}_X t)V = \nabla_X tV - t\nabla_X^\perp V$$

and

$$(\widehat{\nabla}_X n)V = \nabla_X^\perp nV - n\nabla_X^\perp V$$

for any  $X, Y \in \Gamma(TM)$  and  $V \in \Gamma(T^\perp M)$ .

**Definition 1.** Let  $M$  be a Riemannian manifold isometrically immersed in an almost contact metric manifold  $\widehat{M}$ . A submanifold  $M$  of an almost contact metric manifold  $\widehat{M}$  is said to be invariant [2] if  $\phi(T_x M) \subseteq T_x M$ , for every point  $x \in M$ .

**Definition 2.** A submanifold  $M$  of an almost contact metric manifold  $\widehat{M}$  is said to be anti-invariant [13] if  $\phi(T_x M) \subseteq T_x^\perp M$ , for every point  $x \in M$ .

**Definition 3.** Let  $M$  be a submanifold of an almost contact metric manifold  $\widehat{M}$  such that  $\xi$  is tangential to  $M$ . The submanifold  $M$  of an almost contact metric manifold  $\widehat{M}$  is said to be slant [6], if for each non-zero vector  $X$  tangent to  $M$  at  $x \in M$  such that  $X$  is linearly independent to  $\xi_x$ , the angle  $\theta(X)$  between  $\phi X$  and  $T_x M$  is constant, i.e., it does not depend on the choice of the point  $x \in M$  and  $X \in T_x M - \langle \xi \rangle$ . In this case, the angle  $\theta$  is called the slant angle of the submanifold. A slant submanifold  $M$  is called proper slant submanifold if neither  $\theta = 0$  nor  $\theta = \frac{\pi}{2}$ . Here  $TM = D_\theta \oplus \langle \xi \rangle$ , where  $D_\theta$  is slant distribution with slant angle  $\theta$ .

We note that on a slant submanifold  $M$  if  $\theta = 0$ , then it is an invariant submanifold and if  $\theta = \frac{\pi}{2}$ , then it is an anti-invariant submanifold. This means slant submanifold is a generalization of invariant and anti-invariant submanifolds.

**Definition 4.** A submanifold  $M$  of an almost contact metric manifold  $\widehat{M}$  is said to be semi-invariant [1], if there exist two orthogonal complementary distributions  $D$  and  $D^\perp$  on  $M$  such that

$$TM = D \oplus D^\perp \oplus \langle \xi \rangle$$

where  $D$  is invariant and  $D^\perp$  is anti-invariant.

**Definition 5.** A submanifold  $M$  of an almost contact metric manifold  $\widehat{M}$  is said to be semi-slant [15], if there exist two orthogonal complementary distributions  $D$  and  $D_\theta$  on  $M$  such that

$$TM = D \oplus D_\theta \oplus \langle \xi \rangle$$

where  $D$  is invariant and  $D_\theta$  is slant with slant angle  $\theta$ . In this case, the angle  $\theta$  is called semi-slant angle.

**Definition 6.** A submanifold  $M$  of an almost contact metric manifold  $\widehat{M}$  is said to be hemi-slant [20], if there exist two orthogonal complementary distributions  $D_\theta$  and  $D^\perp$  on  $M$  such that

$$TM = D_\theta \oplus D^\perp \oplus \langle \xi \rangle$$

where  $D_\theta$  is slant with slant angle  $\theta$  and  $D^\perp$  is anti-invariant. In this case, the angle  $\theta$  is called hemi-slant angle.

### 3. QUASI HEMI-SLANT SUBMANIFOLDS OF SASAKIAN MANIFOLDS

In the present section of the paper, we introduce the definition of quasi hemi-slant submanifolds of Sasakian manifolds and obtain some related results for later use.

**Definition 7.** Quasi hemi-slant submanifold  $M$  of Sasakian manifold  $\widehat{M}$  is a submanifold that admits three orthogonal complementary distributions  $D$ ,  $D_\theta$  and  $D^\perp$  such that

(i)  $TM$  admits the orthogonal direct decomposition

$$(3.1) \quad TM = D \oplus D_\theta \oplus D^\perp \oplus \langle \xi \rangle .$$

(ii) The distribution  $D$  is invariant, i.e.  $\phi D = D$ .

(iii) The distribution  $D_\theta$  is slant with constant angle  $\theta$ . The angle  $\theta$  is called slant angle.

(iv) The distribution  $D^\perp$  is  $\phi$  anti-invariant, i.e.  $\phi D^\perp \subseteq T^\perp M$ .

In this case, we call  $\theta$  the quasi hemi-slant angle of  $M$ . Suppose the dimension of distributions  $D, D_\theta$  and  $D^\perp$  are  $n_1, n_2$  and  $n_3$  respectively. Then we easily see the following particular cases:

- (i) If  $n_1 = 0$ , then  $M$  is a hemi-slant Submanifold.
- (ii) If  $n_2 = 0$ ; then  $M$  is a semi-invariant submanifold.
- (iii) If  $n_3 = 0$ , then  $M$  is a semi-slant Submanifold.

We say that the quasi hemi-slant submanifold  $M$  is proper if  $D \neq \{0\}$ ,  $D_\theta \neq \{0\}$ ,  $D^\perp \neq \{0\}$  and  $\theta \neq 0, \frac{\pi}{2}$ .

This means quasi hemi-slant submanifold is a generalization of invariant, anti-invariant, semi-invariant, slant, hemi-slant, semi-slant submanifolds and also they are the examples of quasi hemi-slant submanifolds.

**Remark 1.** *It is clear from definition 7 that if  $D \neq \{0\}$ ,  $D_\theta \neq \{0\}$  and  $D^\perp \neq \{0\}$ , then  $\dim D \geq 2$ ,  $\dim D_\theta \geq 2$  and  $\dim D^\perp \geq 1$ . So for proper quasi hemi-slant submanifold  $M$ , the  $\dim M \geq 6$ .*

Let  $M$  be a quasi hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ . We denote the projections of  $X \in \Gamma(TM)$  on the distributions  $D, D_\theta$  and  $D^\perp$  by  $P, Q$  and  $R$  respectively. Then we can write for any  $X \in \Gamma(TM)$

$$(3.2) \quad X = PX + QX + RX + \eta(X)\xi$$

Now, put

$$(3.3) \quad \phi X = TX + NX,$$

where  $TX$  and  $NX$  are tangential and normal components of  $\phi X$  on  $M$ .

Using (3.2) and (3.3), we obtain

$$(3.4) \quad \phi X = TPX + NPX + TQX + NQX + TRX + NRX.$$

Since  $\phi D = D$  and  $\phi D^\perp \subseteq T^\perp M$ , we have  $NPX = 0$  and  $TRX = 0$ . Therefore, we get

$$(3.5) \quad \phi X = TPX + TQX + NQX + NRX.$$

Then for any  $X \in \Gamma(TM)$ , it is easy to see that

$$TX = TPX + TQX$$

and

$$NX = NQX + NRX.$$

Thus from (3.5), we have the following decomposition

$$(3.6) \quad \phi(TM) = D \oplus TD_\theta \oplus ND_\theta \oplus ND^\perp.$$

where ‘ $\oplus$ ’ denotes orthogonal direct sum.

Since  $ND_\theta \subseteq \Gamma(T^\perp M)$  and  $ND^\perp \subseteq \Gamma(T^\perp M)$ , we have

$$(3.7) \quad T^\perp M = ND_\theta \oplus ND^\perp \oplus \mu,$$

where  $\mu$  is the orthogonal complement of  $ND_\theta \oplus ND^\perp$  in  $\Gamma(T^\perp M)$  and it is invariant with respect to  $\phi$ .

For any non-zero vector field  $V \in \Gamma(T^\perp M)$ , we have

$$(3.8) \quad \phi V = tV + nV,$$

where  $tV \in \Gamma(TM)$  and  $nV \in \Gamma(T^\perp M)$ .

**Proposition 1.** *Let  $M$  be a submanifold of Sasakian manifold  $\widehat{M}$ , then for any  $X, Y \in \Gamma(TM)$ , we have*

$$\nabla_X TY - A_{NY}X - T\nabla_X Y - th(X, Y) = g(X, Y)\xi - \eta(Y)X$$

and

$$h(X, TY) + \nabla_X^\perp NY - N(\nabla_X Y) - nh(X, Y) = 0.$$

**Proposition 2.** *Let  $M$  be a quasi hemi-slant submanifold of Sasakian manifold  $\widehat{M}$ . Then we obtain*

$$(3.9) \quad TD = D, TD_\theta = D_\theta, TD^\perp = \{0\}, tND_\theta = D_\theta, tND^\perp = D^\perp.$$

Now, using (3.3) and (3.8) and using the fact that  $\phi^2 = -I + \eta \otimes \xi$ , then on comparing the tangential and normal components, we have the following:



**Proposition 3.** *Let  $M$  be a quasi hemi-slant submanifold of Sasakian manifold  $\widehat{M}$ . Then the endomorphism  $T$  and  $N$ ,  $t$  and  $n$  in the tangent bundle of  $M$ , satisfy the following identities:*

$$(i) \quad T^2 + tN = -I + \eta \otimes \xi \text{ on } TM,$$

$$(ii) \quad NT + nN = 0 \text{ on } TM,$$

$$(iii) \quad Nt + n^2 = -I \text{ on } (T^\perp M),$$

$$(iv) \quad Tt + tn = 0 \text{ on } (T^\perp M),$$

where  $I$  is the identity operator.

**Lemma 1.** *Let  $M$  be a quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ . Then*

$$(1) \quad T^2X = -(\cos^2 \theta)X,$$

$$(2) \quad g(TX, TY) = (\cos^2 \theta)g(X, Y),$$

$$(3) \quad g(NX, NY) = (\sin^2 \theta)g(X, Y),$$

for any  $X, Y \in D_\theta$ .

*Proof.* The proof is the same as in [16]. □

**Proposition 4.** *Let  $M$  be a quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ . Then*

$$(\widehat{\nabla}_X T)Y = A_{NY}X + th(X, Y) + g(X, Y)\xi - \eta(Y)X,$$

$$(\widehat{\nabla}_X N)Y = nh(X, Y) - h(X, TY),$$

$$(\widehat{\nabla}_X t)V = A_{nV}X - TA_VX$$

and

$$(\widehat{\nabla}_X n)V = -h(X, tV) - NA_VX$$

for any  $X, Y \in \Gamma(TM)$  and  $V \in \Gamma(T^\perp M)$ .

**Proposition 5.** *Let  $M$  be a quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ , then*

$$h(X, \xi) = -NX \text{ and } \nabla_X \xi = -TX \text{ for all } X \in \Gamma(TM).$$

**Lemma 2.** *Let  $M$  be a quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ , then*

$$A_{\phi Z}W = A_{\phi W}Z$$

for all  $Z, W \in D^\perp$ .

**Lemma 3.** *Let  $M$  be a quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ , then*

- (i)  $g([X, Y], \xi) = 2g(TX, Y),$
  - (ii)  $g(\widehat{\nabla}_X Y, \xi) = g(TX, Y)$
- for all  $X, Y \in \Gamma(D \oplus D_\theta \oplus D^\perp).$

**4. INTEGRABILITY OF DISTRIBUTIONS AND DECOMPOSITION THEOREMS**

We now examine the integrability conditions for invariant distribution  $D$ , slant distribution  $D_\theta$  and anti-invariant distribution  $D^\perp.$

**Proposition 6.** *Let  $M$  be a proper quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}.$  Then, the invariant distribution  $D$  is not integrable.*

*Proof.* Let  $X, Y \in \Gamma(D),$  using (2.3), (2.5) and (2.6), we have

$$\begin{aligned}
 g([X, Y], \xi) &= 2g(\phi X, Y) \\
 (4.1) \qquad \qquad &\neq 0, \text{ for some } X, Y \in \Gamma(D).
 \end{aligned}$$

Since  $g(\phi X, Y) \neq 0$  therefore  $g([X, Y], \xi) \neq 0.$  Hence the proof. □

**Theorem 1.** *Let  $M$  be a proper quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}.$  Then, the distribution  $D \oplus \langle \xi \rangle$  is integrable if and only if*

$$\begin{aligned}
 (4.2) \qquad g(\nabla_X TY - \nabla_Y TX, TQZ) &= g(h(Y, TX) - h(X, TY), NQZ + NRZ), \\
 \text{for any } X, Y \in \Gamma(D \oplus \langle \xi \rangle) \text{ and } Z &\in \Gamma(D_\theta \oplus D^\perp).
 \end{aligned}$$

*Proof.*  $X, Y \in \Gamma(D \oplus \langle \xi \rangle)$  and  $Z \in \Gamma(D_\theta \oplus D^\perp),$  using (2.2), (2.5) and (3.3), we obtain

$$\begin{aligned}
 g([X, Y], Z) &= g(\widehat{\nabla}_X Y, Z) - g(\widehat{\nabla}_Y X, Z) \\
 &= g(\phi \widehat{\nabla}_X Y, \phi Z) - g(\phi \widehat{\nabla}_Y X, \phi Z)
 \end{aligned}$$

After some computation, we have

$$(4.3) \qquad g([X, Y], Z) = g(\nabla_X TY - \nabla_Y TX, TQZ) - g(h(Y, TX) - h(X, TY), NQZ + NRZ),$$

Hence the proof. □

**Proposition 7.** *Let  $M$  be a proper quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ . Then, the slant distribution  $D_\theta$  is not integrable.*

*Proof.* Let  $X, Y \in \Gamma(D_\theta)$  and using (2.3), (2.5) and (2.6), we have

$$(4.4) \quad \begin{aligned} g([X, Y], \xi) &= 2g(\phi X, Y) \\ &\neq 0, \text{ for some } X, Y \in \Gamma(D_\theta). \end{aligned}$$

Since  $g(\phi X, Y) \neq 0$  therefore  $g([X, Y], \xi) \neq 0$ . Hence the proof.  $\square$

**Theorem 2.** *Let  $M$  be a proper quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ . Then, the distribution  $D_\theta \oplus \langle \xi \rangle$  is integrable if and only if*

$$(4.5) \quad \begin{aligned} g(A_{NTZY} - A_{NTYZ}, W) &= g(A_{NZY} - A_{NYZ}, TPW) \\ &\quad + g(\nabla_Z^\perp NY - \nabla_Y^\perp NZ, NRW), \end{aligned}$$

for any  $Y, Z \in \Gamma(D_\theta \oplus \langle \xi \rangle)$  and  $W \in \Gamma(D \oplus D^\perp)$ .

*Proof.* For any  $Y, Z \in \Gamma(D_\theta \oplus \langle \xi \rangle)$  and  $W = PW + RW \in \Gamma(D \oplus D^\perp)$ , using (2.2), (2.5) and (3.3), we obtain

$$g([Y, Z], W) = g(\phi \widehat{\nabla}_Y Z, \phi W) - g(\phi \widehat{\nabla}_Z Y, \phi W).$$

Then from (2.7), (3.3) and Lemma 1, we have

$$\begin{aligned} \sin^2 \theta g([Y, Z], W) &= g(A_{NTZY} - A_{NTYZ}, W) + g(\nabla_Y^\perp NZ - \nabla_Z^\perp NY, NRW) \\ &\quad - g(A_{NZY} - A_{NYZ}, TPW). \end{aligned}$$

Hence the proof.  $\square$

From above theorem we have the following sufficient conditions for the slant distribution to be integrable:

**Theorem 3.** *Let  $M$  be a proper quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ . If*

$$(4.6) \quad \begin{aligned} \nabla_Z^\perp NW - \nabla_W^\perp NZ &\in ND_\theta \oplus \mu, \\ A_{NTWZ} - A_{NTZW} &\in D_\theta \text{ and} \\ A_{NWZ} - A_{NZW} &\in D^\perp \oplus D_\theta, \end{aligned}$$

for any  $Z, W \in \Gamma(D_\theta \oplus \langle \xi \rangle)$ , then the distribution  $D_\theta \oplus \langle \xi \rangle$  is integrable.

**Theorem 4.** *Let  $M$  be a quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ . Then the anti-invariant distribution  $D^\perp$  is integrable if and only if*

$$\nabla_Z^\perp NW - \nabla_W^\perp NZ \in ND^\perp \oplus \mu$$

for any  $Z, W \in \Gamma(D^\perp)$ .

*Proof.* For any  $Z, W \in \Gamma(D^\perp)$  and  $Y = PY + QY \in \Gamma(D \oplus D_\theta)$ , using (2.2), (2.5), (2.7), (3.3) and Lemma 2, we obtain

$$\begin{aligned} g([Z, W], Y) &= g(\widehat{\nabla}_Z \phi W, \phi Y) - g(\widehat{\nabla}_W \phi Z, \phi Y) \\ &= g(A_{\phi Z} W - A_{\phi W} Z, TPY) + g(\nabla_Z^\perp \phi W - \nabla_W^\perp \phi Z, NQY) \\ &= g(\nabla_Z^\perp NW - \nabla_W^\perp NZ, NQY). \end{aligned}$$

Hence the proof. □

We now obtain a necessary and sufficient condition for a quasi-hemi-slant submanifold to be totally geodesic.

**Theorem 5.** *Let  $M$  be a proper quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ . Then,  $M$  is totally geodesic if and only if*

$$\begin{aligned} g(h(X, PY) + \cos^2 \theta h(X, QY), U) &= g(\nabla_X^\perp NTQY, U) + g(A_{NQY} X + A_{NRY} X, tU) \\ (4.7) \qquad \qquad \qquad &\quad -g(\nabla_X^\perp NY, nU) \end{aligned}$$

for any  $X, Y \in \Gamma(TM)$  and  $U \in \Gamma(T^\perp M)$ .

*Proof.* Since  $X, Y \in \Gamma(TM)$ ,  $U \in \Gamma(T^\perp M)$  and using (2.2) and (2.5), we have

$$\begin{aligned} g(\widehat{\nabla}_X Y, U) &= g(\widehat{\nabla}_X PY, U) + g(\widehat{\nabla}_X QY, U) + g(\widehat{\nabla}_X RY, U) \\ &= g(\widehat{\nabla}_X \phi PY, \phi U) + g(\widehat{\nabla}_X TQY, \phi U) + g(\widehat{\nabla}_X NQY, \phi U) \\ &\quad + g(\widehat{\nabla}_X \phi RY, \phi U). \end{aligned}$$

Using (2.3), (2.6), (2.7), (3.3) and Lemma 1, we have

$$\begin{aligned}
g(\widehat{\nabla}_X Y, U) &= g(\widehat{\nabla}_X P Y, U) - g(\widehat{\nabla}_X T^2 Q Y, U) - g(\widehat{\nabla}_X N T Q Y, U) \\
&\quad + g(\widehat{\nabla}_X N Q Y, \phi U) + g(\widehat{\nabla}_X N R Y, \phi U) \\
&= g(h(X, P Y), U) + \cos^2 \theta g(h(X, Q Y), U) - g(\nabla_X^\perp N T Q Y, U) \\
&\quad + g(-A_{N Q Y} X + \nabla_X^\perp N Q Y, \phi U) + g(-A_{N R Y} X + \nabla_X^\perp N R Y, \phi U) \\
(4.8) \quad g(\widehat{\nabla}_X Y, U) &= g(h(X, P Y) + \cos^2 \theta h(X, Q Y), U) - g(\nabla_X^\perp N T Q Y, U) \\
&\quad - g(A_{N Q Y} X + A_{N R Y} X, t U) + g(\nabla_X^\perp N Y, n U).
\end{aligned}$$

Hence the proof.  $\square$

We now investigate the geometry of leaves of invariant, anti-invariant and slant distribution.

**Proposition 8.** *Let  $M$  be a proper quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ . Then, the invariant distribution  $D$  does not define a totally geodesic foliation on  $M$ .*

*Proof.* Let  $Y, Z \in \Gamma(D)$  and using (2.3), (2.5) and (2.6), we have

$$\begin{aligned}
(4.9) \quad g(\widehat{\nabla}_Y Z, \xi) &= g(\nabla_Y Z, \xi) \\
&= g(\phi Y, Z) \\
(4.10) \quad &\neq 0, \text{ for some } Y, Z \in \Gamma(D).
\end{aligned}$$

Since  $g(\phi Y, Z) \neq 0$  therefore  $g(\widehat{\nabla}_Y Z, \xi) \neq 0$ . Hence the proof.  $\square$

**Theorem 6.** *Let  $M$  be a proper quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ . Then, the distribution  $D \oplus \langle \xi \rangle$  defines a totally geodesic foliation on  $M$  if and only if*

$$\begin{aligned}
(4.11) \quad g(\nabla_X T Y, T Q Z) &= -g(h(X, T Y), N Q Z + N R Z) \text{ and} \\
g(\nabla_X T Y, t V) &= -g(h(X, T Y), n V),
\end{aligned}$$

for any  $X, Y \in \Gamma(D)$ ,  $Z \in \Gamma(D_\theta \oplus D^\perp)$  and  $V \in \Gamma(T^\perp M)$ .

*Proof.* Since  $X, Y \in \Gamma(D), Z = QZ + RZ \in \Gamma(D_\theta \oplus D^\perp)$  and using (2.2), (2.5), (3.3) and  $NY = 0$ , we have

$$\begin{aligned} g(\widehat{\nabla}_X Y, Z) &= g(\widehat{\nabla}_X TY, \phi Z), \\ &= g(\nabla_X TY, TQZ) + g(h(X, TY), NQZ + NRZ). \end{aligned}$$

Now for any  $V \in \Gamma(T^\perp M)$  and  $X, Y \in \Gamma(D)$ , we have

$$\begin{aligned} g(\widehat{\nabla}_X Y, V) &= g(\widehat{\nabla}_X TY, \phi V) \\ &= g(\nabla_X TY, tV) + g(h(X, TY), nV). \end{aligned}$$

Hence the proof. □

**Proposition 9.** *Let  $M$  be a proper quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ . Then, the slant distribution  $D_\theta$  does not define a totally geodesic foliation on  $M$ .*

*Proof.* Let  $Y, Z \in \Gamma(D_\theta)$  and using (2.3), (2.5) and (2.6), we have

$$\begin{aligned} (4.12) \quad g(\widehat{\nabla}_Y Z, \xi) &= g(\nabla_Y Z, \xi) \\ &= g(\phi Y, Z) \\ (4.13) \quad &\neq 0, \text{ for some } Y, Z \in \Gamma(D_\theta). \end{aligned}$$

Since  $g(\phi Y, Z) \neq 0$  therefore  $g(\widehat{\nabla}_Y Z, \xi) \neq 0$ . Hence the proof. □

**Theorem 7.** *Let  $M$  be a proper quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ . Then, the distribution  $D_\theta \oplus \langle \xi \rangle$  defines a totally geodesic foliation on  $M$  if and only if*

$$(4.14) \quad g(\nabla_Z^\perp NW, NRX) = g(A_{NW}Z, TPX) - g(A_{NTW}Z, X) \text{ and}$$

$$(4.15) \quad g(A_{NW}Z, tV) = g(\nabla_Z^\perp NW, nV) - g(\nabla_z^\perp NTW, V),$$

for any  $Z, W \in \Gamma(D_\theta \oplus \langle \xi \rangle), X \in \Gamma(D \oplus D^\perp)$  and  $V \in \Gamma(T^\perp M)$ .

*Proof.* For any  $Z, W \in \Gamma(D_\theta \oplus \langle \xi \rangle), X = PX + RX \in \Gamma(D \oplus D^\perp)$  and using (2.2), (2.5) and (3.3), we have

$$g(\widehat{\nabla}_Z W, X) = g(\widehat{\nabla}_Z \phi W, \phi X) = g(\widehat{\nabla}_Z TW, \phi X) + g(\widehat{\nabla}_Z NW, \phi X)$$

Then using (2.7), (3.3) and Lemma 1, and the fact that  $NPX = 0$ , we have

$$\begin{aligned}
 g(\widehat{\nabla}_Z W, X) &= \cos^2 \theta g(\widehat{\nabla}_Z W, X) - g(\widehat{\nabla}_Z NTW, X) \\
 &\quad + g(\widehat{\nabla}_Z NW, \phi X), \\
 (4.16) \quad \sin^2 \theta g(\widehat{\nabla}_Z W, X) &= g(A_{NTW}Z, X) \\
 &\quad - g(A_{NW}Z, TPX) + g(\nabla_Z^\perp NW, NRX).
 \end{aligned}$$

Similarly, we get

$$(4.17) \quad \sin^2 \theta g(\widehat{\nabla}_Z W, V) = -g(\nabla_Z^\perp NTW, V) - g(A_{NW}Z, tV) + g(\nabla_Z^\perp NW, nV).$$

Thus from (4.16) and (4.17), we have the assertions.  $\square$

**Theorem 8.** *Let  $M$  be a proper quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ . Then, the anti-invariant distribution  $D^\perp$  defines a totally geodesic foliation on  $M$  if and only if*

$$\begin{aligned}
 (4.18) \quad g(A_{NZ}Y, TPW + TQW) &= g(\nabla_Y^\perp NZ, NQW) \quad \text{and} \\
 g(A_{NZ}Y, tV) &= g(\nabla_Y^\perp NZ, nV),
 \end{aligned}$$

for any  $Y, Z \in \Gamma(D^\perp)$ ,  $W \in \Gamma(D \oplus D_\theta)$  and  $V \in \Gamma(T^\perp M)$ .

*Proof.* Since  $Y, Z \in \Gamma(D^\perp)$ ,  $W = PW + QW \in \Gamma(D \oplus D_\theta)$  and using (2.2), (2.5), and (3.3), we have

$$\begin{aligned}
 g(\widehat{\nabla}_Y Z, W) &= g(\widehat{\nabla}_Y \phi Z, \phi W) \\
 &= g(\widehat{\nabla}_Y NZ, \phi W) \\
 (4.19) \quad &= -g(A_{NZ}Y, TPW + TQW) + g(\nabla_Y^\perp NZ, NQW)
 \end{aligned}$$

which gives first part of (4.18).

Now for any  $Y, Z \in \Gamma(D^\perp)$  and  $V \in \Gamma(T^\perp M)$ , and using (2.2), (2.5), (2.7), (2.11), we have

$$\begin{aligned}
 (4.20) \quad g(\widehat{\nabla}_Y Z, V) &= g(\widehat{\nabla}_Y \phi Z, \phi V) = g(\widehat{\nabla}_Y NZ, \phi V), \\
 &= -g(A_{NZ}Y, tV) + g(\nabla_Y^\perp NZ, nV),
 \end{aligned}$$

which gives second part of (4.18).  $\square$

From theorems 6, 7 and 8, we have following decomposition theorem:

**Theorem 9.** *Let  $M$  be a proper quasi-hemi-slant submanifold of a Sasakian manifold  $\widehat{M}$ . Then  $M$  is a local product Riemannian manifold of the form  $M_D \times M_{D_\theta} \times M_{D^\perp}$ , where  $M_D$ ,  $M_{D_\theta}$  and  $M_{D^\perp}$  are leaves of  $D$ ,  $D_\theta$  and  $D^\perp$  respectively, if and only if the conditions (4.18), (4.14), (4.15) and (4.11) hold.*

**5. EXAMPLE**

Now, we construct an example of quasi hami-slant submanifold of a Sasakian manifold.

Let  $(R^{2n+1}, g_{2n+1}, \phi, \xi, \eta)$  denote the manifold  $R^{2n+1}$  with coordinates  $\{(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, z) : x_i, y_i, z \in R\}$  and base field  $\{E_i, E_{n+i}, \xi\}$ , where  $E_i = 2\frac{\partial}{\partial y_i}, E_{n+i} = 2(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z})$  and contravariant vector field  $\xi = 2\frac{\partial}{\partial z}$ . Define usual Sasakian structure on  $R^{2n+1}$  as:

$$\phi\left(\sum_{i=1}^n (X_i\frac{\partial}{\partial x_i} + Y_i\frac{\partial}{\partial y_i}) + Z\frac{\partial}{\partial z}\right) = \sum_{i=1}^n (Y_i\frac{\partial}{\partial x_i} - X_i\frac{\partial}{\partial y_i}) + \sum_{i=1}^n Y_i y_i \frac{\partial}{\partial z}$$

$$g_{2n+1} = \eta \otimes \eta + \frac{1}{4}\left(\sum_{i=1}^n [dx_i \otimes dx_i + dy_i \otimes dy_i]\right),$$

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^n y_i dx_i), \quad \xi = 2\frac{\partial}{\partial z},$$

where  $x_1, \dots, x_i, y_1, \dots, y_i, z$  are the Cartesian coordinates. It is easy to show that  $(R^{2n+1}, \phi, \xi, \eta, g_{2n+1})$  is a Sasakian manifold. Throughout this section we will use these notations.

Let  $F : R^7 \rightarrow R^{11}$  be map defined by

$$F(u, v, w, r, s, t, q) = 2(u, w, 0, s, t, v, r \cos \alpha, r \sin \alpha, 0, 0, q),$$

where  $\alpha$  is a constant.

By direct computation, it is easy to check that the  $F$  defines 7-dimensional submanifold  $M$  in Sasakian manifold in  $R^{11}$  (as defined above). The set  $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7\}$  is a base field



on  $M$ , where

$$\begin{aligned} Z_1 &= 2\left(\frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial z}\right), Z_2 = 2\frac{\partial}{\partial y_1}, \\ Z_3 &= 2\left(\frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial z}\right), Z_4 = 2\left(\cos \alpha \frac{\partial}{\partial y_2} + \sin \alpha \frac{\partial}{\partial y_3}\right), \\ Z_5 &= 2\left(\frac{\partial}{\partial x_4} + y_4 \frac{\partial}{\partial z}\right), Z_6 = 2\frac{\partial}{\partial x_5}, Z_7 = 2\frac{\partial}{\partial z}. \end{aligned}$$

We define the distributions  $D = \langle Z_1, Z_2 \rangle$ ,  $D_\theta = \langle Z_3, Z_4 \rangle$  and  $D^\perp = \langle Z_5, Z_6 \rangle$ . Then, it is clear that  $TM = D \oplus D_\theta \oplus D^\perp \oplus \langle \xi \rangle$  and it can be easily prove that  $D_\theta$  is a slant distribution with angle  $\theta = \alpha$ , where  $\alpha$  is a non zero constant and  $D^\perp$  is anti-invariant distribution.

### CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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