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COSET CAYLEY DIGRAPH STRUCTURES

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Abstract. In this paper, we generalize the results in [9] to produce a new classes of Cayley digraph structures induced by groups.

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1. Introduction

A *binary relation* on a set V is a subset E of $V \times V$. A *digraph* is a pair (V, E) where V is a non empty set (called vertex set) and E is a binary relation on V. The elements of E are called edges. Let V be a non empty set and let $E_1, E_2, \ldots, E_n$ be mutually disjoint binary relations on V. Then the $(n+1)$-tuple $G = (V; E_1, E_2, \ldots, E_n)$ is called a digraph structure[9]. The elements of V are called vertices and the elements of E_i are called E_i-edges. The following definition were introduced in [9].

A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called (i)$E_1E_2\cdots E_n$-*trivial* if $E_i = \emptyset$ for all i, and E_i- *trivial* if $E_i = \emptyset$ (ii)$E_1E_2\cdots E_n$- *reflexive* if for all $x \in G$, $(x, x) \in E_i$ for some i, and E_i- *reflexive* if for all $x \in V$, $(x, x) \in E_i$(iii) $E_1E_2\cdots E_n$- *symmetric* if $E_i = E_i^{-1}$ for all i, and E_i- *symmetric* if $E_i = E_i^{-1}$(iv) $E_1E_2\cdots E_n$- *anti symmetric*, if $(x, y) \in E_i$ and

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$(y, x) \in E_i$ implies $x = y$ for all i, and E_i- anti symmetric if $(x, y) \in E_i$ and $(y, x) \in E_i$ implies $x = y$ (v) $E_1E_2 \cdots E_n$- *transitive* if for every i and j, $E_i \circ E_j \subseteq E_k$ for some k, and E_i transitive if $E_i \circ E_i \subseteq E_i$ (vi) an $E_1E_2 \cdots E_n$- *hasse diagram* if for every positive integer $n \geq 2$ and every $v_0, v_1, \ldots, v_n$ of V, $(v_i, v_{i+1}) \in \cup E_i$ for all $i = 0, 1, 2, \ldots, n-1$, implies $(v_0, v_n) \notin E_i$ for all i, and E_i- *hasse diagram* if for every positive integer $n \geq 2$ and every $v_0, v_1, \ldots, v_n$ of V, $(v_i, v_{i+1}) \in E_i$ for all $i = 0, 1, 2, \ldots, n-1$, implies $(v_0, v_n) \notin E_i$, (viii)$E_1E_2 \cdots E_n$- *complete* if $\cup E_i = V \times V$, and E_i *complete* if $E_i = V \times V$.

A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called (i) an $E_1E_2 \cdots E_n$- quasi ordered set if it is both $E_1E_2 \cdots E_n$- reflexive and $E_1E_2 \cdots E_n$ -transitive (ii)an $E_1E_2 \cdots E_n$ - partially ordered set if it is $E_1E_2 \cdots E_n$- anti symmetric and $E_1E_2 \cdots E_n$- quasi ordered set. Similarly, we can define E_i quasi ordered set and E_i partially ordered set as in the case of ordinary relations.

An $E_1E_2 \cdots E_n$- walk of length k in a digraph structure is an alternating sequence $W = v_0, e_0, v_1, \ldots, e_{k-1}, v_k$, where $e_i = (v_i, v_{i+1}) \in \cup E_i$. An $E_1E_2 \cdots E_n$ -walk W is called a $E_1E_2 \cdots E_n$- path if all the internal vertices are distinct. We use notation $(v_0, v_1, v_2, \ldots, v_n)$ for the $E_1E_2 \cdots E_n$ - path W. As in digraphs, we define E_i- walk and E_i- path. For example, an E_i- path between two vertices u and v consists of only E_i- edges.

A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called (i) $E_1E_2 \cdots E_n$- *connected* if there exits at least one $E_1E_2 \cdots E_n$- path from v to u for all $u, v \in V$, (ii)$E_1E_2 \cdots E_n$- quasi connected if for every pair of vertices x, y there is a vertex z such that there is an $E_1E_2 \cdots E_n$-path from z to x and an $E_1E_2 \cdots E_n$-path from z to y, (iii) $E_1E_2 \cdots E_n$- *locally connected* iff for every pair of vertices $u, v \in V$ there is an $E_1E_2 \cdots E_n$ - path from v to u whenever there is an $E_1E_2 \cdots E_n$ - path from u to v and (iv) $E_1E_2 \cdots E_n$- *semi connected* for every pair of vertices u, v, there is an $E_1E_2 \cdots E_n$- path from u to v or an $E_1E_2 \cdots E_n$- path from v to u.

A digraph structure $(V; E_1, E_2, \ldots, E_n)$ is called E_i -*connected* if there exits at least one E_i path from v to u for all $u, v \in V$. Similarly we can define E_i quasi connected, E_i -locally connected and E_i - semi connected digraph structures.

The $E_1E_2\cdots E_n$ - distance between two vertices x and y in a digraph structure G is the length of the shortest $E_1E_2\cdots E_n$- path between x and y, denoted by $d_{1,2,3,\ldots,n}(x,y)$. Let $G=(V;E_1,E_2,\ldots,E_n)$ be a finite $E_1E_2\cdots E_n$- connected digraph structure. Then the $E_1E_2\cdots E_n$ diameter of G is defined as $d(G)=\max_{x,y\in G}\{d_{1,2,3,\ldots,n}(x,y)\}$. Similarly we can define E_i distance and E_i diameter as in digraphs.

Two digraph structures $(V_1;E_1,E_2,\ldots,E_n)$ and $(V_2;R_1,R_2,\ldots,R_m)$ are said to be isomorphic if (i) $m=n$ and (ii) there exits a bijective function $f\colon V_1\longmapsto V_2$ such that $(x,y)\in E_i \Leftrightarrow (f(x),f(y))\in R_i$. This concept of isomorphism is a generalization of isomorphism between two digraphs. An isomorphism of a digraph structure onto itself is called an automorphism. A digraph structure $(V;E_1,E_2,\ldots,E_n)$ is said to be vertex-transitive if, given any two vertices a and b of V, there is some digraph automorphism $f:V\to V$ such that $f(a)=b$. Let $(V;E_1,E_2,\ldots,E_n)$ be a digraph structure and let $v\in V$. Then the $E_1E_2\cdots E_n$ out-degree of u is $|\{v\in V:(u,v)\in\cup E_i\}|$ and $E_1E_2\cdots E_n$ in-degree of u is $|\{v\in V:(v,u)\in\cup E_i\}|$. Similarly we can define the E_i out- degree and E_i in- degree as in the case of digraphs.

Let $(V_1;E_1,E_2,\ldots,E_n)$ be a digraph structure. A vertex $v\in G$ is called an $E_1E_2\cdots E_n$ -source if for every vertex $x\in G$, there is an $E_1E_2\cdots E_n$ - path from v to x. Similarly a vertex $u\in G$ is called an $E_1E_2\cdots E_n$- sink if for very vertex $y\in G$ there is an $E_1E_2\cdots E_n$- path from y to u. As in digraphs, we define E_i - source and E_i - sink. Let $(V_1;E_1,E_2,\ldots,E_n)$ be a digraph structure and let $v\in G$. Then the $E_1E_2\cdots E_n$ reachable set $R_{1,2,3,\cdots,n}(u)$ is $\{x\in G:$ there is an $E_1E_2\cdots E_n$- path from u to $x\}$. Similarly, the $E_1E_2\cdots E_n$- antecedent set $Q_{1,2,\ldots,n}(u)$ is defined as

$$Q_{1,2,\ldots,n}(u)=\{x\in G:\text{ there is an } E_1E_2\cdots E_n\text{- path from } x \text{ to } u\}.$$

As in the case of digraphs, we can define the E_i- reachable set and E_i-antecedent set of a vertex.

2. Coset Cayley Digraph Structures

In [9] the authors introduced a class of Cayley digraph structures induced by groups. In this paper, we introduce a class of coset Cayley digraph structures induced by groups and prove that every vertex transitive digraph structure is isomorphic to the coset Cayley digraph structure . These class of Cayley digraphs structures can be viewed as a generalization of those obtained in [9].

We start with the following definition:

Definition 2.1. *Let G be a group and $S_1, S_2, \ldots, S_n$ be mutually disjoint subsets of G and H be a subgroup of G. Then coset Cayley digraph structure of G with respect to $S_1, S_2, \ldots, S_n$ is defined as the digraph structure $(G/H; E_1, E_2, \ldots, E_n)$, where*

$$E_i = \{(xH, yH) : x^{-1}y \in HS_iH\}.$$

The sets $S_1, S_2, \ldots, S_n$ are called connection sets of $(G/H; E_1, E_2, \ldots, E_n)$. We denote the coset Cayley digraph structure of G with respect to $S_1, S_2, \ldots, S_n$ by

$$\mathscr{C} = \text{Cay}(G/H; HS_1H, HS_2H, \ldots, HS_nH).$$

In this paper, we may use the following notations: Let $\mathscr{C}$ be a coset Cayley digraph structure induced by the group G with respect to the connection sets $S_1, S_2, \ldots, S_n$.

(1) Let A_k be the union of set of all k products of the form $(HS_{i1}H)(HS_{i2}H)\cdots(HS_{ik}H)$ from the set $\{HS_1H, HS_2H, \ldots, HS_nH\}$. Then $\bigcup_k A_k$. is denoted by $[HSH]$.

(2) Let A_k^{-1} be the union of set of all k products of the form:

$$(HS_{i_1}^{-1}H)(HS_{i_2}^{-1}H)\cdots(HS_{i_k}^{-1}H).$$

Then $\bigcup_k A_k^{-1}$ is denoted by $[HS^{-1}H]$.

(3) Let A be a subset of a group G, then the semigroup generated by A is denoted by $< A >$.

2.1 Main Theorems

Theorem 2.1.1 *If G is a group and let $S_1, S_2, \ldots, S_n$ are mutually disjoint subsets of G and H is a subgroup of G, then the coset Cayley digraph structure $\mathscr{C}$ is vertex transitive.*

Proof. To see that $\text{Cay}(G/H; HS_1H, HS_2H, \ldots, HS_nH)$ is a vertex transitive digraph structure, we first need only show that E_i's are well defined. Let x, y, x', y' be any four elements of G with $xH = x'H$ and $yH = y'H$. Then $x = x'h_1$ and $y = y'h_2$ for some

$h_1, h_2 \in H$. Observe that

$$\begin{aligned}(xH, yH) \in E_i &\Leftrightarrow x^{-1}y \in HS_iH \\ &\Leftrightarrow (x'h_1)^{-1}(y'h_2) \in HS_iH \\ &\Leftrightarrow h_1^{-1}(x')^{-1}y'h_2 \in HS_iH \\ &\Leftrightarrow (x')^{-1}y' \in HS_iH \\ &\Leftrightarrow (x'H, y'H) \in HS_iH.\end{aligned}$$

Hence each E_i's are well defined and hence $\mathrm{Cay}(G/H; HS_1H, HS_2H, \ldots, HS_nH)$ is a digraph structure. Let aH and bH be any two arbitrary elements in G/H. Define a mapping $\varphi : G \longmapsto G$ by

$$\varphi(xH) = ba^{-1}xH \text{ for all } xH \in G/H.$$

This mapping defines a permutation of the vertices of $\mathrm{Cay}(G/H; HS_1H, HS_2H, \ldots, HS_nH)$. It is also an automorphism. Note that

$$\begin{aligned}(xH, yH) \in E_i &\Leftrightarrow x^{-1}y \in HS_iH \\ &\Leftrightarrow (ba^{-1}x)^{-1}(ba^{-1}y) \in HS_iH \\ &\Leftrightarrow (ba^{-1}xH, ba^{-1}yH) \in E_i \\ &\Leftrightarrow (\varphi(xH), \varphi(yH)) \in E_i.\end{aligned}$$

Also we note that

$$\varphi(aH) = ba^{-1}aH = bH.$$

Hence $\mathrm{Cay}(G/H; HS_1H, HS_2H, \ldots, HS_nH)$ is vertex transitive digraph structure.

Theorem 2.1.2

Let $(V; W_1, W_2, \cdots, W_n)$ be any vertex transitive digraph structure such that $|V| \geq n$. Then $(V; W_1, W_2, \cdots, W_n)$ is isomorphic to Cay$(G/H; HS_1H, HS_2H, \ldots, HS_nH)$.

Proof. Let G be the automorphism group of the digraph structure $(V; W_1, W_2, \cdots, W_n)$. Let $q_1, q_2, \cdots, q_n$ be fixed elements in V. For $i = 1, 2, \ldots, n$, define the following:

$$H_i := \{\theta \in G : \theta(q_i) = q_i\},$$

$$S_i := \{\theta \in G : (q_i, \theta(q_i)) \in W_i\}.$$

Note that $H = \cap_{i=1}^{n} H_i$ is a subgroup of G. Construct the Cayley digraph structure $\mathrm{Cay}(G/H; HS_1H, HS_2H, \ldots, HS_nH)$ as in theorem 2.2.1.

Define a map $\varphi : G/H \longmapsto V$ by

$$(xH)\varphi = x(q_i) \text{ for all } xH \in G/H.$$

where q_i is a fixed element in the set $\{q_1, q_2, \ldots, q_n\}$.

(i)φ is well defined:

Let $xH = yH$. Then $y = xh_1,$ for some $\quad h_1 \in H$. Observe that

$$\begin{aligned} \varphi(yH) = &\ y(q_i) \\ &= (xh_1)(q_i) \\ &= x[h_1(q_i)] \\ &= x(q_i) \\ &= \varphi(xH) \end{aligned}$$

(ii) φ is one to one:

$$\begin{aligned} \varphi(xH) = \varphi(yH) &\Leftrightarrow x(q_i) = y(q_i) \\ &\Leftrightarrow y^{-1}x(q_i) = q_i \\ &\Leftrightarrow y^{-1}x \in H \\ &\Leftrightarrow xH = yH. \end{aligned}$$

(iii) φ is onto:

Let v be any element in V. Since $(V; W_1, W_2, \cdots, W_n)$ is vertex transitive, there exists an automorphism θ such that $\theta(v) = q_i$. This implies that $v = \theta^{-1}(q_i)$. That is, $v = \varphi(\theta^{-1}H)$.

(iv) φ preserves adjacency relation :

Observe that

$$
\begin{aligned}
(xH, yH) \in E_i &\Leftrightarrow x^{-1}y \in HS_iH \\
&\Leftrightarrow x^{-1}y = h_1 s_i h_2 \\
&\Leftrightarrow h_1^{-1}x^{-1}yh_2^{-1} = s_i \in S_i \\
&\Leftrightarrow (q_i, (h_1^{-1}x^{-1}yh_2^{-1})(q_i)) \in W_i \\
&\Leftrightarrow (h_1(q_i), x^{-1}y(q_i)) \in W_i \\
&\Leftrightarrow (x(q_i), y(q_i)) \in W_i \\
&\Leftrightarrow (\varphi(xH), \varphi(yH)) \in W_i.
\end{aligned}
$$

2.2 Corollaries

In this section we can prove many graph theoretic properties in terms of algebraic properties. Moreover, these results can be considered as the generalization of those obtained in [9].

Proposition 2.3 *The coset Cayley graph structure $\mathscr{C}$ is an $E_1E_2\cdots E_n$ -trivial digraph structure $\Leftrightarrow S_i = \emptyset$ for all i.*

Proof. By definition, $\mathscr{C}$ is $E_1E_2\cdots E_n$ - trivial $\Leftrightarrow E_i = \emptyset$ for all i. This implies that $S_i = \emptyset$ for all i.

Proposition 2.4 *The coset Cayley graph structure $\mathscr{C}$ is an E_i -trivial digraph structure $\Leftrightarrow S_i = \emptyset$.*

Proposition 2.5 *The coset Cayley graph structure $\mathscr{C}$ is $E_1E_2\cdots E_n$- reflexive $\Leftrightarrow 1 \in S_i$ for some i.*

Proof. Assume that $\mathscr{C}$ is an $E_1E_2\cdots E_n$ -reflexive digraph structure. Then for every $xH \in G/H$, $(xH, xH) \in E_i$ for some i. This implies that $1 \in HS_iH$ for some i. Conversely, assume that $1 \in S_i$ for some i. This implies for each $xH \in G/H$, $(xH, xH) \in E_i$ for some i. That is, $(xH, xH) \in \cup E_i$ for all $x \in G$.

Proposition 2.6 *The coset Cayley graph structure* $\mathscr{C}$ *is* E_i*- reflexive* $\Leftrightarrow 1 \in HS_iH$.

Proposition 2.7 *The coset cayley graph structure* $\mathscr{C}$ *is* $E_1E_2\cdots E_n$*- symmetric if and only if* $HS_iH = HS_i^{-1}H$ *for all* i.

Proof. First, assume that $\mathscr{C}$ is an $E_1E_2\cdots E_n$ -symmetric digraph structure. Let $a \in HS_iH$. Then $(H, aH) \in E_i$. Since $\mathscr{C}$ is symmetric $(a, 1) \in E_i$. This implies that $a^{-1} \in HS_iH$. That is $a \in HS_i^{-1}H$. Hence $HS_iH \subseteq HS_i^{-1}H$. Similarly, we can prove that $HS_i^{-1}H \subseteq HS_iH$.

Conversely, if $HS_iH = HS_i^{-1}H$, we can prove that $\mathscr{C}$ is an $E_1E_2\cdots E_n$ -symmetric digraph structure.

Proposition 2.8 $\mathscr{C}$ *is* E_i *symmetric if and only if* $HS_iH = HS_i^{-1}H$.

Proposition 2.9 $\mathscr{C}$ *is an* $E_1E_2\cdots E_n$ *- transitive if and only if for every* i, j, $HS_iHS_jH \subseteq HS_kH$ *for some* k.

Proof. First, assume that $\mathscr{C}$ is $E_1E_2\cdots E_n$ - transitive. We will show that for all (i, j), $HS_iHS_jH \subseteq HS_kH$ for some k. Let $x \in HS_iHS_jH = HS_iHHS_jH$. Then

$$x = z_1z_2 \text{ for some } \quad z_1 \in HS_iH, z_2 \in HS_jH$$

This implies that $(H, z_1H) \in E_i$ and $(z_1H, z_1z_2H) \in E_j$. Since $\mathscr{C}$ is $E_1E_2\cdots E_n$ - transitive, $(H, z_1z_2H) \in HS_kH$ for some k. That is $z_1z_2 \in HS_kH$. Hence $HS_iHS_jH \subseteq HS_kH$.

Conversely, assume that all (i, j), $HS_iHS_jH \subseteq HS_kH$ for some k. We will show that $\mathscr{C}$ is $E_1E_2\cdots E_n$ - transitive. Let $(H, xH) \in E_i$, $(xH, yH) \in E_j$. Then $x \in HS_iH$ and $x^{-1}y \in HS_jH$. This implies that $y = xx^{-1}y \in HS_iHS_jH$. Since $HS_iHS_jH \subseteq HS_kH$, we have $y \in HS_kH$. It follows that $(H, yH) \in E_k$.

Proposition 2.10 *$\mathscr{C}$ is an $E_1E_2\cdots E_n$ -k- transitive if and only if for every $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, k\}$, we have*

$$(HS_{i_1}H)(HS_{i_2}H)\cdots(HS_{i_k}H) \subseteq (HS_{j_1}H) \text{ for some } j_1;$$
$$(HS_{i_1}H)(HS_{i_2}H)\cdots(HS_{i_{k-1}}H) \subseteq (HS_{j_2}H) \text{ for some } j_2;$$
$$\vdots$$
$$(HS_{i_1}H)(HS_{i_2}H) \subseteq (HS_{j_{k-1}}H) \text{ for some } j_1.$$

Proof. First, assume that $\mathscr{C}$ is an $E_1E_2\cdots E_n$ -k- transitive. Let $x \in (HS_{i_1}H)(HS_{i_2}H)\cdots(HS_{i_k}H)$. Then there exits $z_j \in (HS_{i_j}H)$, $j = 1, 2, \ldots, k$ such that $x = z_1z_2\cdots z_k$. This implies that

$$(H, z_1H, z_1z_2H, z_1z_2z_3H, \ldots, z_1z_2z_3\ldots z_kH)$$

is a path from 1 to x. Since $\mathscr{C}$ is an $E_1E_2\cdots E_n$ -k- transitive, we have

$$(H, z_1z_2z_3\ldots z_kH) \in E_{j_1} \text{ for some } j_1,$$
$$(H, z_1z_2z_3\ldots z_{k-1}H) \in E_{j_1} \text{ for some } j_2,$$
$$\vdots$$
$$(H, z_1z_2H) \in E_{j_{k-1}}\text{for some } j_{k-1}.$$

The above statements tells us that

$$(HS_{i_1}H)(HS_{i_2}H)\cdots(HS_{i_k}H) \subseteq (HS_{j_1}H) \text{ for some } j_1;$$
$$(HS_{i_1}H)(HS_{i_2}H)\cdots(HS_{i_{k-1}}H) \subseteq (HS_{j_2}H) \text{ for some } j_2;$$
$$\vdots$$
$$(HS_{i_1}H)(HS_{i_2}H) \subseteq (HS_{j_{k-1}}H) \text{ for some } j_{k-1}.$$

Conversely, assume that the above conditions holds. Let $x_1H, x_2H, \ldots, x_nH \in G/H$ such that $(x_1H, x_2H) \in E_{i_1}, (x_2H, x_3H) \in E_{i_2}, \ldots, (x_{k-1}H, x_nH) \in E_{i_k}$. Then

$$x_2 = x_1t_1, x_3 = x_2t_2, \ldots, x_k = x_{k-1}t_{k-1}$$

for some $t_i \in HS_{i_1}H$.

The above equations can be written as:

$$x_3 = x_1(t_1t_2)$$

$$x_4 = x_1(t_1t_2t_3)$$

$$\vdots$$

$$x_{k-1} = x_1(t_1t_2\cdots t_n)$$

The above equations tells as that $(x_1H, x_3H) \in E_{i_1}, (x_1H, x_4H) \in E_{i_2}, \ldots, (x_1H, x_{k-1}H) \in E_{i_{k-1}}$. This completes the proof.

Proposition 2.11 *$\mathscr{C}$ is an E_i-k- transitive if and only if $(HS_iH)^n \subseteq (HS_iH)$ for $n = 2, 3, \ldots, k$.*

Proposition 2.12 *$\mathscr{C}$ is $E_1E_2\cdots E_n$ -complete if and only if $G = \cup HS_iH$.*

Proof. Suppose $\mathscr{C}$ is $E_1E_2\cdots E_n$ complete. Then for every $xH \in G/H$, we have $(H, xH) \in \cup E_i$. This implies that $x \in HS_iH$ for some i. This implies that $G = \cup HS_iH$. Conversely, assume that $G = \cup HS_iH$. Let xH and yH be two arbitrary elements in G/H such that $y = xz$. Then $z \in G$. This implies that $z \in HS_iH$ for some i. That is, $(H, zH) \in \cup E_i$. That is $(xH, xzH) = (xH, yH) \in \cup E_i$. This shows that $\mathscr{C}$ is complete.

Proposition 2.13 *$\mathscr{C}$ is E_i complete if and only if $G = HS_iH$.*

Proposition 2.14 *$\mathscr{C}$ is $E_1E_2\cdots E_n$ connected if and only if $G = [HSH]$.*

Proof. Suppose $\mathscr{C}$is $E_1E_2\cdots E_n$ connected and let $xH \in G/H$.

Let $(H, y_1H, y_2H, \ldots, y_nH, xH)$ be an $E_1E_2\cdots E_n$- path leading from H to xH. Then

$$y_1 \in HS_{i_1}H \quad \text{for some } i_1;$$

$$y_1^{-1}y_2 \in HS_{i_2}H \quad \text{for some } i_2;$$

$$y_2^{-1}y_3 \in HS_{i_3}H \quad \text{for some } i_3;$$

$$\vdots$$

$$y_n^{-1}x \in HS_{i_{n+1}}H \quad \text{for some } i_{n+1}.$$

Note that $x = y_1 y_1^{-1} y_2 y_2^{-1} y_3 \cdots y_n^{-1} x$. Hence from the above equations, we have: $x \in (HS_{i_1}H)(HS_{i_2}H)(HS_{i_3}H) \cdots (HS_{i_n}H) \subseteq [HSH]$. Since x is arbitrary, $G = [HSH]$.

Conversely, assume that $G = [HSH]$. Let x and y be any arbitrary elements in G. Let $y = xz$. Then $z \in G$. That is, $z \in (HS_iH)(HS_jH) \cdots (HS_kH)$ for some $i, j, \ldots$ and k. This implies that $z = s_i s_j \ldots s_k$ for some $i, j \ldots$ and k. Then clearly, $(H, s_iH, s_is_jH, \ldots, s_is_j \ldots s_kH)$ is an $E_1E_2 \cdots E_n$- path from H to zH. That is $(xH, xs_iH, xs_is_jH, \ldots, xs_is_j \ldots s_kH)$ is a $E_1E_2 \cdots E_n$- path from xH to yH. Hence $\mathscr{C}$ is connected.

Proposition 2.15 *$\mathscr{C}$ is E_i connected if and only if $G =< HS_iH >$, where $< HS_iH >$ is the semigroup generated by HS_iH.*

Proposition 2.16 *$\mathscr{C}$ is $E_1E_2 \cdots E_n$ quasi connected if and only if $G = [HSH]^{-1}[HSH]$.*

Proof. First, assume that $\mathscr{C}$ is quasi strongly connected. Let xH be any arbitrary element in G/H. Then there exits a vertex $yH \in G$ such that there is a path from yH to xH, say: $(yH, y_1H, y_2H, \cdots, y_nH, H)$ and a path from yH to H, say: $(yH, x_1H, x_2H, \ldots, x_mH, xH)$. Then we have the following system of equations:

$$
\begin{aligned}
& y^{-1}y_1 \in HS_{i_1}H; \\
& y_1^{-1}y_2 \in HS_{i_2}H; \\
(1) \qquad & y_2^{-1}y_3 \in HS_{i_3}H; \\
& \vdots \\
& y_n^{-1} \in HS_{i_{n+1}}H.
\end{aligned}
$$

and

$$
\begin{aligned}
& y^{-1}x_1 \in HS_{i_1}H; \\
& x_1^{-1}x_2 \in HS_{i_2}H; \\
(2) \qquad & x_2^{-1}x_3 \in HS_{i_3}H; \\
& \vdots \\
& x_m^{-1}x \in HS_{i_{m+1}}H.
\end{aligned}
$$

From equation (1) we obtain the following:

$$y^{-1} = (y^{-1}y_1)(y_1^{-1}y_2)(y_2^{-1}y_3)\cdots(y_n^{-1}) \in S_{i_2} \in (HS_{i_1}H)(HS_{i_2}H)\cdots(HS_{i_{n+1}}H).$$

This implies that

$$y \in (HS_{i_1}^{-1}H)(HS_{i_2}^{-1}H)\cdots(HS_{i_{n+1}}^{-1}H) \in [HS^{-1}H]. \tag{3}$$

Similarly, from equation (2) we obtain the following:

$$y^{-1}x = (y^{-1}x_1)(x_1^{-1}x_2)\cdots(x_m^{-1}x) \in (HS_{i_1}H)(HS_{i_2}H)\cdots(HS_{i_{m+1}}H). \tag{4}$$

That is

$$y^{-1}x \in [HSH].$$

That is

$$x \in y[HSH] \subseteq [HS^{-1}H][HSH].$$

Since x is arbitrary, we have

$$G = [HS^{-1}H][HSH].$$

Conversely, assume that $G = [HS^{-1}H][HSH]$. Let x and y be two arbitrary vertices in G. Let $y = xz$. Then $z \in G$. This implies that $z \in [HS^{-1}H][HSH]$. Then there exits $z_1 \in [HS^{-1}H]$ and $z_2 \in [HSH]$ such that $z = z_1z_2$. $z_1 \in [HS^{-1}H]$ implies that there exits $t_k \in HS_{i_k}H$ such that

$$z_1 = t_1t_2\dots t_n \text{ for some } t_k \in HS_{i_k}^{-1}H, k = 1, 2, \dots, n.$$

This implies that

$$(z_1H, t_1t_2H\dots t_{n-1}, \dots, H)$$

is a path from z_1H to H. That is

$$(yz_1H, yt_1t_2H\dots t_{n-1}H, \dots, yH)$$

is a path from yz_1H to yH.

Similarly, $z_2 \in [HSH]$ implies that there exits $a_k \in S_{i_k}$ such that

$$z_2 = a_1a_2\dots a_m.$$

Observe that

$$(z_2H, a_1a_2H, a_1a_2a_3H, \ldots, H)$$

is a path from z_2H to H. That is,

$$(z_1z_2H, z_1a_1a_2H, a_1a_2a_3H, \ldots, z_1H)$$

is a path from zH to z_1H. That is

$$(yzH, yz_1a_1a_2H, ya_1a_2a_3H, \ldots, z_1H)$$

is a path from xH to z_1H.

Proposition 2.17 *$\mathscr{C}$ is E_i- quasi connected if and only if $G =< HS_i^{-1}H >< HS_iH >$.*

Proposition 2.18 *$\mathscr{C}$ is $E_1E_2\cdots E_n$ - locally connected if and only if $[HSH] = [HS^{-1}H]$.*

Proof.

Assume that $\mathscr{C}$is $E_1E_2\cdots E_n$ - locally connected. Let $x \in [HSH]$. Then $x \in A_m$ for some m. Then $x = s_is_j \ldots s_m$. Let $x_0 = 1, x_1 = s_i, x_2 = s_is_j, \ldots, x_m = s_is_j \ldots s_m$. Then

$$(x_0H, x_1H, x_2H, \ldots, x_mH)$$

is a path leading from 1 to x. Since $\mathscr{C}$ is locally connected, there exits a path from xH to H, say:

$$(xH, y_1H, y_2H, \ldots, y_mH, H)$$

This implies that

$$x^{-1}y_1 \in S_{i_1}$$

$$y_1^{-1}y_2 \in S_{i_2}$$

$$\vdots$$

$$y_m^{-1} \in S_{i_n}$$

The above equations tells us that $x^{-1} \in [HSH]$. That is $x \in [HS^{-1}H]$. Hence $[HSH] = [HS^{-1}H]$. Conversely, if $[HSH] = [HS^{-1}H]$, one can easily verify that $\mathscr{C}$ is $E_1E_2\cdots E_n$ - locally connected.

Proposition 2.19 *$\mathscr{C}$ is E_i- locally connected if and only if $< HS_i^{-1}H >=< HS_iH >$.*

Proposition 2.20 *$\mathscr{C}$ is $E_1E_2\cdots E_n$- semi connected if and only if $G = [HSH] \cup [HS^{-1}H]$.*

Proof. Assume that $\mathscr{C}$ is $E_1E_2\cdots E_n$ - semi connected and let $xH \in G/H$. Then there is a path from H to xH, say

$$(H, x_1H, x_2H, \cdots, x_nH, xH)$$

or a path from xH to H, say

$$(xH, y_1H, y_2H, \cdots, y_mH, H)$$

This implies that $x \in [HSH]$ or $x \in [HS^{-1}H]$. This implies that $G = [HSH] \cup [HS^{-1}H]$. Similarly, if $G = [HSH] \cup [HS^{-1}H]$, then one can prove that $\mathscr{C}$ is $E_1E_2\cdots E_n$- semi connected.

Proposition 2.21 *$\mathscr{C}$ is E_i - semi connected if and only if $G =< HS_iH > \cup < HS_i^{-1}H >$.*

Proposition 2.22 *$\mathscr{C}$ is an $E_1E_2\cdots E_n$- quasi ordered set if and only if*

$$(i) 1 \in (HS_1H) \cup (HS_2H) \cdots \cup (HS_nH),$$

$$(ii) \text{for every} (i, j), HS_iHS_jH \subseteq HS_kH \text{ for some } k.$$

Proposition 2.23 *$\mathscr{C}$ is an E_i quasi ordered set if and only if*

$$(i) 1 \in HS_iH,$$

$$(ii) (HS_iH)^2 \subseteq HS_iH.$$

Proposition 2.24 *$\mathscr{C}$ if an $E_1E_2\cdots E_n$- partially ordered set if and only if*

$$(i) 1 \in (HS_1H) \cup (HS_2H) \cdots \cup (HS_nH),$$

$$(ii) \text{for every} (i, j), (HS_iH)(HS_jH) \subseteq (HS_kH) \text{ for some } k,$$

$$(iii) \cup (HS_iH) \cap (HS_i^{-1}H) = \{1\}.$$

Proof. Observe that

$$x \in \cup(HS_iH) \cap H(S_i)^{-1}H \Leftrightarrow x \in (HS_iH) \cap (HS_i^{-1}H) \text{ for some } i$$

$$\Leftrightarrow x \in HS_iH \text{ and } x \in HS_i^{-1}H$$

$$\Leftrightarrow (H, xH) \in E_i \text{ and } (xH, H) \in E_i$$

$$\Leftrightarrow x = 1.$$

Proposition 2.25 *$\mathscr{C}$ if an E_i partially ordered set if and only if*

$$(i) 1 \in HS_iH,$$

$$(ii)(HS_iH)^2 \subseteq HS_iH$$

$$(iii)(HS_iH) \cap (HS_i^{-1}H) = \{1\}$$

Proposition 2.26 *Let $A_m (m \geq 2)$ is the set of m products of the form $S_{i_1}S_{i_2}\cdots S_{i_m}$. Then $\mathscr{C}$ is an $E_1E_2\cdots E_n$ - hasse diagram if and only if $C \cap S_i = \emptyset$ for all i and for all $C \in A_m$.*

Proof. Suppose the condition holds. Let $x_0H, x_1H, \ldots, x_mH$ be $(m+1)$ elements in G/H such that $(x_iH, x_{i+1}H) \in \cup E_i$ for $i = 0, 1, \ldots, m-1$. This implies that

$$x_0^{-1}x_1 \in S_{i_1};$$

$$x_1^{-1}x_2 \in S_{i_2};$$

$$x_2^{-1}x_33 \in S_{i_3};$$

$$\vdots$$

$$x_{m-1}^{-1}x_m \in S_{i_m}.$$

The above equation tells us that $x_0^{-1}x_m \in A_m$. Since $C \cap S_i = \emptyset$ for all i and for all $C \in A_m$, $(x_0, x_m) \notin \cup E_i$.

Conversely assume that $\mathscr{C}$ is an $E_1E_2\cdots E_n$ hasse diagram. We will show that $C \cap S_i = \emptyset$ for all i and for all $C \in A_m$. Let $S_{i_1}S_{i_2}S_{i_3}\cdots S_{i_m}$ be any element in A_m. Let $x \in S_{i_1}S_{i_2}S_{i_3}\cdots S_{i_m}$. Then $x = s_{i_1}s_{i_2}s_{i_3}\ldots s_{i_n}$ for some $s_{i_k} \in S_{i_k}$. This implies that

$$(H, s_{i_1}H, s_{i_2}s_{i_3}H, \ldots, xH)$$

is a path from H to xH. Since $\mathscr{C}$ is an $E_1E_2\cdots E_n$ hasse- diagram, $x \notin S_i$ for any i. That is, $A_m \cap S_i = \emptyset$ for all i.

Proposition 2.27 *The $E_1E_2\cdots E_n$ out-degree of $\mathscr{C}$ is the cardinal number $|S_1 \cup S_2 \cup \cdots \cup S_n/H|$.*

Proof. Since $\mathscr{C}$ is vertex- transitive it suffices to consider the out degree of the vertex $H \in G/H$. Observe that

$$\begin{aligned}\rho(H) &= \{uH : (H, uH) \in E\}\\ &= \{uH : u \in HS_iH \text{ for some } i\}\\ &= (HS_1H) \cup (HS_2H) \cup \cdots \cup (HS_nH)/H\end{aligned}$$

Hence $|\rho(H)| = |(HS_1H) \cup (HS_2H) \cup \cdots \cup (HS_nH)/H|$.

Proposition 2.28 *The E_i out-degree of $\mathscr{C}$ is the cardinal number $|HS_iH/H|$.*

Proposition 2.29 *The $E_1E_2\cdots E_n$ in-degree of $\mathscr{C}$ is the cardinal number $|(HS_1^{-1}H) \cup (HS_2^{-1}H) \cup \cdots \cup (HS_n^{-1}H)/H|$.*

Proof. Since $\mathscr{C}$ is vertex- transitive it suffices to consider the in degree of the vertex $H \in G/H$. Observe that

$$\begin{aligned}\sigma(H) &= \{uH : (uH, H) \in E\}\\ &= \{uH : (uH, H) \in E_i\}\\ &= \{uH : u^{-1} \in HS_iH\}\\ &= \{uH : u \in HS_i^{-1}H\}\end{aligned}$$

Hence $|\sigma(H)| = |(HS_1^{-1}H) \cup (HS_2^{-1}H) \cup \cdots \cup (HS_n^{-1}H)/H|$.

Proposition 2.30 *The E_i in-degree of $\mathscr{C}$ is the cardinal number $|HS_i^{-1}H/H|$.*

Proposition 2.31 *For $k = 1, 2, 3, \ldots$ let A_k be the set of all k products of the form $(HS_{i_1}H)(HS_{i_2}H)\cdots(HS_{i_k}H)$. If $\mathscr{C}$ has finite diameter, then the diameter of $\mathscr{C}$ is the least positive integer m such that*

$$G = A_m$$

Proof. Let m be the smallest positive integer such that $G = A_m$. We will show that the diameter of $\mathscr{C}$ is m. Let xH and yH be any two arbitrary elements in G such that $y = xz$. Then $z \in G$. This implies that $x \in A_m$. But then z has a representation of the form $x = s_{i_1}s_{i_2}\cdots s_{i_m}$. This implies that

$$(H, s_{i_1}H, s_{i_1}s_{i_2}H, \ldots, zH)$$

is path of m edges from H to zH. That is

$$(xH, xs_{i_1}H, xs_{i_1}s_{i_2}H, \ldots, yH)$$

is a path of length m from xH to yH. This shows that $d(xH, yH) \leq n$. Since xH and yH are arbitrary,

$$\max_{xH,yH\in G}\{d_{1,2,\cdots,n}(xH, yH)\} \leq m$$

Therefore the diameter of $\mathscr{C}$ is less than or equal to n. On the other hand let the diameter of $\mathscr{C}$ be k. Let $x \in G$ and $d_{1,2,\cdots,n}(H, xH) = k$. Then we have $x \in B$ for some $B \in A_k$. That is

$$G = A_k$$

Now by the minimality of k, we have $m \leq k$. Hence $k = m$.

Proposition 2.32 *The vertex H is an $E_1E_2\cdots E_n$- source of $\mathscr{C}$ if and only if $G = [HSH]$.*

Proof. First, assume that H is an $E_1E_2\cdots E_n$ -source of $\mathscr{C}$. Then for any vertex $xH \in G/H$, there is an $E_1E_2\cdots E_n$- path from H to xH. This implies that $G = [HSH]$. Conversely, if $G = [HSH]$, one can prove that H is an $E_1E_2\cdots E_n$- source.

Proposition 2.33 *The vertex H is an E_i- source of $\mathscr{C}$ if and only if $G =< HS_iH >$.*

Proposition 2.34 *The vertex H is an $E_1E_2\cdots E_n$- sink of $\mathscr{C}$ if and only if $G = [HS^{-1}H]$.*

Proof. First, assume that H is an $E_1E_2\cdots E_n$ -sink of $\mathscr{C}$. Then for each $xH \in G/H$, there is an $E_1E_2\cdots E_n$- path from xH to H. This implies that $x \in [HS^{-1}H]$. Hence $G = [HS^{-1}H]$.

Conversely, if $G = [HS^{-1}H]$, one can easily prove that H is an $E_1E_2\cdots E_n$ -sink of $\mathscr{C}$.

Proposition 2.35 *The vertex H is an E_i sink of $\mathscr{C}$ if and only if $G =< HS_i^{-1}H >$.*

Proposition 2.36 *The $E_1E_2\cdots E_n$- reachable set $R_{1,2,\ldots,n}(H)$ of the vertex H is the set $[HSH]$.*

Proof. By definition,

$$R(H) = \{xH : \text{there exits an } E_1E_2\cdots E_n \text{ - path from } H \text{ to } xH\}$$

Observe that

$$xH \in R_{1,2,\ldots,n}(H) \Leftrightarrow \text{there exits an } E_1E_2\cdots E_n \text{ -path from } H \text{ to } xH, \text{ say}$$

$$(H, x_1H, x_2H, \ldots, x_nH, xH)$$

$$\Leftrightarrow x \in [HSH]$$

Therefore, $R_{1,2,3,\cdots,n}(H) = [HSH]$.

Proposition 2.37 *The E_i reachable set $R_i(H)$ of the vertex H is the set $< S_i >$.*

Proposition 2.38 *The $E_1E_2\cdots E_n$- antecedent set $Q_{1,2,\ldots,n}(H)$ of the vertex H is the set $[HS^{-1}H]$.*

Proof. *Observe that*

$$x \in Q_{1,2,\ldots,n}(H) \Leftrightarrow \textit{there exits an } E_1E_2\cdots E_n \textit{ path from } xH \textit{ to } H, \textit{ say}$$

$$(xH, x_1H, x_2H, \ldots, x_nH, H)$$

$$\Leftrightarrow x \in [HS^{-1}H].$$

$$\therefore \quad Q_{1,2,\ldots,n}(H) = [HS^{-1}H].$$

Proposition 2.39 *The E_i antecedent set $Q_i(H)$ of the vertex H is the set $< HS_i^{-1}H >$.*

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