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CONNECTED k-FORCING SETS OF GRAPHS AND SPLITTING GRAPHS

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Abstract. The notion of k-forcing number of a graph was introduced by Amos et al. For a given graph G and a given subset I of the vertices of the graph G, the vertices in I are known as initially colored black vertices and the vertices in $V(G)-I$ are known as not initially colored black vertices or white vertices. The set I is a k-forcing set of a graph G if all vertices in G eventually colored black after applying the following color changing rule: If a black colored vertex is adjacent to at most k-white vertices, then the white vertices change to be colored black. The cardinality of a smallest k-forcing set is known as the k-forcing number $Z_k(G)$ of the graph G. If the sub graph induced by the vertices in I are connected, then I is called the connected k-forcing set. The minimum cardinality of such a set is called the connected k-forcing number of G and is denoted by $Z_{ck}(G)$. This manuscript is intended to study the connected k-forcing number of graphs and the splitting graphs.

Keywords: zeroforcing number; k-forcing number; connected k-forcing number.

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1. Introduction

Through out this manuscript, we consider graphs without loops and multiple edges. That is we consider only simple graph $G=(V,E)$ with vertex set $V(G)$ and edge set $E(G)$. The

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splitting graph $\mathbb{S}(G)$ of a graph G is the graph derived from a simple graph G by taking a vertex v' corresponds to each vertex $v \in G$ and join $v^{'}$ to all vertices which are adjacent to v. The concept of Splitting graph was first defined by E. Sampathkumar et al. in [14]. In [5] and [4] the authors studied about the zero forcing number of the splitting graph of a graph and the k-forcing number of graphs and their splitting graphs.

Zero forcing number of graphs were introduced by the AIM Special Work Group (See[11]). The zero forcing number have applications in power network monitoring [10] and quantum physics [3].

In this paper, we introduce the concept of connected k-forcing number. This can be regarded as a generalization of connected zero forcing number.

Definition 1. *k-color-changing rule: Let G be a graph in which each vertex is colored either black or white. If a black colored vertex has at most k white neighbors, then change the colors of k white neighbors to black. When the k-color changing rule is applied to an arbitrary vertex v to alter the colors of some vertices* $w_1, w_2, \ldots, w_k$ *to black, then we say the vertex v k-forces the vertices* $w_1, w_2, \ldots, w_k$ *and we denote it as* $v \to w_1, v \to w_2, \ldots, v \to w_k$.

The k-forcing number of a graph was introduced by D Amos, Y Caro, R Davila and R Pepper in [1].

Definition 2. *A k-forcing set of a graph G is a subset* Z_k *of vertices such that if at first the vertices in* Z_k *are colored black and* $V(G) - Z_k$ *are colored white, the whole graph G may be colored black by continuously applying the k-color changing rule. The k-forcing number of G, denoted by* $Z_k(G)$*, is the minimum cardinality of a k-forcing set in G. If the subgraph induced by the vertices in* Z_k *(that is* $\langle Z_k \rangle$*) is connected, then* Z_k *is known as the k-conneted zero forcing set. The minimum size of such a set is called the connected k-forcing number of G and is denoted by* $Z_{ck}(G)$.

The connected zero forcing set was introduced by M. Khosravi, S. Rashidi and A. Sheikhhosseni (See[13]). When $k = 1$, the definition of connected 1-forcing set is equivalent to the definition of connected zero forcing set, $Z_c(G)$ (See [11]). In this article, we deal with connected k-forcing number of some graphs and their splitting graphs. We use the following definitions for the further development of this article.

- Corona Product: For any two graphs G and H, the Corona product $G \circ H$ of the graphs G and H is the graph detemined by taking one copy of G and $\mid V(G) \mid$ copies of H and by connecting each vertex of the j^{th} copy of H to the j^{th} vertex of G, $1 \leq j \leq\mid V(G) \mid$.
- Rooted Product: Let G be a connected graph with vertices $v_1, v_2, \ldots, v_n$ and let H be a sequence of n-rooted graphs $H_1, H_2, ..., H_n$. The rooted product of G and H is defined as the graph obtained by identifying the root of $H_i, 1 \leq i \leq n$ with the i^{th} vertex of G for all i. This graph is denoted by $G(H)$ and is known as the rooted product of G by H (See[8]).
- Square of a Graph: Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Then the square of G, denoted by G^2, is the graph having the vertex set same as that of G and such that two vertices in G^2 are adjacent if the distance between them is at most two in G.
- When the k-color changing rule is applied to an arbitrary vertex u to change the color of the vertex v, we say u, k-forces (if it is zero forcing, then we say u forces v) v and write $u \to v$.

For more definitions on graphs, we refer to [9]. From the definitions above, we have the following proposition.

Proposition 3. *Let P_n, $n \geq 3$ be a path on n vertices. Then*

$$Z_{ck}[\mathbb{S}(P_n)] = \begin{cases} 3 \; if \, k = 1 \\ \\ 1 \; if \, k \geq 2 \end{cases}$$

Proof. **Case 1** Assume that $k = 1$. It can be easily observed that if we color any two adjacent vertices as black, it is not possible to obtain a derived coloring. Therefore, $Z_{c1}[\mathbb{S}(P_n)] \geq 3$. Now, let $u_1, u_2, \ldots, u_n$ be the vertices of P_n and $u_1^{'}, u_2^{'}, \ldots, u_n^{'}$ be the corresponding vertices in $\mathbb{S}(P_n)$. Color the vertices u_1, u_2 and $u_1^{'}$ as black. Clearly, the vertex u_1 forces $u_2^{'}$ as black, the vertex $u_2^{'}$ forces the vertex u_3 as black, the vertex u_2 forces the vertex $u_3^{'}$ as black and so on. Therefore,

$Z = \{u_1, u_2, u_1^{'}\}$ forms a connected zero forcing set for the path P_n. So, $Z_{c1}\mathbb{S}(P_n) \leq 3$. Hence the result follows.

Case 2 Assume that $k \geq 2$. In this case, the vertex u_1 forms a connected zero forcing set and hence the result follows. □

It can be observed that any conncetd k-forcing set is a k-forcing set. Therefore, we have the following

Proposition 4. *For any simple graph G, and for any fixed k, $Z_k(G) \leq Z_{ck}(G)$, where Z_k is the k-forcing number and $Z_{ck}(G)$ is the connected k-forcing number of G.*

We consider the next proposition from [5] to prove the result concerning the splitting graph of the cycle C_n.

Proposition 5 ([5])**.** *If G is the cycle C_n on $n \geq 4$ vertices, then $Z[S(G)] = 4$.*

In the succeeding proposition, we consider the spliting graph of the cycle $C_n, n \geq 4$.

Proposition 6. *Let $\mathbb{S}(C_n)$ be the splitting graph of the cycle C_n. Then*

$$Z_{ck}[\mathbb{S}(C_n)] = \begin{cases} 4 \text{ } if\, k = 1 \\ 3 \text{ } if\, k = 2 \\ 1 \text{ } if k \geq 3 \end{cases}$$

Proof. **Case 1** Assume that $k = 1$. From Proposition-4 and Proposition-5, we have the following:

$$4 \leq Z_{c1}[\mathbb{S}(C_n)]$$

To prove the reverse part, let us consider the vertices of the cycle C_n as $v_1, v_2, \ldots, v_n$ and $v'_1, v'_2, \ldots, v'_n$ be the corresponding vertices of $v_1, v_2, \ldots, v_n$ in $\mathbb{S}(C_n)$. Consider the set of vertices $\{v_1, v'_1, v_2, v'_2\}$ as black. Now the vertex $v'_2 \to v_3$ to black, the vertex $v_2 \to v'_3$ to black, the vertex $v_3' \to v_4$ to black and so on. Therefore, we can obtain a derived coloring with the set of black vertices $\{v_1, v_1', v_2, v_2'\}$. Clearly,

$$4 \geq Z_{c1}[\mathbb{S}(C_n)]$$

Hence the result follows.

Case 2 Let us assume that $k = 2$ and $Z_{c2}[\mathbb{S}(C_n)] = 2$. Consider a connected 2-forcing set consisting of two vertices. Let u and v be the two adjacent vertices in the connected 2-forcing set of $\mathbb{S}(C_n)$. Then we have two sub cases:

Subcase 2.1 $deg(u) = deg(v) = 4$. Since u and v are adjacent to three white neighbors, color changing rule is not applicable in this case, we get a contradiction to our assumption that $Z_{c2}[\mathbb{S}(C_n)] = 2$.

Subcase 2.2 $deg(u) = 2$ and $deg(v) = 4$. In this case the vertex u can force one more adjacent vertex of degree 4 to black. Therefore, in this case it is not possible to obtain a derived coloring. Hence from subcases 2.1 and 2.2, we have $Z_{c2}[\mathbb{S}(C_n)] \geq 3$.
It can be easily observed that the vertices $\{v_1, v_1', v_2\}$ forms a zero forcing set for $\mathbb{S}(C_n)$ and hence the result follows. For $k = 3$ the result is obvious. □

The Friendship graph F_p is the graph obtained by identifying p copies of the cycle graph C_3 with a common vertex.

Proposition 7. *Let F_p denote the friendship graph with $p \geq 2$ triangles. Then* $Z_{ck}(F_p) =$

$$\begin{cases} p+1-\frac{k}{2} \; if \; k \; is \; even \; and \; k < \Delta - 2 \\ \frac{2p-k+3}{2} \; if \; k \; is \; odd \; and \; k < \Delta - 2 \\ 1 \; if \; k \geq \Delta - 2 \end{cases}$$

Proof. **Case 1** Assume that k is even and $k < \Delta - 2$. Let v be the vertex with maximum degree Δ. It can be noted that v should be a member of any connected zeroforcing set. Otherwise, the zero forcing set will not be connected. Therefore, assume that the vertex

v is there in any connected zero forcing set of F_p. If we take one vertex from each of the $p-\frac{k}{2}-1$ triangles, then it is not possible to obtain a derived coloring since $deg(v)=2p$ and by using color changing rule we get $2(p-\frac{k}{2}-1)=2p-k-2$ black vertices which are adjacent to the vertex v. Now we have $2p-(2p-k-2)=k+2$ white vertices remains. It is not possible to force these $k+2$ vertices by using the vertex v. Therefore, we must take one black vertex from each of the $p-\frac{k}{2}$ triangles since the vertex v is black, $p+1-\frac{k}{2}\leq Z_{ck}(F_p)$.

Let us take one vertex from each of the $p-\frac{k}{2}$ triangles as black. Since the vertex v is black, these $p-\frac{k}{2}$ vertices will force the remaining vertices in the $p-\frac{k}{2}$ triangles as black. Now we have $2(p-\frac{k}{2})$ black vertices together with the black vertex v in the connected k-forcing set. It can be observed that at this stage we have $2p-(2p-k)=k$ white vertices adjacent to the vertex v. Now the vertex v can force these k-vertices as black. Therefore we get a derived coloring with $p-\frac{k}{2}+1$ black vertices. Hence $Z_{ck}(F_p)\leq p-\frac{k}{2}+1$.

Case 2 Assume that k is odd and $k<\Delta-2$. Let v be the vertex with maximum degree Δ. It can be noted that v should be a member of any connected zeroforcing set. Otherwise, the zero forcing set will not be connected. Therefore, assume that the vertex v is there in any connected zero forcing set of F_p. Now let us assume that there exist a zero forcing set consisting of $\frac{2p-k+1}{2}$ vertices. Since the vertex v is black we can distribute the remaining $\frac{2p-k-1}{2}$ vertices along the triangles. To force the maximum number of vertices as black, we need to distribute one black vertex for each $\frac{2p-k-1}{2}$-triangles. Now we have $2(\frac{2p-k-1}{2})+1=2p-k$ black vertices and $2p+1-(2p-k)=k+1$ white vertices. All these white vertices are adacent to v. Therefore, color changing rule is not applicable since $k+1$ white vertices are adjacent to the black v. Therefore, $\frac{2p-k+3}{2}\leq Z_{ck}(F_p)$.

Let us take one vertex from each of the $\frac{2p-k+3}{2}-1$ triangles as black. Since the vertex v is back, these $\frac{2p-k+3}{2}-1$ will force the remaining vertices in the $\frac{2p-k+3}{2}-1$ triangles as black. At this stage we have $2(\frac{2p-k+3}{2}-1)+1=2p-k+2$ black vertices remains. Therefore the total number of white vertices remains in this stage is $2p+1-(2p-k+2)=k-1$. All these $k-1$ white vertices are adjacent to v. Therefore, v k forces all these $k-1$ white vertices as black. Hence $\frac{2p-k+3}{2} \geq Z_{ck}(F_p)$.

It can be easily obseved that if $k \geq \Delta - 2$, then $Z_{ck}(F_p) = 1$. □

Theorem 8. *Let G be a connected graph with $\mid V(G) \mid = p_1$ and let H be another connected graph with $Z_{ck}(H) = p_2$. Let $\mathscr{G}$ be the graph obtained by taking the corona product of G and H, that is $\mathscr{G} \equiv G \circ H$. Then $Z_{ck}(\mathscr{G}) \leq p_1(1+p_2)$.*

Proof. With out loss of generality, assume that G is connected, $\mid V(G) \mid = p_1$ and $Z_{ck}(H) = p_2$. Color all vertices of G black. To form the k-forcing set for the sub graph induced by $v_1 \cup H_1$, we need a maximum of $1+p_2$ black vertices. That is, $Z_k(\langle v_1 \cup H_1 \rangle) \leq 1+p_2$, where H_1 is the first copy of H corresponds to the vertex v_1 in $\mathscr{G}$. $Z_k(\langle v_2 \cup H_2 \rangle) \leq 1+p_2$, where H_2 is the second copy of H corresponds to the vertex v_2 in $\mathscr{G}$. Proceeding like this, we can observe that $Z_k(\langle v_{p_1} \cup H_{p_1} \rangle) \leq 1+p_2$. Now the graph $\mathscr{G} \equiv \langle v_1 \cup H_1 \rangle \cup \langle v_2 \cup H_2 \rangle \cup, \ldots, \cup \langle v_{p_1} \cup H_{p_1} \rangle$. Therefore, $Z_k(\mathscr{G}) \leq (1+p_2)+(1+p_2)+\ldots+(1+p_2)$-$p_1$ times. This follows that $Z_k(\mathscr{G}) \leq p_1(1+p_2)$. Since each vertex in G is connected to the vertices of all copies of H, the k-forcing set obtained here forms a connected k-forcing set. Therefore, $Z_{ck}(\mathscr{G}) \leq p_1(1+p_2)$. □

Proposition 9. *Let G be the complete bipartite graph $K_{m,n}$, and $n \geq 2, m \geq 2$. Then the connected zero forcing number of G is $m+n-2$. That is, $Z_c(G) = m+n-2$.*

Proof. Since G is a complete bipartite graph, therefore, the vertex set of G can be partitioned into two sets X and Y. Let $u_1, u_2, \ldots, u_m$ be the vertices in X and $v_1, v_2, \ldots, v_n$ be the vertices in Y. Note that the vertices in X are non-adjacent. The vertices in Y are also non-adjacent. To start the color changing rule , color any vertex, say u_1, in X as black. Since each vertex in X is connected to every vertex in Y, we have to color $n-1$ vertices in Y as black. Let the only white vertex in Y be v_n. Now $u_1 \to v_n$ to black. In X there are $m-1$ white vertices. Each vertex in Y is joined to $m-1$ white vertices in X. Assign black color to $m-2$ white vertices in X. Then any black vertex in Y, say v_1 forces the remaining white vertex in X as black. Now the zero forcing set consists of $1+m-2+n-1$ black vertices, which are connected. Hence the connected zero forcing number of G is $m+n-2$. That is, $Z_c(G) = m+n-2$. □

We use the following results from [2] and [11] to prove the next result

Proposition 10. [2] *For any connected graph G, $Z(G) \leq Z_c(G)$, where $Z(G)$ is the zero forcing number of G.*

Proposition 11. [11] *Let G be the graph obtained by taking the Cartesian product of the cycle C_n with the path P_m. Then $Z(G) = min\{n, 2m\}$*

Proposition 12. *Let G be the graph obtained by taking the Cartesian product of the cycle C_n with the path P_m and let $n \geq 2m$. Then $Z_c(G) = 2m$.*

Proof. Let v_1 and v_2 be the two adjacent vertices in the cycle C_n. Let $A = \{v_1^1, v_1^2, \ldots, v_1^m\}$ be the vertices corresponding to the vertex v_1 in G and let $B = \{v_2^1, v_2^2, \ldots, v_2^m\}$ be the vertices corresponding to the vertex v_2 in G. Now consider the vertices in the set $A \cup B$ and color these vertices as black in G. The vertices in $A \cup B$ forces the remaining vertices in G as black. Clearly these vertices are connected in G and thus forms a connected zero forcing set in G. Hence

$$Z_c(G) \leq 2m \tag{1}$$

Also we have from proposition-10 and proposition-11 that

$$Z_c(G) \geq 2m \tag{2}$$

From (1) and (2) the result follows. □

Proposition 13. *Let G be the star graph $k_{1,n}$ on $n+1$ vertices and $n > 2$. Then $Z_c(G) = n$. In general, if $n \geq k \geq 2$, then $Z_{ck}(G) = n-k+1$.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of the star graph $k_{1,n}$ with degree 1. Assume that v is the vertex having degree n. We generate the connected zero forcing set as follows. Since $deg(v) = n$, to apply the color changing rule, we have to color $n-1$ vertices in G adjacent to v as black. Then v forces the only remaining white vertex to black. Therefore, $Z_c(G) = n-1+1 = n$. If $k = 2$,we can easily show that the connected zero forcing number of G is $n-2+1 = n-1$. Proceeding like this, we obtain $Z_{ck}(G) = n-k+1$ for any positive integer $n \geq k \geq 2$. □

2. CONNECTED k-FORCING NUMBER OF ROOTED PRODUCT OF GRAPHS

In this section, we deal with the connected k-forcing number of rooted product of cycle with paths, cycle with cycles.

Proposition 14. *Let $P_1, P_2, \ldots, P_n$ be n-paths (each path is of length $n \geq 3$) rooted at the pendant vertex and C_n be a cycle on $n \geq 3$ vertices. Let G be the graph obtained by taking the rooted product of the cycle C_n with the paths $P_1, P_2, \ldots, P_n$. Then*

$$Z_{ck}(G) = \begin{cases} n \; if \; k = 1 \\ 1 \; if \; 2 \leq k \leq \Delta, \end{cases}$$

where $\Delta(G) = 3$.

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle $C_n, n \geq 3$ and $P_1, P_2, \ldots, P_n$ be the paths rooted at the vertices $u_1, u_2, \ldots, u_n$ respectively. Each path is of length $n, n \geq 3$.

Represent the vertices of P_1 by $p_1^1, p_2^1, \ldots, p_n^1$, the vertices of P_2 by $p_1^2, p_2^2, \ldots, p_n^2$ and the vertices of P_n by $p_1^n, p_2^n, \ldots, p_n^n$. Let u_1 be the vertex identified with the vertex p_1^1 in G, u_2 be the vertex identified with the vertex p_1^2 in $G., \ldots, u_n$ be the vertex identified with the vertex p_1^n.

Case 1. Assume that $k = 1$. This case is similar to that of the connected zero forcing number of G. Color the vertices $u_1, u_2, \ldots, u_n$ in G black. Now one can easily infer that

$$Z_c(G) \leq n \tag{3}$$

It can be worth mentioning that if we start the color changing rule with vertices of $P_i, 1 \leq i \leq n$ other than the vertices identified with the vertices $u_1, u_2, \ldots, u_n$ of C_n, we cannot obtain a connected zero forcing set with at least n black vertices. Therefore, we need to consider the vertices in the cycle to force the remaining vertices in G.

Now assume that we have a connected zero forcing set consisting of $n-1$ black vertices. From the above it can be noted that these vertices must be from the cycle C_n. Without loss of generality, assume that the vertices are $u_1, u_2, \ldots, u_{n-1}$. Clearly the black vertex u_2

can force the vertices of the path P_2, u_3 can force the vertices of the path P_3, ..., the vertex u_{n-2} can force the vertices of the path P_{n-2}. Since the black vertex u_1 is adjacent to two white vertices u_n and p_2^1, u_1 cannot force the vertices u_n and p_2^1. Similarly the vertex u_{n-1} is adjacent to two white vertices u_n and p_2^{n-1}. Therefore, the vertex u_{n-1} cannot force u_n and p_2^{n-1}, this contradicts our assumption that $Z_c(G) = n-1$. Therefore,

$$Z_c(G) \geq n \tag{4}$$

Now from (3) and (4) the result follows.

Case 2. Assume that $k \geq 2$. In this case, if we consider any pendant vertex of G as a black vertex , then it can force the remaining white vertices of G as black. Hence $Z_{ck}(G) = 1$. □

Proposition 15. *Let $D_1, D_2, \ldots, D_n$ be the cycles C_n of order $n \geq 3$ rooted at a vertex and C_n be another cycle of order $n > 3$. Let G be the graph derived from the rooted product of C_n with the cycles $D_1, D_2, \ldots, D_n$. Then*

$$Z_{ck}(G) = \begin{cases} 2n \; if \; k = 1 \\ n \; if \; k = 2 \\ 1 \; if \; 3 \leq k \leq \Delta, \end{cases}$$

where $\Delta(G) = 4$.

Proof. Without loss of generality, assume that $u_1, u_2 \ldots, u_n$ be the vertices of the cycle C_n in G and let $D_1, D_2, \ldots, D_n$ be the cycles rooted at $u_1, u_2, \ldots, u_n$ respectively. Represent the vertices of the cycle D_1 in G by $d_1^1, d_2^1, \ldots, d_n^1$. Similarly the vertices of D_2 in G by $d_1^2, d_2^2, \ldots, d_n^2$ and the vertices of D_n by $d_1^n, d_2^n, \ldots, d_n^n$. Assume that the vertex d_1^1 be rooted at u_1 , the vertex d_1^2 be rooted at u_2, ..., the vertex d_1^n be rooted at u_n.

Case 1. Let us suppose that $k = 1$. This case is similar to that of the connected zero forcing number of G. Color the vertices $u_1, u_2, \ldots, u_n, d_2^1, d_2^2, \ldots, d_2^n$ as black. Now we can easily see that these black vertices forms a connected zero forcing set. Hence

(5) $$Z_c(G) \leq 2n$$

It can be easily infer that to form a minimum connected zero forcing set for G, we need to color the vertices $u_1, u_2, \ldots, u_n$ as black and color at least one vertex from each of the cycles $D_i, 1 \leq i \leq n$ adjacent to each $u_i, 1 \leq i \leq n$ as black, otherwise we cannot form a connected zero forcing set with at least $2n$ black vertices . Clearly,

(6) $$Z_c(G) \geq 2n$$

(5) and (6) concludes the result.

Case 2. Let us suppose that $k = 2$. Color all vertices of C_n in G black. Each vertex $u_i, 1 \leq i \leq n$, is adjacent to exactly two white vertices of D_i and $k = 2$. Therefore, these vertices forms a 2- forcing set for G. The sub graph induced by these black vertices are connected and hence it forms a connected 2-forcing forcing set for G. Therefore,

(7) $$Z_{c2}(G) \leq n$$

It can be easily infer that to form a minimum connected 2- forcing set for G, we need to color the vertices $u_1, u_2, \ldots, u_n$ as black, otherwise we cannot form a connected 2-forcing set with at least n black vertices. Clearly,

(8) $$Z_{c2}(G) \geq n$$

Therefore from (7) and (8), the result follows.

Case 3. Let us suppose that $k \geq 3$. In this case any arbitrary vertex from the cycle $D_i, 1 \leq i \leq n$ will k-forces the remaining vertices as black in G. Therefore, $Z_{ck}(G) = 1$. □

Proposition 16. *Let G be the rooted product of $P_n \square P_2$ (the Ladder graph) with $P_t, t \geq 3$ rooted at the pendant vertex. Then*

$$Z_{ck}(G) = \begin{cases} 2n \;\; if \; k = 1 \\ n \;\; if \; k = 2 \\ 1 \;\; if \; 3 \leq k \leq \Delta(G) \\ where \; \Delta(G) = 4 \end{cases}$$

Proof. Represent the vertices of the graph $P_n \square P_2$ by $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$. Let $P_1, P_2, \ldots, P_n$ be the paths rooted at the vertices $u_1, u_2, \ldots, u_n$ respectively. Also let $Q_1, Q_2, \ldots, Q_n$ be the paths rooted at the vertices $v_1, v_2, \ldots, v_n$ respectively. The vertices of the paths $P_1, P_2, \ldots, P_n$ and $Q_1, Q_2, \ldots, Q_n$ in G can be named as follows:

Consider

$$P_1 = \{p_1^1, p_2^1, \ldots, p_t^1\}, \quad Q_1 = \{q_1^1, q_2^1, \ldots, q_t^1\}$$

$$P_2 = \{p_1^2, p_2^2, \ldots, p_t^2\}, \quad Q_2 = \{q_1^2, q_2^2, \ldots, q_t^2\}$$

$$\cdots \qquad \cdots$$

$$\cdots \qquad \cdots$$

$$\cdots \qquad \cdots$$

$$P_n = \{p_1^n, p_2^n, \ldots, p_t^n\}, \quad Q_n = \{q_1^n, q_2^n, \ldots, q_t^n\}$$

Now color the vertices $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ as black. Clearly these vertices forms a connected zero forcing set for G and hence

$$Z_c(G) \leq 2n \tag{9}$$

Refer Figure 1.

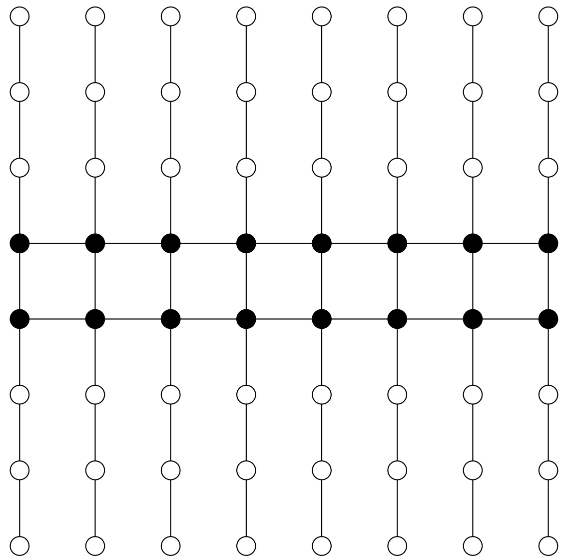

FIGURE 1. Rooted product of $P_8 \square P_2$ with P_4.

There exists three types of minimum connected zero forcing sets with $Z_c(G) = 2n$. Consider these three sets as follows. We denote them as A, B and C

$$A = \{u_1, u_2, \ldots, u_n, v_1, v_2 \ldots, v_n\}$$

$$B = \{u_1, u_2, \ldots, u_n, p_2^1, p_2^2, \ldots, p_2^n\}$$

$$C = \{v_1, v_2, \ldots, v_n, q_2^1, q_2^2, \ldots, q_2^n\}$$

It can be easily observed that if we take 2n vertices other than these three sets, then it will not form a minimum connected zero forcing set. Now Assume that there exists a connected zero forcing set consisting of $2n-1$ black vertices.

Case 1. Consider the black vertices as depicted in Figure 2. Assume that the black vertices are from the set A, except the vertex u_n. Consider the vertex $u_n = u_8$ as white. The blue colored vertices represent the vertices which are forced by the black vertices. In this case, if we consider G, then there are $3t-2$ vertices remain as white. Therefore, we cannot obtain a derived coloring, a contradiction to our assumption that there exists a connected zero forcing set consisting of $2n-1$ vertices. The case is similar if we consider u_1, v_1 and $v_n = v_8$ as white vertices.

Case 2. Consider the black vertices as depicted in Figure 1. If we choose any black vertex other than $u_1, v_1, u_n = u_8, v_n = v_8$ as white, then one can observe that there are $4t-3$ white vertices remains in G, a contradiction to our assumption that $Z_c(G) = 2n-1$.

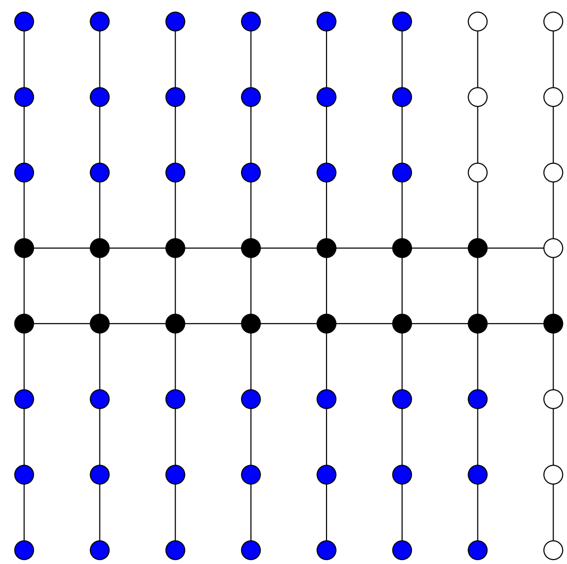

FIGURE 2. Rooted product of $P_8 \square P_2$ with P_4.

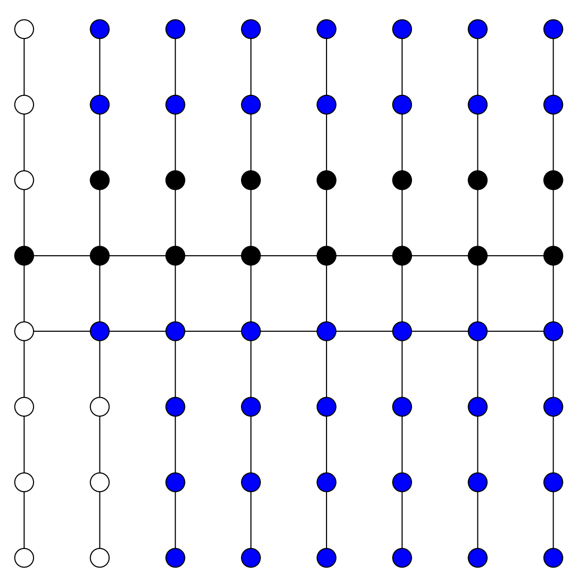

FIGURE 3. Rooted product of $P_8 \square P_2$ with P_4.

Case 3. Consider the black vertices as depicted in Figure- 3. Assume that the black vertices are from the set B, except the vertex p_2^1. Consider the vertex p_2^1 as white. The blue colored vertices represent the vertices which are forced by the black vertices. In this case if we consider the graph G, then there are $3t-2$ vertices remain as white. Therefore, we cannot obtain a derived coloring with $Z_c(G) = 2n-1$, a contradiction. The case is similar if we consider the vertex p_2^n as white.

Sub Case 3.1. Consider the black vertices as depicted in Figure- 3. Assume that the black vertices are from the set B, except one the vertices $p_2^i, 2 \leq i \leq n-1$. Consider the vertex p_2^i as white. In this case if we consider G, then there are $4t-3$ vertices remain as white. Color changing rule is not applicable at this stage, a contradiction to our assumption that $Z_c(G) = 2n-1$.

Sub Case 3.2. Assume that the black vertices are from the set B, except the vertex $u_i, 1 \leq i \leq n$. In this case we loose the connectivity of the zero forcing set. That is the zero forcing set is not connected, again a contradiction.

Case 4. Assume that the black vertices are from the set C, except one. This case is similar to that of Case 3. Since the sub graph induced by the connected zero forcing sets B and C are isomorphic. Combining cases $1,2,3$ and 4,

$$Z_c(G) \geq 2n \tag{10}$$

From (9) and (10), $Z_{ck}(G) = 2n$, if $k = 1$.

Case 5. Let $k > 1$. If we color any one of the pendant vertices from G as black, then the pendant vertex forms a connected zero forcing set for G. Hence $Z_{ck}(G) = 1$ if $1 < k \leq 4$, where $\Delta(G) = 4$. □

Proposition 17. *Let G be the rooted product of $P_n \Box P_n$ (The Grid graph) with $P_t, t \geq 3$ rooted at the pendant vertex. Then*

$$Z_{ck}(G) \begin{cases} \leq n^2 \; if \; k = 1 \\ \leq \; n \; if \; k{=}2 \\ = 1 \; if \; 3 \leq k \leq 5. \end{cases}$$

Proof. **Case 1.** Assume that $k = 1$. In this case color all the vertices of the Cartesian product $P_n \Box P_n$ in G as black. One can easily observe that these n^2- black vertices forms a connected zero forcing set for G. Thus $Z_c(G) \leq n^2$.

Case 2. Assume that $k = 2$. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path P_n in $P_n \Box P_n$ of G. Color these vertices as black in G. Now one can easily verify that these vertices form a connected zero forcing set for G. Thus, $Z_{c2}(G) \leq n$, if $k = 2$.

Case 3. Assume that $3 \leq k \leq 5$. Let P_t be the path identified at the vertex u_1 in G. Now color the pendant vertex of the path P_t in G as black. Let it be the vertex v. Clearly the vertex v forces the remaining vertices in G as black. Therefore we can form a derived coloring for G. Thus $Z_{ck}(G) = 1$, as desired. □

We strongly believe that the bounds in the above proposition is sharp.

Proposition 18. *Let G be the rooted product of $P_n \Box P_2$ with the cycle C_n. Then*

$$Z_{ck}(G) = \begin{cases} 4n \; if \; k = 1 \\ 2n \; if \; k = 2 \\ 1 \; if \; 3 \leq k \leq 5 \end{cases}$$

Proof. **Case 1.** Assume that $k = 1$. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path P_n in G and let $v_1, v_2, \ldots, v_n$ be the vertices corresponding to the copy of the path P_n in G. Note that $deg(u_1) = deg(v_1) = deg(u_n) = deg(v_n) = 4$. The remaining vertices of $P_n \Box P_2$ in G have degree 5. It can be noted that any connected zero forcing set of G must contain all the vertices of $P_n \Box P_2$. Otherwise the zero forcing set will be disconnected. Without loss of generality, assume that we have a set consisting of $2n$ connected black vertices from $P_n\Box P_2$ in G. To force the white vertices in each cycle, we must select a vertex adjacent to the rooted vertex of each $C_i, 1 \leq i \leq 2n$. Therefore we need to choose $2n$ black vertices from the cycle C_n. Now we have a set of 4n black vertices which forces the remaining vertices of G, which is connected. Therefore, $Z_{ck}(G) = 4n$.

Case 2. Assume that $k = 2$. It can be observed that the connected zero forcing set of G must contain all the vertices of $P_n\Box P_2$, Otherwise, the zero forcing set will be disconnected. If we take the $2n$ black vertices of $P_n\Box P_2$ in G, then these black vertices will 2-forces the remaining white vertices as black and hence $Z_{ck}(G) = 2n$.

Case 3. Assume that $5 \geq k \geq 3$. Consider the cycle identified with the vertex u_1, say C_1. Choose a vertex from C_1 of degree 2 as black. This vertex will 3-force the remaining vertices in G as black. Hence $Z_{ck}(G) = 1$. □

Definition 19 ([1]). *A connected graph G is defined as a cycle-path graph (CP-graph) if it contains r vertex disjoint cycles that are connected by $r-1$ edges of the path P_r. Thus a CP-graph with n vertices contains $m = n+r-1$ edges and edge between two cycles is a cut edge.*

The zero forcing number of *CP*- graph was studied in some detail in [5]. Here we study the connected zeroforcing number of the *CP*- graph considered in [5].

Proposition 20. *Let G be the CP-graph $C_3P_r, r \geq 3$. Then $Z_c(G) = 2r$. Moreover $Z_{ck}(G) = 1$ if $k = 2,3$.*

Proof. Denote the cycles by $C_1, C_2, \ldots, C_r$. Let the vertex sets of the cycles in C_3P_r be

$$V(C_1) = \{c_1^1, c_1^2, c_1^3\}$$

$$V(C_2) = \{c_1^2, c_2^2, c_3^2\}$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$V(C_r) = \{c_1^r, c_2^r, c_3^r\}$$

Case 1. Assume that $k = 1$. We prove the result by mathematical induction on the number of cycles r on the CP-graph. Assume that $r = 1$. In this case G is the cycle C_3 therefore, $Z_c(C_3) = 2$ and the result is true for $r = 1$.

Assume that the result is true for all C_3P_r graphs with $r-1$ cycles C_3, where $r \geq 2$. Let $\mathbb{C}$ be the end cycle connected to the rest of the C_3P_r graph by an edge $e = ab$, where $a \in V(C_{3P_r}) - V(\mathbb{C})$ and $b \in V(\mathbb{C})$. Let $Y = \{a, b\}$ be the cut set where $a \in \langle V(C_3P_r) - V(C)\rangle$ and $b \in V(C)$.

Assume that the result is true for the sub graph induced by $\langle V(C_3P_r) - V(C)\rangle$. That is $Z_c(\langle V(C_3P_r) - V(C)\rangle) = 2r - 2 = 2(r-1)$.

Let W be the minimum zero forcing set of $\langle V(C_3P_r) - V(C)\rangle$ with $\mid W \mid = 2r-2$. Let u_1 and u_2 be two white neighbors of the vertex b in $\mathbb{C}$. Since the vertex a is black it forces the vertex b to black. Since the vertex b has two white neighbors, further forcing is not possible. In order to make the zero forcing set connected, we have to include the black vertex b in the connected zero forcing set of G. Therefore, our new connected zero forcing set is $W \cup \{b\}$. The set $W \cup \{b\}$ cannot force the remaining two white vertices (u_1 and u_2) adjacent to b. Therefore, we need to include either u_1 or u_2 in the connected zero forcing set of G. Let it be u_1. Hence by induction

$$Z_c(G) = \mid W \cup \{b, u_1\} \mid$$

$$= 2r - 2 + 2 = 2r.$$

□

Case 2. Assume that either $k = 2$ or $k = 3$. In this case any vertex of degree 2 will form a connected k-forcing set.

The Cartesian product $C_n \Box K_2$ is known as the Prism graph or the circular ladder graph. The length of the shortest cycle in a graph G is called the girth of G. We recall the following observation from [6].

Proposition 21. [6] *Let G be a graph with girth at least 4 and minimum degree* $\delta(G) \geq 3$. *Then* $Z_c(G) \geq \delta(G)+1$.

Proposition 22. *Let G be the circular ladder graph of order* $n \geq 10$. *Then* $Z_c(G) = 4$. *Also,* $Z_{ck}(G) = 2$, *if* $k = 2$, $Z_{ck}(G) = 1$, *if* $k = 3$.

Proof. Let $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ be the vertices of the circular ladder graph G, $u_1, u_2, \ldots, u_n$ being the vertices of the inner circle. By the proposition 21, since $\Delta(G) = \delta(G) = 3$ and the girth is at least 4 ,we have $Z_c(G) \geq 3+1 = 4$.

To establish the reverse inequality, we proceed as follows.
Without loss of generality, choose four vertices u_1, u_2, u_3 and v_1. Allow these vertices to have black color. Then clearly the black vertex $u_2 \to v_2$ to black. Now the black vertex $v_2 \to v_3$ to black. Again,the black vertex $u_3 \to u_4$ to black,$v_3 \to v_4$ to black. Apply color changing rule step by step, the black vertex $u_{n-1} \to u_n$ to black and $v_{n-1} \to v_n$ to black.Hence $Z = \{u_1, u_2, u_3, v_1\}$ forms a connected zero forcing set for G. Here the cardinality of the set Z is 4. So, $Z_c(G) \leq 4$. This concludes the result.

Case 1 Assume that $k = 2$. In this case , clearly a set consisting of any two adjacent black vertices forms a connected zero forcing set for G. Hence, $Z_{c2}(G) = 2$.

Case 2. Assume that $k = 3$. It is obvious that any single black vertex gives a derived coloring for G. Therefore, the result follows. That is $Z_{c3}(G) = 1$. □

Proposition 23. *Let G be the rooted product of the circular ladder graph,* $C_n \Box K_2$ *with the path* P_t, *a path of length t,* $t \geq 4$ *rooted at the pendent vertex. Then,* $Z_c(G) = 2n$.

Proof. Represent the vertices of $C_n \Box K_2$ as $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ in G and the paths rooted at the pendent vertex by $P_1, P_2, \ldots, P_n$ of length t. Let $v_1 = p_1^1$, $v_2 = p_1^2, \ldots, v_n = p_1^n$, where

$p_1^1, p_1^2, \ldots, p_1^n$ are the pendent vertices of the paths identified at the vertices $v_1, v_2, \ldots, v_n$ respectively, where

$$P_1 = \{p_1^1, p_2^1, \ldots, p_t^1\}$$

$$P_2 = \{p_1^2, p_2^2, \ldots, p_t^2\}$$

$$\ldots$$

$$\ldots$$

$$\ldots$$

$$P_n = \{p_1^n, p_2^n, \ldots, p_t^n\}$$

We examine the different possibilities of forming a connected zero forcing set as follows.

Case 1. Assume that we have a connected zero forcing set consisting of $2n-1$ black vertices $\{u_1, u_2, \ldots, u_n v_1, v_2, \ldots, v_{n-1}\}$ for G. Then , the black vertex u_n has two white neighbors v_n and a vertex of the path rooted at u_n. So, the further forcing from the black vertex u_n is not possible, a contradiction.

Case 2. Suppose that $Z = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots u_{n-1}\}$ is a connected zero forcing set for G. Then we can easily observe that further forcing from the black vertex v_n is not possible, since it has two white neighbors, a contradiction to our assumption.

Case 3. The case of forming a connected zero forcing set by taking the $2n-1$ black pendent vertices of the paths only is ruled out, since the pendent vertices do not form a connected induced sub graph in G.

Case 4. Consider a connected zero forcing set of $2n-1$ black vertices having the following combinations.

Sub case 4.1. Combination of the vertices of $u_i, i = 1, 2, \ldots, n$ and the vertices of the path P_i, $i = 1, 2, \ldots, n$, rooted at u_i.

Sub case 4.2. Combination of the vertices of v_i and the vertices of the path $P_i, i = 1,2,\ldots,n$, rooted at v_i.

Sub case 4.3. Combination of the vertices u_i and v_i, and the vertices of $P_i, i = 1,2,\ldots,n$. Note that the combination of the vertices u_i and the vertices of P_i is not considered, since that combination does not form a connected induced sub graph in G. It is easy to verify that none of the above combinations will never form a connected zero forcing set for G. Hence from the above cases, we can infer that

$$Z_c(G) \geq 2n. \tag{11}$$

To claim $Z_c(G) \leq 2n$, we proceed as follows. Select $2n$ black vertices $u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n$. Then the black vertex $v_1 \to p_2^1$ to black, the black vertex $p_2^1 \to p_3^1$ to black, $\ldots,p_{t-1}^1 \to p_t^1$ to black. Similarly, all the vertices of the paths rooted at the black vertices $v_2,v_3,\ldots,v_n$ are colored black. The same argument holds good for the vertices of the paths rooted at the black vertices $u_1,u_2,\ldots,u_n$. Therefore, $Z = \{u_1,u_2,\ldots,u_n,v_1,v_2,\ldots,v_n\}$ generates a connected zero forcing set for G. Cardinality of Z is $2n$. So,

$$Z_c(G) \leq 2n \tag{12}$$

From (11) and (12), the result follows. □

Proposition 24. *Let G be the rooted product of the circular ladder graph with the cycle C_k, $k \geq 4$. Then $Z_c(G) \leq 4n$.*

Proof. Let $A = \{u_1,u_2,\ldots,u_n\}$ and $B = \{v_1,v_2,\ldots,v_n\}$ be the vertex set of the graph G, A being the vertex set of the inner cycle. Suppose that $C_1,C_2,\ldots.C_n$ be the cycles rooted at the vertices $v_1,v_2,\ldots.v_n$ and $D_1,D_2,\ldots,D_n$ be the cycles rooted at the vertices $u_1,u_2,\ldots,u_n$. Represent the vertices of cycles $C_1,C_2,\ldots,C_n$ and $D_1,D_2,\ldots,D_n$ as follows.

$$C_1 = \{c_1^1,c_2^1,\ldots,c_k^1,c_1^1\}$$
$$C_2 = \{c_1^2,c_2^2,\ldots,c_k^2,c_1^2\}$$
$$\ldots$$

$$\cdots$$

$$C_n = \{c_1^n, c_2^n, \ldots, c_k^n, c_1^n\}$$

$$D_1 = \{d_1^1, d_2^1, \ldots, d_k^1, d_1^1\}$$

$$D_2 = \{d_1^2, d_2^2, \ldots, d_k^2, d_1^2\}$$

$$\cdots$$

$$\cdots$$

$$D_n = \{d_1^n, d_2^n, \ldots, d_k^n, d_1^n\}$$

Let $v_1 = c_1^1, v_2 = c_1^2, \ldots, v_n = c_1^n$ and $u_1 = d_1^1, u_2 = d_1^2, \ldots, u_n = d_1^n$.

We generate a zero forcing set for the graph G as follows . Consider the set $\mathscr{Z} = \{v_1, c_2^1, v_2, c_2^2, \ldots, v_n, c_2^n, u_1, u_2, \ldots, u_n, d_2^1, d_2^2, \ldots, d_2^n\}$. Color the vertices in $\mathscr{Z}$ as black. Now the vertices in $\mathscr{Z}$ can force the remaining white vertices of the cycles $C_1, C_2, \ldots, C_n$ and $D_1, D_2, \ldots, D_n$ as black by repeatedly applying the color changing rule. Thus, the set

$$\mathscr{Z} = \{v_1, c_2^1, v_2, c_2^2, \ldots, v_n, c_2^n, u_1, u_2, \ldots, u_n, d_2^1, d_2^2, \ldots, d_2^n\}$$

generates a connected zero forcing set for G. The cardinality of the set $\mathscr{Z}$ is $4n$. Hence, $Z_c(G) \leq 4n$. $\square$

We strongly believe that the above bound is sharp.

Proposition 25. *Let G be the rooted product of the path $P_n, n \geq 3$, with P_t, a path of length t, $t \geq 4$ rooted at the pendant vertex. Then $Z_c(G) = n$.*

Proof. Denote the vertices of the path P_n by $u_1, u_2, \ldots, u_n$ in G. Let $u_1 = P_1^1, u_2 = P_1^2, \ldots, u_n = P_1^n$, where $P_1^1, P_1^2, \ldots, P_1^n$ are the vertices of the path rooted at $u_1, u_2, \ldots, u_n$.

Claim: Any set consisting of $(n-1)$ black vertices will never form a connected zero forcing set for the graph G. For, consider the following cases.

Case 1. Select the pendant vertex of each path rooted at the vertices

$$u_1, u_2, \ldots, u_{n-1}.$$

Clearly they cannot form a connected zero forcing set for G.

Case 2: Form a set of $n-1$ black vertices from the vertices of the paths rooted at the vertices $u_1, u_2, \ldots, u_n$. We can easily observe that this set will not form a connected zero forcing set for G.

Case 3: Assume that $Z = \{u_1, u_2, \ldots, u_{n-1}\}$. Color the vertices in the set Z as black. Then we can see that the vertices of the paths rooted at the vertices $u_1, u_2, \ldots, u_{n-2}$ can be colored as black by applying color changing rule. Note that the forcing from the black vertex u_{n-1} is not possible, since u_{n-1} has two white neighbours. So the set Z cannot generate a zero forcing set for G. In view of the above cases, we have $Z_c(G) \geq n$.

To prove the reverse part, let $Z_1 = \{u_1, u_2, \ldots, u_n\}$. Assign black color to the vertices in the set Z_1. Then it can be seen that the set Z_1 generates a connected zero forcing set for G. Therefore, $Z_c(G) \leq n$. Hence the result follows.

Again, when $k = 2$, any black vertex of the graph G, other than the vertex having degree 3 , gives a derived coloring for G. Hence, $Z_c(G) = 1$.

When, $k = 3$, any vertex of G forms a connected zero forcing set, as we wish. □

3. CONNECTED k-FORCING NUMBER OF SQUARE OF GRAPHS

In this section, we deal with the connected k-forcing number of square of path graph $P_n, n \geq 4$, the cycle graph $C_n, n \geq 5$.

Proposition 26. *Let G denotes the square of the path P_n,$n \geq 3$. Then the connected zero forcing number of G is* 2.

Proof. Represent the vertices of G by $u_1, u_2, \ldots, u_n$ and let u_1 and u_n be the pendant vertices in G. The vertices in G and G^2 are the same. It is obvious that with one black vertex, we cannot get a derived coloring for G. Since $\delta(G) = 2 \leq Z(G) \leq Z_c(G)$. So, $Z_c(G) \geq 2$.

On the other hand, without loss of generality, color the vertices u_1 and u_2 as black. Then the black vertex u_1 forces u_3 to black, u_2 forces u_4 to black, u_3 forces u_5 to black and so on till all the vertices of G are colored black. So, $Z = \{u_1, u_2\}$ forms a connected zero forcing set for G. $\mid Z \mid = 2$. Therefore, we have $Z_c(G) \leq 2$. Hence the result follows. □

Proposition 27. *The connected zero forcing number of the square of a cycle C_n, $n \geq 5$, is* 4.

Proof. Let G denotes the square of the cycle C_n, $n \geq 5$. It is clear that G is a 4-regular graph. That is, $\Delta(G) = \delta(G) = 4$. This implies that $Z_c(G) \geq 4$.

In order to establish the reverse inequality, choose any four connected vertices of G. Let they be u_1, u_2, u_3 and u_n. Color them as black. In G, the white vertices adjacent to the vertex u_1 are u_2, u_3, u_n and u_{n-1}. So the black vertex u_1 forces the vertex u_{n-1} to black. Now consider the black vertex u_2. The adjacent vertices of u_2 are u_1, u_n, u_3,and u_4. Of these vertices, u_1, u_n, u_3 are already black. So, the vertex u_2 forces u_4 to black. Again, consider the black vertex u_3. At this stage, the vertex u_3 has only one white vertex u_5. Hence u_3 forces u_5 to black and so on. Finally, consider the black vertex u_{n-4}. The vertex u_{n-4} has 4 neighbours $u_{n-5}, u_{n-6}, u_{n-3}, u_{n-2}$ of which the only one white vertex is u_{n-2}. Therefore, the vertex u_{n-4} forces u_{n-2} to black. The vertex u_{n-3} is already colored black by the vertex u_{n-5}. Therefore, the set $Z = \{u_1, u_2, u_3, u_n\}$ yields a connected zero forcing set for the graph G. Hence, we have $Z_c(G) \leq 4$. This completes the proof. □

4. Conclusion and Open Problems

In this paper we addressed the problem of determining the connected k-forcing number of certain graphs. Also we found the exact value of connected zero forcing number of some classes of graphs. In Section 1, we found an upper bound of $Z_{ck}(\mathscr{G})$ for the corona product of two graphs G and H. It is an open problem to charaterize the connected graphs for which $Z_{ck}(\mathscr{G}) = p_1(1+p_2)$. In Section 2, we found the exact values of the connected k-forcing number of rooted product of cycles with paths and cycle with cycles. Section 3, deals with the connected k-forcing number of square of graphs such as the paths and cycles.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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