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ATOMIC SOLUTION OF SECOND ORDER VECTOR VALUED FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. Some times it is not easy to find the exact solution of certain differential equations. In this paper we study atomic solutions of fractional vector valued differential equations.

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1. Introduction.

In [4], a new definition called $\alpha-$conformable fractional derivative was introduced:

Let $\alpha \in (0,1)$, and $f : E \subseteq (0,\infty) \to R$. For $x \in E$ let:

$$D^{\alpha} f(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}.$$

If the limit exists then it is called the $\alpha-$ conformable fractional derivative of f at x.

For $x = 0$, $D^{\alpha} f(0) = \lim_{x \to 0} D^{\alpha} f(0)$ if such limit exists.

The new definition satisfies:

1.$T_{\alpha}(af+bg) = aT_{\alpha}(f) + bT_{\alpha}(g)$, for all $a,b \in \mathbb{R}$.

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2.$T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

Further, for $\alpha \in (0,1]$ and and f, g be $\alpha-$differentiable at a point t, with $g(t) \neq 0$. Then

3. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.

4. $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$

We list here the fractional derivatives of certain functions,

(1) 5.$T_\alpha(t^p) = p\, t^{p-\alpha}$.

6.$T_\alpha(\sin \frac{1}{\alpha} t^\alpha) = \cos \frac{1}{\alpha} t^\alpha$.

7.$T_\alpha(\cos \frac{1}{\alpha} t^\alpha) = -\sin \frac{1}{\alpha} t^\alpha$.

8.$T_\alpha(e^{\frac{1}{\alpha} t^\alpha}) = e^{\frac{1}{\alpha} t^\alpha}$.

On letting $\alpha = 1$ in these derivatives, we get the corresponding ordinary derivatives.

One should notice that a function could be $\alpha-$conformable differentiable at a point but not differentiable, for example, take $f(t) = 2\sqrt{t}$. Then $T_{\frac{1}{2}}(f)(t) = 1$. Hence $T_{\frac{1}{2}}(f)(0) = 1$. But $T_1(f)(0)$ does not exist. This is not the case for the known classical fractional derivatives.

For more on fractional calculus and its applications we refer to [1], [8] and [9].

2. Atomic Solution

Let X and Y be two Banach spaces and X^* be the dual of X. Assume $x \in X$ and $y \in Y$. The operator $T : X^* \to Y$, defined by $T(x^*) = x^*(x)y$ is a bounded one rank linear operator. We write $x \otimes y$ for T. such operators are called atoms. Atoms are among the main ingredient in the theory of tensor product. Atoms are used in theory of best approximation in Banach spaces, [6], and [7].

It is a known result, [5], and we need it in our paper that: If the sum of two atoms is an atom, the either the first component are dependent or the second are dependent.

An equation of the form

$$T_\alpha T_\alpha v + AT_\alpha v = f(t) \tag{1}$$

Is called a fractional vector valued differential equation, where, v and f are nice functions from $[0,\infty)$ to the Banach space X, and A is a closed linear operator on X.

A solution of this equation of the form $v = u \otimes x$ is called an atomic solution. In this paper we are interested in finding an atomic solution to equation (1).

That is, we will find solution to the equation:

$$u^{(2\alpha)}(t)\otimes x+u^{(\alpha)}(t)\otimes Ax=f(t)\otimes z\,,\ \text{where}\ \ u(0)=1,u^{(\alpha)}(0)=1 \tag{2}$$

Here $u(t)$ and x are the unknowns, while A,z, and f are given. Further, we assume without loss of generality that $f(0)=1$.

Theorem 2.1. Let z be a unique image in the range of the operator $I+A$**,** and A has a unique fixed point. Then equation (2) has a unique solution.

Proof. Now , $u^{(2\alpha)}\otimes x$ and $u^{(\alpha)}\otimes Ax$ are two atoms whose sum is also an atom $f\otimes z$.

Hence, [5], we have two cases:

Case (i): $u^{(2\alpha)}=\beta u^{(\alpha)}$.

Since $x\otimes y=\beta x\otimes\frac{1}{\beta}y$, then with no loss of generality, we can assume $\beta=1$. So we have

$$T_\alpha T_\alpha u=T_\alpha u \tag{3}$$

Using result in [10] and property 8 that the conformable derivative satisfies we get

$$u(t)=C_1+C_2e^{(\frac{\beta}{\alpha})t^\alpha}$$

But from (2), $u(0)=1$ and $u^{(\alpha)}(0)=1$. Hence

$C_1+C_2=1$ and $C_2=1$.

Consequently

$$u(t)=e^{(\frac{1}{\alpha})t^\alpha} \tag{4}$$

Now, we go back to (2), to get:

$$e^{\frac{t^\alpha}{\alpha}}(x+Ax)=f(t)z.$$

The conditions on u and f give a unique x such that $x+Ax=z$. Thus equation (2) has a unique solution.

Case (ii): $Ax=\beta x$. Again with no loss of generality we can assume that $\beta=1$. Thus $Ax=x$. By the assumption on A, there is a unique x such that $Ax=x$

Now, substitute in (2) to get

$$u^{(2\alpha)}(t)\otimes x+u^{(\alpha)}(t)\otimes x=f(t)\otimes z.$$

So

$$\left(u^{(2\alpha)}+u^{(\alpha)}\right)\otimes x=f\otimes z$$

By the condition on u, we get $x=z$. Consequently, we get

$$u^{(2\alpha)}+u^{(\alpha)}=f$$

Being a linear fraction differential equation, we can use a result in [10] to obtain:

$u_g=u_h+u_p$, the general solution is the sum of the homogenous part plus the particular part. Using the same result in [10], to get

$$u_h=C_1+C_2e^{\frac{-t^\alpha}{\alpha}}$$

The conditions on u imply

$$u_h=2-e^{\frac{-t^\alpha}{\alpha}}$$

As for the particular solution, we use variation of parameters introduced in [8]. Thus we have

$$u_p(t)=\int_b^t \frac{\begin{vmatrix} u_1(t) & u_2(t) \\ u_1(x) & u_2(x)\end{vmatrix}}{\begin{vmatrix} u_1(t) & u_2(t) \\ T_\alpha u_1(t) & T_\alpha u_2(t)\end{vmatrix}} f(t)\frac{dt}{t^{1-\alpha}}$$

which can be evaluated for a given function f.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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