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FRACTIONAL DISTRIBUTIONS AND PROBABILITY DENSITY FUNCTIONS OF RANDOM VARIABLES GENERATED USING FDE

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Abstract. Distributions have many applications in probability and other applied sciences. The aim of this research is to generate probability density functions of random variables using fractional differential equations (FDE).

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1. Introduction

Stochastic and partial differential equations are both useful tools when modeling problems in reality. Uncertainties in data are natural in many models of real world problems that use partial differential equations, specially in physical sciences. Calculus is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. Probability density function (PDF) can be expressed as ODE whose solution is the PDF. Some of which are available. They include:

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Laplace distribution ([4, 5]), beta distribution ([3]), raised cosine distribution ([10]), Lomax distribution ([2]), beta prime distribution or Inverted beta distribution ([6]).

Various systems of distributions have been constructed to provide approximations to a wide variety of distributions, see, e.g., Johnson et al. 1994. One of these systems is the Pearson system. A continuous distribution belongs to this system if its probability density function (PDF) $f(x)$ satisfies a differential equation of the form

$$\frac{df(x)}{dx} = \frac{x+\lambda}{\beta x^2 + \gamma x + \eta} f(x)$$

where λ, β, γ, and η are real parameters such that $f(x)$ is a PDF. Based on these parameters , Pearson ([7, 8]) classified these distributions into a number of types known as Pearson Types I – VI. Later in another paper, Pearson ([9]) defined more special cases and subtypes, known as Pearson Types VII - XII. Many well-known distributions are special cases of Pearson Type distributions which include Normal and Student t distributions (Pearson Type VII), Beta distribution (Pearson Type I), Gamma distribution (Pearson Type III), among others.

There are many types of fractional derivatives known in the literature. In this paper we will use the most recent one: The conformable fractional derivative:

Definition 1.1. *Let* $\alpha \in (0,1]$ *and* $f : E \subseteq (0,\infty) \to \mathbb{R}$. *For* $x \in E$ *let*

$$D^\alpha f(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}.$$

If the limit exists then it is called the $\alpha-$ *conformable fractional derivative of f at x.*
For $x = 0$, $D^\alpha f(0) = \lim_{x\to 0} D^\alpha f(0)$ *if such limit exists.*

This definition satisfies:

1.$D^\alpha(af+bg) = aD^\alpha(f) + bD^\alpha(g)$, for all $a, b \in \mathbb{R}$.

2.$D^\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

Further, for $\alpha \in (0,1]$.Let f, g be $\alpha-$differentiable at a point t, with $g(t) \neq 0$. Then

3. $D^\alpha(fg) = fD^\alpha(g) + gD^\alpha(f)$.

4. $D^\alpha(\frac{f}{g}) = \frac{gD^\alpha(f) - fD^\alpha(g)}{g^2}$

We list here the fractional derivatives of certain functions,

(1) 5.$D^{\alpha}(t^{p}) = p\, t^{p-\alpha}$.
6.$D^{\alpha}(\sin \frac{1}{\alpha} t^{\alpha}) = \cos \frac{1}{\alpha} t^{\alpha}$.
7.$D^{\alpha}(\cos \frac{1}{\alpha} t^{\alpha}) = -\sin \frac{1}{\alpha} t^{\alpha}$.
8.$D^{\alpha}(e^{\frac{1}{\alpha} t^{\alpha}}) = e^{\frac{1}{\alpha} t^{\alpha}}$.

By letting $\alpha = 1$ in these derivatives, we get the corresponding ordinary derivatives.

One should notice that a function could be $\alpha-$conformable differentiable at a point but not differentiable. For example, take $f(t) = 2\sqrt{t}$. Then $D_{\frac{1}{2}}(f)(t) = 1$. Hence $D_{\frac{1}{2}}(f)(0) = 1$. But $D_1(f)(0)$ does not exist. This is not the case for the known classical fractional derivatives.

For more on fractional calculus and its applications we refer to [11] to [15].

2. Fractional Distributions

In this section we introduce many fractional distributions that can be used in many applications in probability and applied sciences.

(1) **α- Chi-Square Distribution**

Consider the following conformable differential equation

$$2x^{\alpha}D^{\alpha}y + (-k + x^{\alpha} + 2)\,y = 0, \qquad \text{where } 0 < \alpha < 1, k > 2(1-\alpha), x > 0$$

If y has ordinary derivative y' with respect to x, then the above equation is equivalent to the ordinary differential equation

$$2x^{\alpha}x^{1-\alpha}y' + (-k + x^{\alpha} + 2)\,y = 0$$

Thus,

$$\frac{y'}{y} = \frac{k - x^{\alpha} - 2}{2x^{\alpha}} = \frac{k-2}{2x} - \frac{x^{\alpha-1}}{2}$$

$$\ln y = \frac{k-2}{2}\ln x - \frac{x^{\alpha}}{2\alpha} + c$$

$$y = Ax^{(k/2)-1}\, e^{-x^{\alpha}/2\alpha}, \qquad \text{where } A = e^{c} > 0$$

Set $f_{\alpha}(x) = Ax^{(k/2)-1}\, e^{-x^{\alpha}/2\alpha}$. For $f_{\alpha}(x)$ to be a conformable probability distribution function (CPDF) with support $(0,\infty)$, we need $\int_0^{\infty} f_{\alpha}(x)\, d^{\alpha}x = 1$.

Thus,

$$A = \frac{1}{2^{((k-2)/2\alpha)+1}\ \alpha^{(k-2)/2\alpha}\ \Gamma\left(\dfrac{k-2}{2\alpha}+1\right)}\ , k > 2(1-\alpha).$$

Hence,

$$f_\alpha(x) = \frac{1}{2^{((k-2)/2\alpha)+1}\ \alpha^{(k-2)/2\alpha}\ \Gamma\left(\dfrac{k-2}{2\alpha}+1\right)} x^{(k/2)-1}\, e^{-x^\alpha/2\alpha},$$

$$x > 0, \alpha \in (0,1), k > 2(1-\alpha).$$

Consequently,

$$\lim_{\alpha \to 1^-} f_\alpha(x) = \frac{1}{2^{(k/2)}\ \Gamma\left(\dfrac{k}{2}\right)} x^{(k/2)-1}\, e^{-x/2}, \qquad x > 0, k > 0.$$

This is the PDF of a gamma distribution that is denoted by $G\left(\left(\dfrac{k}{2}\right), 2\right)$, so the CPDF $f_\alpha(x)$ is a generalization of the PDF of a chi-square distribution with k degrees of freedom of χ^2_k.

(2) **α-Rayleigh Distribution**

Consider the following conformable differential equation

$$\sigma^2 x^\alpha D^\alpha y + \left(x^{2\alpha} - \sigma^2\right) y = 0, 0 < \alpha < 1, \sigma > 0$$

So,

$$\frac{y'}{y} = \frac{1}{x} - \frac{x^{2\alpha-1}}{\sigma^2}$$

$$\ln y = \ln x - \frac{x^{2\alpha}}{2\alpha\sigma^2} + c$$

$$y = Ax\, e^{-x^{2\alpha}/2\alpha\sigma^2}$$

Set $f_\alpha(x) = Ax\, e^{-x^{2\alpha}/2\alpha\sigma^2}$, where $A = e^c > 0$, $x > 0$. For $f_\alpha(x)$ to be a CPDF. we need $\int_0^\infty f_\alpha(x)\, d^\alpha x = 1$.

Thus,

$$A = \frac{\alpha}{2^{1/2((1/\alpha)-1)}\ \left(\alpha\sigma^2\right)^{(1+\alpha)/2\alpha}\ \Gamma\left(\dfrac{1+\alpha}{2\alpha}\right)}$$

Hence,

$$f_\alpha(x) = \frac{\alpha}{2^{1/2((1/\alpha)-1)}\ (\alpha\sigma^2)^{(1+\alpha)/2\alpha}\ \Gamma\left(\dfrac{1+\alpha}{2\alpha}\right)}\, x\, e^{-x^{2\alpha}/2\alpha\sigma^2}$$

Consequently,

$$\lim_{\alpha\to 1^-} f_\alpha(x) = \frac{1}{\sigma^2}\, x\, e^{(-x^2/2\sigma^2)}.$$

This is the PDF of Rayleigh distribution, so the CPDF $f_\alpha(x)$ is a generalization of the PDF of Rayleigh distribution.

(3) **α-Inverse-Gamma Distribution**

Consider the following conformable differential equation

$$x^{2\alpha} D^\alpha y + (bx^\alpha - \beta + x^\alpha)\, y = 0$$

So,

$$\begin{aligned}
\frac{y'}{y} &= \frac{\beta}{x^{\alpha+1}} - \frac{b+1}{x} \\
\ln y &= \frac{-\beta}{\alpha x^\alpha} - (b+1) - (b+1)\ln x \\
y &= Ax^{-(b+1)}\ e^{-\beta/(\alpha x^\alpha)}
\end{aligned}$$

Set $f_\alpha(x) = Ax^{-(b+1)}\ e^{-\beta/(\alpha x^\alpha)}$, where $A = e^c > 0$. For $f_\alpha(x)$ to be a CPDF, we need $\int_0^\infty f_\alpha(x)\, d^\alpha x = 1$.
Thus,

$$A = \frac{\alpha^2 \left(\dfrac{\beta}{\alpha}\right)^{(b+1)/\alpha}}{\beta\,\Gamma\left(\dfrac{b-\alpha-1}{\alpha}\right)}\ , b > \alpha + 1$$

Hence,

$$f_\alpha(x) = \frac{\alpha^2 \left(\dfrac{\beta}{\alpha}\right)^{(b+1)/\alpha}}{\beta\,\Gamma\left(\dfrac{b-\alpha+1}{\alpha}\right)} x^{-(b+1)}\ e^{-\beta/(\alpha x^\alpha)}$$

Consequently,

$$\lim_{\alpha\to 1^-} f_\alpha(x) = \frac{\beta^b}{\Gamma(b)} x^{-(b+1)} e^{-\beta/x}.$$

This is the PDF of Inverse Gamma distribution. So, the CPDF $f_\alpha(X)$ is a generalization of the PDF of Inverse Gamma distribution.

(4) **α- Gamma Distribution**

Consider the following conformable differential equation

$$x^\alpha D^\alpha y + (-k+1+\theta x^\alpha)\, y = 0$$

$$xy' = ((k-1)-\theta x^\alpha)\, y$$

So,

$$\frac{y'}{y} = \frac{k-1}{x} - \theta x^{\alpha-1}$$

$$\ln y = (k-1)\ln x - \frac{\theta}{\alpha}x^\alpha + c$$

$$y = Ax^{k-1}\, e^{-(\theta x^\alpha)/\alpha}, \qquad \text{where } A = e^c > 0$$

Set $f_\alpha(x) = Ax^{k-1}\, e^{-(\theta x^\alpha)/\alpha}$.In order for $f_\alpha(x)$ to be a CPDF, we need $\int_0^\infty f_\alpha(x)\, d^\alpha x = 1$.

Hence,

$$A = \frac{\theta}{\left(\frac{\theta}{\alpha}\right)^{(1-k)/\alpha} \Gamma\left(\frac{k+\alpha-1}{\alpha}\right)} \quad , k > 1-\alpha$$

Thus,

$$f_\alpha(x) = \frac{\theta\, x^{k-1}\, e^{(-\theta x^\alpha)/\alpha}}{\left(\frac{\theta}{\alpha}\right)^{(1-k)/\alpha} \Gamma\left(\frac{k+\alpha-1}{\alpha}\right)}$$

Consequently,

$$\lim_{\alpha\to 1^-} f_\alpha(x) = \frac{x^{k-1} e^{-\theta x}}{\theta^{-k}\Gamma(k)}.$$

This is the PDF of Gamma distribution with parameters k and $\frac{1}{\theta}$. So, the CPDF $f_\alpha(X)$ is a generalization of the PDF of two parameters Gamma distribution.

Special case : If $k=1$,then $f_\alpha(x) = \theta\, e^{(-\theta x^\alpha)/\alpha}$, which is called **α-exponential distribution.**

Since $\lim_{\alpha\to 1^-} f_\alpha(x) = \theta\, e^{-\theta x}$, is the PDF of exponential distribution with parameter θ.

(5) **α-Nagakami Distribution**

Consider the following conformable differential equation

$$bx^\alpha D^\alpha y + \left(2mx^{2\alpha} - 2mb + b\right) y = 0, x > 0, m >= \frac{1}{2}$$

Then,

$$\begin{aligned} & bx\frac{y'}{y} = b(2m-1) - 2mx^{2\alpha} \\ & \frac{y'}{y} = \frac{2m-1}{x} - \frac{2m}{b}x^{2\alpha-1} \\ & \ln y = (2m-1)\ln x - \frac{m}{b\alpha}x^{2\alpha} + c \\ & y = A\,x^{2m-1} e^{(-m/(b\alpha))x^{2\alpha}}, \qquad \text{where } A = e^c > 0 \end{aligned}$$

Set $f_\alpha(x) = A\,x^{2m-1} e^{(-m/(b\alpha))x^\alpha}$.In order for $f_\alpha(x)$ to be a CPDF, we need $\int_0^\infty f_\alpha(x)\, d^\alpha x = 1$.
Hence,

$$A = \frac{2\alpha}{\left(\frac{m}{b\alpha}\right)^{(2m+\alpha-1)/2\alpha}\Gamma\left(\frac{2m+\alpha-1}{2\alpha}\right)} \ , m > \frac{1-\alpha}{2}$$

Thus,

$$f_\alpha(x) = \frac{2\alpha}{\left(\frac{m}{b\alpha}\right)^{(2m+\alpha-1)/2\alpha}\Gamma\left(\frac{2m+\alpha-1}{2\alpha}\right)} x^{2m-1}\ e^{-mx^{2\alpha}/(b\alpha)},$$

$$\text{where } x > 0, 0 < \alpha < 1, m > \frac{1-\alpha}{2}.$$

Consequently,

$$\lim_{\alpha\to 1^-} f_\alpha(x) = \frac{2\,x^{2m-1}\ e^{-mx^2/b}}{\left(\frac{m}{b}\right)^m \Gamma(m)}.$$

This is the PDF of Nagakami distribution. So the CPDF $f_\alpha(x)$ is a generalization of the PDF of Nagakami distribution.

(6) **α-Arcsine Distribution**

Consider the following conformable differential equation

$$2\left(1-x^{\alpha}\right)x^{\alpha}D^{\alpha}y-\left(2\alpha x^{\alpha}-\alpha\right)y=0$$

Then,

$$\begin{aligned}
\frac{y'}{y} &= \frac{2\alpha x^{\alpha}-\alpha}{2x\left(1-x^{\alpha}\right)} \\
\frac{y'}{y} &= \frac{-\alpha}{2x}+\frac{\alpha x^{\alpha-1}}{2\left(1-x^{\alpha}\right)} \\
\ln y &= -\frac{\alpha}{2}\ln x-\frac{1}{2}\ln\left(1-x^{\alpha}\right)+c \\
\ln y &= \frac{-1}{2}\ln\left(x^{\alpha}\left(1-x^{\alpha}\right)\right)+c, \qquad x\in(0,1) \\
y &= \frac{A}{\sqrt{\left(x^{\alpha}\left(1-x^{\alpha}\right)\right)}}, \qquad \text{where } A=e^{c}>0.
\end{aligned}$$

Set $f_{\alpha}(x)=\frac{A}{\sqrt{\left(x^{\alpha}\left(1-x^{\alpha}\right)\right)}}$. In order for $f_{\alpha}(x)$ to be a CPDF, we need $\int_0^1 f_{\alpha}(x)\,d^{\alpha}x=1$.

Hence,

$$A=\frac{\alpha}{\pi}$$

Thus,

$$f_{\alpha}(x)=\frac{\alpha}{\pi\sqrt{\left(x^{\alpha}\left(1-x^{\alpha}\right)\right)}}, \qquad \text{where } 0<x<1, 0<\alpha<1.$$

Consequently,

$$\lim_{\alpha\to 1^-} f_{\alpha}(x)=\frac{1}{\pi\sqrt{x(1-x)}}.$$

This is the PDF of Arcsine distribution. So the CPDF $f_{\alpha}(x)$ is a generalization of the PDF of Arcsine distribution.

(7) **α- Beta Distribution**

Consider the following conformable differential equation

$$(1-x^{\alpha})\,x^{\alpha}D^{\alpha}y-(\alpha(b-1)-\alpha(b+\beta-2)\,x^{\alpha})\,y=0,\;\;0<x<1$$

Then,

$$\frac{y'}{y}=\frac{\alpha(b-1)}{x\,(1-x^{\alpha})}-\frac{\alpha(b+\beta-2)\,x^{\alpha-1}}{1-x^{\alpha}}$$

So,

$$\frac{y'}{y}=\frac{\alpha(b-1)}{x}+\frac{\alpha(b-1)x^{\alpha-1}}{1-x^{\alpha}}-\frac{\alpha(b+\beta-2)\,x^{\alpha-1}}{1-x^{\alpha}}$$

$$\ln y=\alpha(b-1)\ln x-(b-1)\ln(1-x^{\alpha})+(b+\beta-2)\ln(1-x^{\alpha})+c$$

$$\ln y=\alpha(b-1)\ln x+(\beta-1)\ln(1-x^{\alpha})+c$$

$$y=A\Big(x^{\alpha}\Big)^{b-1}\Big(1-x^{\alpha}\Big)^{\beta-1}, \qquad \text{where } A=e^{c}>0$$

Set $f_{\alpha}(x)=A\Big(x^{\alpha}\Big)^{b-1}\Big(1-x^{\alpha}\Big)^{\beta-1}$.In order for $f_{\alpha}(x)$ to be a CPDF, we need $\int_0^1 f_{\alpha}(x)\,d^{\alpha}x=1.$
Hence,

$$A=\frac{\alpha\,\Gamma(b+\beta)}{\Gamma(b)\,\Gamma(\beta)}$$

Thus,

$$f_{\alpha}(x)=\frac{\alpha\Gamma(b+\beta)}{\Gamma(b)\Gamma(\beta)}\Big(x^{\alpha}\Big)^{b-1}\Big(1-x^{\alpha}\Big)^{\beta-1}, \quad x\in(0,1),\alpha\in(0,1],b>0,\beta>0.$$

Consequently,

$$\lim_{\alpha\to1^-} f_{\alpha}(x)=\frac{\Gamma(b+\beta)}{\Gamma(b)\,\Gamma(\beta)}x^{b-1}(1-x)^{\beta-1}.$$

This is the PDF of Beta(b,β) distribution.So the CPDF $f_{\alpha}(x)$ is a generalization of the PDF of Beta(b,β) distribution.

Remark. Notice that by letting $b = \frac{1}{2}$ and $\beta = \frac{1}{2}$ in the α-Beta Distribution above, we get that the α-Arcsine distribution is a special case of the α- Beta distribution, since

$$f_\alpha(x) = \frac{\alpha\Gamma(1)}{\Gamma(1/2)\Gamma(\frac{1}{2})}\left(x^\alpha\right)^{\frac{-1}{2}}\left(1-x^\alpha\right)^{\frac{-1}{2}} = \frac{\alpha}{\pi\sqrt{\left(x^\alpha\left(1-x^\alpha\right)\right)}}$$

(8) **α- Levy Distribution**

Consider the following conformable differential equation

$$2\left(x^\alpha-\mu\right)^2 D^\alpha y + \alpha\left(3x^\alpha - c - 3\mu\right)y = 0\ ,\ x > \mu\ ,\ c > 0$$

$$\frac{y'}{y} = \frac{\alpha(c+3\mu)x^{\alpha-1}}{2(x^\alpha-\mu)^2} - \frac{3\alpha x^{2\alpha-1}}{2(x^\alpha-\mu)^2}$$

Hence,

$$\ln y = \frac{-(c+3\mu)}{2(x^\alpha-\mu)} - \frac{3}{2}\ln\left(x^\alpha-\mu\right) + \frac{3\mu}{2(x^\alpha-\mu)} + d$$

$$\ln y = \frac{-c}{2(x^\alpha-\mu)} - \frac{3}{2}\ln\left(x^\alpha-\mu\right) + d$$

$$y = A\frac{e^{-c/2(x^\alpha-\mu)}}{(x^\alpha-\mu)^{3/2}}, \qquad \text{where } A = e^d > 0.$$

Set $f_\alpha(x) = A\frac{e^{-c/2(x^\alpha-\mu)}}{(x^\alpha-\mu)^{3/2}}$. In order for $f_\alpha(x)$ to be a CPDF, we need $\int_\mu^\infty f_\alpha(x)\,d^\alpha x = 1$. Hence,

$$A = \frac{\alpha\sqrt{c}}{\sqrt{2\pi}}$$

Thus,

$$f_\alpha(x) = \frac{\alpha\sqrt{c}}{\sqrt{2\pi}}\frac{e^{-c/2(x^\alpha-\mu)}}{\sqrt{\left(x^\alpha\left(1-x^\alpha\right)\right)}}, \qquad \text{where } x \geq \mu, \alpha \in (0,1]\ ,\ c > 0$$

Consequently,

$$\lim_{\alpha\to1^-} f_\alpha(x) = \frac{\sqrt{c}}{\sqrt{2\pi}}\frac{e^{-c/2(x-\mu)}}{\sqrt{\left(x(1-x)\right)}}.$$

This is the PDF of Levy distribution.So the CPDF $f_\alpha(x)$ is a generalization of the PDF of Levy distribution.

(9) **Beta- Prime Distribution**

Consider the following conformable differential equation

$$x^{\alpha}(x^{\alpha}+1)D^{\alpha}y+(-\alpha b+\alpha+\alpha x^{\alpha}(\beta+1))y=0$$

$$x^{\alpha}(x^{\alpha}+1)x^{1-\alpha}y'+(-\alpha b+\alpha+\alpha x^{\alpha}(\beta+1))y=0$$

$$\frac{y'}{y}=\frac{\alpha(b-1)}{x(x^{\alpha}+1)}-\frac{\alpha(\beta+1)x^{\alpha-1}}{(x^{\alpha}+1)}$$

Hence,

$$\frac{y'}{y}=\frac{\alpha(b-1)}{x}+\frac{\alpha(1-b)x^{\alpha-1}}{(x^{\alpha}+1)}-\frac{\alpha(\beta+1)x^{\alpha-1}}{(x^{\alpha}+1)}$$

$$\ln y=\alpha(b-1)\ln x+(1-b)\ln(x^{\alpha}+1)-(\beta+1)\ln(x^{\alpha}+1)+c$$

$$\ln y=(b-1)\ln x^{\alpha}+(-b-\beta)\ln(x^{\alpha}+1)+c$$

$$y=A\left(x^{\alpha}\right)^{b-1}\left(x^{\alpha}+1\right)^{-b-\beta}, \qquad \text{where } A=e^{c}>0.$$

Set $f_{\alpha}(x)=A\left(x^{\alpha}\right)^{b-1}\left(x^{\alpha}+1\right)^{-b-\beta}$. In order for $f_{\alpha}(x)$ to be a CPDF, we need $\int_{0}^{\infty}f_{\alpha}(x)\,d^{\alpha}x=1$.
Hence,

$$A=\frac{\alpha\,\Gamma(b+\beta)}{\Gamma(b)\,\Gamma(\beta)}$$

Thus,

$$f_{\alpha}(x)=\frac{\alpha\,\Gamma(b+\beta)}{\Gamma(b)\,\Gamma(\beta)}\left(x^{\alpha}\right)^{b-1}\left(x^{\alpha}+1\right)^{-b-\beta}, \qquad x>0,\alpha\in(0,1].$$

Consequently,

$$\lim_{\alpha\to1^{-}}f_{\alpha}(x)=\frac{\Gamma(b+\beta)}{\Gamma(b)\,\Gamma(\beta)}\,x^{b-1}\left(x+1\right)^{-b-\beta}.$$

This is the PDF of beta prime distribution. So the CPDF $f_{\alpha}(x)$ is a generalization of the PDF of beta prime distribution.

(10) **α-Lomax Distribution**

Consider the following conformable differential equation

$$(\lambda + x^{\alpha}) D^{\alpha} y + (b+1) y = 0$$

$$(\lambda + x^{\alpha}) x^{1-\alpha} y' + (b+1) y = 0$$

$$\frac{y'}{y} = \frac{-(b+1)x^{\alpha-1}}{\lambda + x^{\alpha}}$$

$$\ln y = \frac{-(b+1)}{\alpha} \ln\left(\lambda + x^{\alpha}\right) + c$$

$$y = A\left(\lambda + x^{\alpha}\right)^{\left(-(b+1)/\alpha\right)}, \qquad \text{where } A = e^{c} > 0.$$

Set $f_{\alpha}(x) = A\left(\lambda + x^{\alpha}\right)^{(-(b+1)/\alpha)}$. In order for $f_{\alpha}(x)$ to be a CPDF, we need $\int_0^{\infty} f_{\alpha}(x)\, d^{\alpha}x = 1$.
Hence,

$$A = \frac{1 + b - \alpha}{\lambda^{(\alpha - b - 1)/\alpha}}$$

Thus,

$$f_{\alpha}(x) = \frac{b - \alpha + 1}{\lambda}\left(1 + \frac{x^{\alpha}}{\lambda}\right)^{(-(b+1)/\alpha)}, \qquad x \geq 0, \alpha \in (0,1].$$

Consequently,

$$\lim_{\alpha \to 1^{-}} f_{\alpha}(x) = \frac{b}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-(b+1)}.$$

This is the PDF of lomax distribution. So the CPDF $f_{\alpha}(x)$ is a generalization of the PDF of lomax distribution.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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