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# ON 2 ACYCLIC SIMPLE GRAPHOIDAL COVERING OF BICYCLIC GRAPHS

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**Abstract.** A $2-$simple graphoidal cover of $G$ is a set $\psi$ of (not necessarily open) paths in $G$ such that every edge is in exactly one path in $\psi$ and every vertex is an internal vertex of at most two paths in $\psi$ and any two paths in $\psi$ has at most one vertex in common. The minimum cardinality of the $2-$simple graphoidal cover of $G$ is called the $2-$simple graphoidal covering number of $G$ and is denoted by $\eta_{2s}$. A $2-$simple graphoidal cover $\psi$ of a graph $G$ is called $2-$acyclic simple graphoidal cover if every member of $\psi$ is a path. The minimum cardinality of a $2-$acyclic simple graphoidal cover of $G$ is called the $2-$acyclic graphoidal covering number of $G$ and is denoted by $\eta_{2as}$. This paper discusses $2-$acyclic simple graphoidal cover on bicyclic graphs.

**Keywords:** bicyclic graphs; 2-simple graphoidal cover; 2-acyclic simple graphoidal cover.

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## 1. Introduction

In graph theory, Graph Decomposition is the one of the fastest-growing research topics and plays a major role in Road Network, Block design and so on. A decomposition of a graph $G$ is a collection of edge disjoint subgraphs $H_1, H_2, \ldots, H_n$ of $G$ such that every edge of $G$ is in exactly one $H_i$. Several authors [1][3][7][9][12] impose different conditions and parameters to

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find out different types of decomposition of the graphs. The above motivates the definition of Graphoidal covers.

## 2. Preliminaries

All the graph $G = (V,E)$ in this paper is a nontrivial, simple-connected, and undirected graphs. The number of elements of $V$ is said to be the order of $G$ is expressed by $p$ and the number of elements in the $E$ are said to be the size of $G$ is expressed by $q$. For graph theoretic terminology, Harary [8] is referred. The vertices $u_0$ and $u_l$ are called external vertices of $P$ and $u_1, u_2, \ldots, u_{l-1}$ are internal vertices of $P$, where $P = (u_0, u_1, u_2, \ldots, u_{l-1}, u_l)$ is a path or cycle in $G$. Two paths $P_1$ and $P_2$ are said to be internally disjoint if no vertex of $G$ is an internal vertex of both $P_1$ and $P_2$. The graphoidal cover introduced and discussed by Acharya and Sampath Kumar [1] [2]. $2-$graphoidal path cover introduced by Nagarajan et.al [9]. $2-$graphoidal cover extensively studied and discussed by Das and Singh [7]. The authors [12] discuss about $2-$acyclic simple graphoidal cover and discusses $2-$acyclic simple graphoidal covering on standard graph. In this paper, the authors determine the $2-$acyclic simple graphoidal cover on bicyclic graphs.

**Definition 2.1** (1)**.** A graphoidal cover of $G$ is a set $\psi$ of (not necessarily open) paths in $G$ satisfying the following conditions.

(i) Every path in $\psi$ has at least two vertices.
(ii) Every vertex of $G$ is an internal vertex of at most one path in $\psi$.
(iii) Every edge of $G$ is in exactly one path in $\psi$.

The minimum cardinality of a graphoidal cover of $G$ is called the graphoidal covering number of $G$ and is denoted by $\eta(G)$.

**Definition 2.2** (9)**.** An $2-$graphoidal cover of a graph $G$ is a collection $\psi$ of paths (not necessarily open) in $G$ such that

(i) Every path in $\psi$ has atleast two vertices.
(ii) Every edge is exactly in one path $\psi$.
(ii) Every vertex is an internal vertex of at most two paths in $\psi$.

The minimum cardinality of a $2-$graphoidal cover of $G$ is called the $2-$ graphoidal covering number of $G$ and is denoted by $\eta_2(G)$.

**Definition 2.3** (12)**.** A $2-$simple graphoidal cover of a graph $G$ is a $2-$graphoidal cover $\psi$ of $G$ such that any two paths in $\psi$ have at most one vertex in common. The minimum cardinality of a $2-$simple graphoidal cover of $G$ is called the $2-$simple graphoidal covering number of $G$ and is denoted by $\eta_{2s}(G)$.

**Definition 2.4** (12)**.** A $2-$acyclic simple graphoidal cover of $G$ is said to be $2-$ simple graphoidal cover $\psi$ of $G$ such that every member $\psi$ of $G$ is a path. The minimum cardinality of a $2-$acyclic simple graphoidal cover of $G$ is called the $2-$acylic simple graphoidal covering number of $G$ and is denoted by $\eta_{2as}(G)$.

**Definition 2.5.** Let $\psi$ be a collection of internally disjoint paths in $G$. A vertex $v$ of $G$ is said to be an interior vertex of $\psi$ if it is an internal vertex of some path in $\psi$. Otherwise, it is said to be an exterior vertex.

**Notations 2.6 (9).** *Let $\psi$ be a $2-$acyclic simple graphoidal cover of $G$. The following notations are used in the theorems. Here $i_\psi(P), t_1(\psi), t_2(\psi), t_\psi$ denotes the number of internal vertices of the path $P$, the number of vertices appear as internal vertex exactly in one path $\psi$ , the number of vertices appears as internal vertex exactly in two paths of $\psi$ and the number of vertices are not internal in $\psi$ respectively.*

*If $2-$acyclic simple graphoidal cover $\psi$ of $G$ is minimum, then it is clear that $t_1(\psi), t_2(\psi)$ should be maximum and $t(\psi)$ should be minimum. We define $t_i = max\, t_i(\psi)$ $(i = 1,2)$ where the maximum is taken over all $2-$acyclic simple graphoidal covers of $\psi$ of $G$ and $t = min\, t_\psi$ where the minimum is taken over all $2-$acyclic simple graphoidal cover $\psi$ of $G$.*

**Theorem 2.7** (12)**.** *For any $(p,q)$ graph, $\eta_{2as}(G) = q - p - t_2 + t$.*

**Corollary 2.8** (12)**.** *There exists a $2-$acyclic simple graphoidal cover $\psi$ of $G$ in which every vertex is internal vertex in exactly $2$ paths in $\psi$ of $G$ if and only if $\eta_{2as}(G) = q - 2p$.*

**Theorem 2.9** (12). *Let G be a unicycle graph with n pendent vertices. Let C be the unique cycle on G. Let l be the number of vertices of degree greater than 2 on C. Then*

$$\eta_{2as}(G) = \begin{cases} 3 & \textit{if } l{=}0 \\ (n+1)-m & \textit{if } l{=}2 \\ (n+2)-m & \textit{if } l{=}1 \\ n-m & \textit{otherwise} \end{cases}$$

*where m is the total number of vertices of degree* $\geq 4$ *on G.*

**Definition 2.10** (6). A connected $(p, p+1)-$ graph $G$ is called a Bicyclic graph.

**Definition 2.11** (6). A one-point union of two cycles is a simple graph obtained from two cycles, say $C_l$ and $C_m$ where $l, m \geq 3$, by identifying one and the same vertex from both cycles. Without loss of generality, we assume $C_l = (u_0, u_1, \ldots, u_{l-1}, u_0)$ and $C_m = (u_0, u_l, u_{l+1}, \ldots, u_{m+l-2}, u_0)$. This graph is denoted by $U(l : m)$.

**Definition 2.12** (6). A long dumbbell graph is a simple graph obtained by joining two cycles $C_l$ and $C_m$ where $l, m \geq 3$, with a path of length $i, i \geq 1$. Without loss of generality, we may assume $C_l = (u_0, u_1, \ldots, u_{l-1}, u_0)$, $P_i = (u_{l-1}, u_l, \ldots, u_{l+i-1})$ and $C_m = (u_{l+i-1}, u_{l+i}, .., u_{l+m+i-2}, u_{l+i-1})$. This graph is denoted by $D(l : m : i)$.

## 3. Main Results

**Theorem 3.1.** *Let G be a bicyclic graph with n pendant vertices. Also let* $U(l : m)$ *be the unique bicycle in G and let l be the number of vertices of degree greater than* 2 *on C . Then*

$$\eta_{2as}(G)=\begin{cases} 4 & \text{if } G=U(l:m) \\ (n+5)-m & \text{if } (l=1 \text{ and } deg(u_0)\geq 5) \\ (n+4)-m & \text{if } (l=2 \text{ and } deg(u_0)\geq 4, deg(v)\geq 3) \\ (n+3)-m & \text{if } (\ l=3 \text{ and } deg(u_0)\geq 4, deg(v)\geq 3, deg(u)\geq 3) \text{ Or } (l=4 \text{ and } deg(u_0)\geq 4, deg(u)\geq 3, deg(v)\geq 3, deg(w)\geq 3, u,v,w\in C_l) \text{ Or } (l\geq 5 \text{ and } deg(u_0)\geq 4, deg(u)\geq 3, deg(v)\geq 3, deg(w)\geq 3, deg(x)\geq 3, u,v,w,x\in C_l) \\ (n+2)-m & \text{if } (\ l=4 \text{ and } deg(u_0)\geq 4, deg(u)\geq 3, deg(v)\geq 3, deg(w)\geq 3, u,v\in C_l, w\in C_m) \text{ Or } (l\geq 5 \text{ and } deg(u_0)\geq 4, deg(u)\geq 3, deg(v)\geq 3, deg(w)\geq 3, deg(x)\geq 3, u,v,w\in C_l, x\in C_m) \\ (n+1)-m & \text{otherwise} \end{cases}$$

*Proof.* Let $C_l=(u_0,u_1,\dots,u_{l-1},u_0)$ and $C_m=(u_0,u_l,\dots,u_{l+m-2},u_0)$ be two cycles sharing a common vertex say $u_0$ with $q=p+1$.

**Case 1.** Suppose $G=U(l:m)$

Let $P_1=(u_2,u_3,...,u_0,u_l,...,u_{l+m-5}), P_2=(u_{l+m-5},u_{l+m-4}), P_3=(u_{l+m-4},u_{l+m-3},u_{l+m-2}, u_0,u_1), P_4=(u_1,u_2)$ is a minimum $2-$acyclic simple graphoidal cover of $G$ so that $\eta_{2as}(G)=4$.

**Case 2.** When $l=1$ and $deg(u_0)\geq 5$

Let $P=(w,x)$, where $P$ be a path on $U(l:m)$ and $w,x\in C_m$. Take $G_1=G-P$ is a unicyclic graph with $(n+2)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l=1$. Hence by theorem 2.9, $\eta_{2as}(G_1)=(n+2)+2-m=(n+4)-m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Then $\psi=\psi_1\cup\{P\}$ is a minimum $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G)\leq(n+5)-m$. For any $2-$acyclic simple graphoidal cover of $G$,

$n$ pendent vertices and atleast four vertices in $U(l:m)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+4), t_2(\psi) \leq m$. Hence $t \geq (n+4), t_2 \leq m$ so that $\eta_{2as}(G) = q-p-t_2+t = (p+1)-p-m+(n+4) \geq (n+5)-m$. Thus $\eta_{2as}(G) = (n+5)-m$.

**Case 3.** When $l=2$ and let $u_0, v$ be the two vertices is of degree greater than two on $U(l:m)$ and $deg(u_0) \geq 4, deg(v) \geq 3, v \in C_l$

Take $G_1 = G-P$ where $P = (w,x)$ be a path on $U(l:m)$ and $w,x \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n+2)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l=2$. By theorem 2.9, $\eta_{2as}(G_1) = (n+2)+1-m = (n+3)-m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P\}$ is a minimum $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+4)-m$. For any $2-$acyclic simple graphoidal cover of $G$, $n$ pendent vertices and atleast three vertices in $U(l:m)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+3), t_2(\psi) \leq m$. Hence $t \geq (n+3), t_2 \leq m$ so that $\eta_{2as}(G) = q-p-t_2+t = (p+1)-p-m+(n+3) \geq (n+4)-m$. Thus $\eta_{2as}(G) = (n+4)-m$.

**Case 4.** When $l=3$ and let $u_0, u, v$ be the vertices is of degree greater than two on $U(l:m)$ and $deg(u_0) \geq 4, deg(u) \geq 3, deg(v) \geq 3$. Then there are two subcases.

**Subcase 4.1.** Suppose $u, v \in C_l$

Take $G_1 = G-P$ where $P = (x,y)$ be a path on $U(l:m)$ and $x,y \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n+2)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l=3$. By theorem 2.9, $\eta_{2as}(G_1) = (n+2)-m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P\}$ is a minimum $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+3)-m$. For any $2-$acyclic simple graphoidal cover of $G$, $n$ pendent vertices and atleast two vertices in $U(l:m)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ so that $\eta_{2as}(G) = q-p-t_2+t = (p+1)-p-m+(n+2) \geq (n+3)-m$. Thus $\eta_{2as}(G) = (n+3)-m$.

**Subcase 4.2.** Suppose $u \in C_l, v \in C_m$, then there are two subcases.

**Subcase 4.2.1.** Suppose $deg(v) = 3$ (Or) $deg(v) \geq 5$

Take $G_1 = G-P$ where $P = (v,x)$ be a path on $U(l:m)$ and $x \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n+1)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l=2$. By theorem 2.9, $\eta_{2as}(G_1) = (n+2)-m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P\}$ is a minimum $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+3)-m$. For any $2-$acyclic simple graphoidal cover of $G$, $n$ pendent vertices and atleast two vertices in $U(l:m)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ so that $\eta_{2as}(G) = q-p-t_2+t = (p+1)-p-m+(n+2) \geq (n+3)-m$. Thus $\eta_{2as}(G) = (n+3)-m$.

**Subcase 4.2.2.** Suppose $deg(v) = 4$

Take $G_1 = G-P$ where $P = (v,x)$ be a path on $U(l:m)$ and $x \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n+1)$ pendent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l = 2$. By theorem 2.9, $\eta_{2as}(G_1) = (n+1)+1-(m-1) = (n+3)-m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_1$ be a path in $\psi_1$ in which $v$ is an external. Then $\psi = (\psi_1 - P_1) \cup \{P_1P\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq ((n+3)-m)-1+1 = (n+3)-m$. For any $2-$acyclic simple graphoidal cover of $G$, $n$ pendent vertices and atleast two vertices in $U(l:m)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ so that $\eta_{2as}(G) = q-p-t_2+t = (p+1)-p-m+(n+2) \geq (n+3)-m$. Thus $\eta_{2as}(G) = (n+3)-m$.

**Case 5.** Suppose $l = 4$ and let $u_0, u, v, w$ be the vertices is of degree greater than two on $U(l:m)$ and $deg(u_0) \geq 4, deg(u) \geq 3, deg(v) \geq 3, deg(w) \geq 3$, then there are two subcases.

**Subcase 5.1.** Suppose $u, v, w \in C_l$

Take $G_1 = G-P$, where $P = (y,z)$ be a path on $U(l:m)$ and $y, z \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n+2)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l = 3$. By theorem 2.9, $\eta_{2as}(G_1) = (n+2)-m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+3)-m$. For any $2-$acyclic simple graphoidal cover of $G$, $n$ pendent vertices and atleast two vertices in $U(l:m)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ so that $\eta_{2as}(G) = q-p-t_2+t = (p+1)-p-m+(n+2) \geq (n+3)-m$. Thus $\eta_{2as}(G) = (n+3)-m$.

**Subcase 5.2.** Suppose $u, v \in C_l$ and $w \in C_m$, then there are two subcases.

**Subcase 5.2.1.** Suppose $deg(w) = 3$ (Or) $deg(w) \geq 5$

Take $G_1 = G - P$, where $P = (w, y)$ be a path on $U(l : m)$ and $y \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n+1)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l = 3$. By theorem 2.9, $\eta_{2as}(G_1) = (n+1) - m$. Let $\psi_1$ be the minimum 2−acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P\}$ is a 2−acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+2) - m$. For any 2−acyclic simple graphoidal cover of $G$, $n$ pendent vertices and atleast one vertex in $U(l : m)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+1), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+1) \geq (n+2) - m$. Thus $\eta_{2as}(G) = (n+2) - m$.

**Subcase 5.2.2.** Suppose $deg(w) = 4$

Take $G_1 = G - P$, where $P = (w, y)$ be a path on $U(l : m)$ and $y \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n+1)$ pendent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l = 3$. By theorem 2.9, $\eta_{2as}(G_1) = (n+1) - (m-1) = (n+2) - m$. Let $\psi_1$ be the minimum 2−acyclic simple graphoidal cover of $G_1$. Let $P_1$ be a path in $\psi_1$ in which $x$ is an external vertex. Then $\psi = (\psi_1 - P_1) \cup \{P_1 P\}$ is a 2−acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq ((n+2) - m) - 1 + 1 = (n+2) - m$. For any 2−acyclic simple graphoidal cover of $G$, $n$ pendent vertices and atleast one vertex in $U(l : m)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+1), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+1) \geq (n+2) - m$. Thus $\eta_{2as}(G) = (n+2) - m$.

**Case 6.** When $l \geq 5$ and let $u_0, u, v, w, x$ be the vertices is of degree greater than two on $U(l : m)$ and Suppose $deg(u_0) \geq 4, deg(u) \geq 3, deg(v) \geq 3, deg(w) \geq 3, deg(x) \geq 3$, then there are three subcases.

**Subcase 6.1.** Suppose $u, v, w, x \in C_l$

Take $G_1 = G - P$, where $P = (y, z)$ be a path on $U(l : m)$ and $y, z \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n+2)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l \geq 5$. By theorem 2.9, $\eta_{2as}(G_1) = (n+2) - m$. Let $\psi_1$ be the minimum 2−acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P\}$ is a 2−acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+3) - m$. For any 2−acyclic simple graphoidal cover of $G$, $n$ pendent vertices and atleast two vertices in $U(l : m)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t =$

$(p+1)-p-m+(n+2) \geq (n+3)-m$. Thus $\eta_{2as}(G)=(n+3)-m$.

**Subcase 6.2.** When $u,v,w \in C_l$ and $x \in C_m$ and let $P=(x,y)$ be a path in $U(l:m)$ and $y \in C_m$. Then there are two subcases.

**Subcase 6.2.1.** Suppose $deg(x)=3$ (Or) $deg(x) \geq 5$

Take $G_1 = G-P$, is a unicyclic graph with $(n+1)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l \geq 4$. By theorem 2.9, $\eta_{2as}(G_1)=(n+1)-m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+2)-m$. For any $2-$acyclic simple graphoidal cover of $G$, $n$ pendent vertices and atleast one vertex in $U(l:m)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+1), t_2(\psi) \leq m$. Hence $t \geq (n+1), t_2 \leq m$ so that $\eta_{2as}(G)=q-p-t_2+t=(p+1)-p-m+(n+1) \geq (n+2)-m$. Thus $\eta_{2as}(G)=(n+2)-m$.

**Subcase 6.2.2.** Suppose $deg(x)=4$

Take $G_1 = G-P$ is a unicyclic graph with $(n+1)$ pendent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l \geq 4$. By theorem 2.9, $\eta_{2as}(G_1)=(n+1)-(m-1)=(n+2)-m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_1$ be a path in $\psi_1$ in which $x$ is an external vertex. Then $\psi=(\psi_1-P_1)\cup\{P_1 \cup P\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq ((n+2)-m)-1+1=(n+2)-m$. For any $2-$acyclic simple graphoidal cover of $G$, $n$ pendent vertices and atleast one vertex in $U(l:m)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+1), t_2(\psi) \leq m$. Hence $t \geq (n+1), t_2 \leq m$ so that $\eta_{2as}(G)=q-p-t_2+t=(p+1)-p-m+(n+1) \geq (n+2)-m$. Thus $\eta_{2as}(G)=(n+2)-m$.

**Subcase 6.3.** Suppose $u,v \in C_l$ and $w,x \in C_m$ and let $P=(w,x)$ be a path in $U(l:m)$. Then there are three subcases.

**Subcase 6.3.1.** When $(deg(w)=deg(x)=3$ (Or) $5)$ (Or) $(deg(w)=3$ and $deg(x) \geq 5)$ (Or) $(deg(w) \geq 5$ and $deg(x) \geq 5)$

Take $G_1 = G-P$ is a unicyclic graph with $n$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l=3$. By theorem 2.9, $\eta_{2as}(G_1)=n-m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Then $\psi=\psi_1 \cup \{P\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+1)-m$. For any $2-$acyclic simple graphoidal cover of $G$, $n$ pendent vertices are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq n, t_2(\psi) \leq m$.

Hence $t \geq n, t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + n \geq (n+1) - m$. Thus $\eta_{2as}(G) = (n+1) - m$.

**Subcase 6.3.2.** When $(deg(w) = 3$ and $deg(x) = 4)$ (Or) $(deg(w) \geq 5$ and $deg(x) = 4)$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l = 3$. By theorem 2.9, $\eta_{2as}(G_1) = n - (m-1) = (n+1) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_1$ be a path in $\psi_1$ in which $x$ is an external vertex. Then $\psi = (\psi_1 - P_1) \cup \{PP_1\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq ((n+1) - m) - 1 + 1 = (n+1) - m$. For any $2-$acyclic simple graphoidal cover of $G$, $n$ pendent vertices are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq n, t_2(\psi) \leq m$. Hence $t \geq n, t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + n \geq (n+1) - m$. Thus $\eta_{2as}(G) = (n+1) - m$.

**Subcase 6.3.3.** When $deg(w) = 4$ and $deg(x) = 4$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendent vertices and $(m-2)$ vertices is of degree $\geq 4$ with $l = 3$. By theorem 2.9, $\eta_{2as}(G_1) = n - (m-2) = (n+2) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_1$ be a path in $\psi_1$ in which $x$ is an external vertex and $P_2$ be a path in $\psi_1$ in which $w$ is an external vertex. Then $\psi = (\psi_1 - P_1 - P_2) \cup \{P_2PP_1\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq ((n+2) - m) - 2 + 1 = (n+1) - m$. For any $2-$acyclic simple graphoidal cover of $G$, $n$ pendent vertices are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq n, t_2(\psi) \leq m$. Hence $t \geq n, t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + n \geq (n+1) - m$. Thus $\eta_{2as}(G) = (n+1) - m$. $\square$

**Example 3.2.** Consider the Bicyclic graph $U(l:m)$ shown in Figure 1

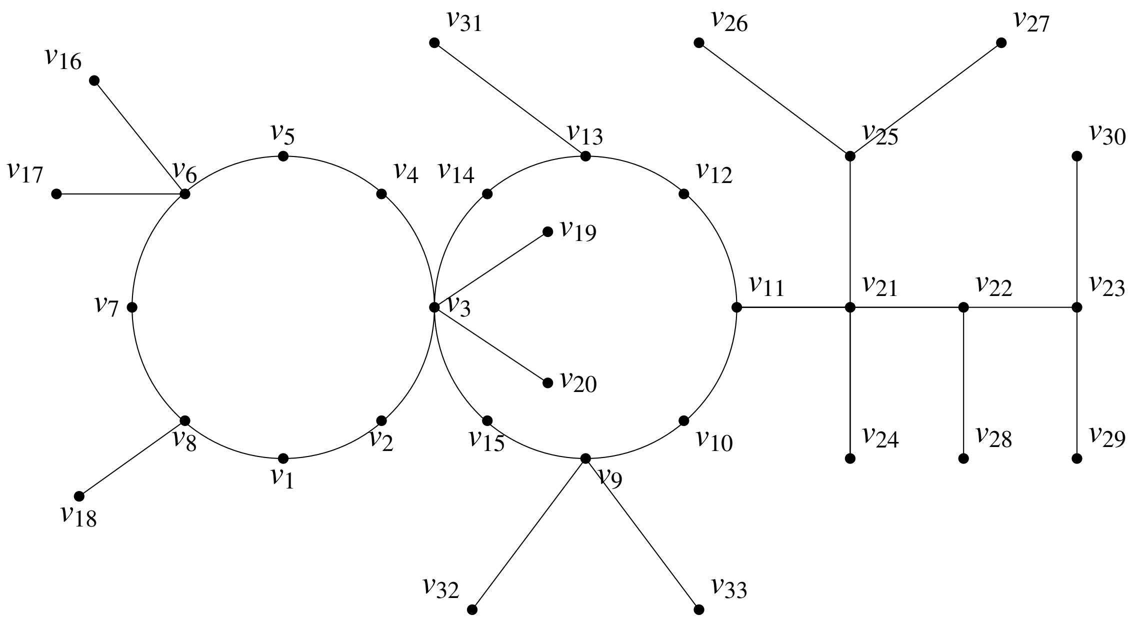


FIGURE 1. Bicyclic Graphs $U(l:m)$

Here $p = 33$, $q = 34$, $n = 14$, $l = 6$ and $m = 4$, then
$\eta_{2as}(G) = \{\ (v_{16}, v_6, v_5, v_4, v_3, v_{19}), (v_{20}, v_3, v_{14}, v_{13}, v_{31}), (v_{13}, v_{12}, v_{11}, v_{21}, v_{25}, v_{26}), (v_{25}, v_{27}),$
$(v_{24}, v_{21}, v_{22}, v_{28}), (v_{22}, v_{23}, v_{29}), (v_{23}, v_{30}), (v_{11}, v_{10}, v_9, v_{33}), (v_{32}, v_9, v_{15}, v_3), (v_3, v_2, v_1, v_8, v_{18}),$
$(v_8, v_7, v_6, v_{17})\} = 11 = (n+1) - m$

**Theorem 3.3.** *Let $G$ be a bicyclic graph with $n$ pendant vertices. Also let $D(l:m:i)$ be the unique bicycle in $G$ and let $l$ be the number of vertices of degree greater than 2 on cycles on $D(l:m:i)$ Then*

$$\eta_{2as}(G) = \begin{cases} 5 & \text{if } G = D(l:m:i) \\ (n+5)-m & \text{if } (l = 2 \text{ and } deg(u_{l-1}) \geq 4, deg(u_{l+i-1}) \geq 3) \\ (n+4)-m & \text{if } (\ l = 3 \text{ and } deg(u_{l-1}) \geq 3,\ deg(u_{l+i-1}) \geq 3, \\ & deg(u) \geq 3) \\ (n+3)-m & \text{if } (\ l = 4 \text{ and } deg(u_{l-1}) \geq 3, deg(u_{l+i-1}) \geq 3, deg(u) \geq \\ & 3, deg(v) \geq 3) \text{ Or } (l \geq 5 \text{ and } deg(u_{l-1}) \geq 3, deg(u_{l+i-1}) \geq \\ & 3, deg(u) \geq 3, deg(v) \geq 3, u,v,w \in C_l) \\ (n+2)-m & \text{if } (l = 5 \text{ and } deg(u_{l-1}) \geq 3, deg(u_{l+i-1}) \geq 3, deg(u) \geq \\ & 3, deg(v) \geq 3, deg(w) \geq 3, u,v \in C_l, w \in C_m) \text{ Or } (l > \\ & 5, deg(u_{l-1}) \geq 3, deg(u_{l+i-1}) \geq 3, deg(u) \geq 3, deg(v) \geq \\ & 3, deg(w), deg(x) \geq 3, u,v,x \in C_l, w \in C_m)) \\ (n+1)-m & \text{otherwise} \end{cases}$$

*Proof.* Let $C_l = (u_0, u_1, \ldots\ldots, u_{l-1}, u_0), P_i = (u_{l-1}, u_l, \ldots\ldots, u_{l+i-1})$ and $C_m = (u_{l+i-1}, u_{l+i}, \ldots, u_{l+m+i-2}, u_{l+i-1})$.

**Case 1.** Suppose $G = D(l:m:i)$

Let $P_1 = (u_2, u_3, \ldots\ldots, u_{l-3})$, $P_2 = (u_{l-3}, u_{l-2}, u_{l-1}, u_l, \ldots\ldots, u_{l+i-1}, u_{l+i})$, $P_3 = (u_{l+i}, u_{l+i+1})$, $P_4 = (u_{l+i+1}, u_{l+i+2}, \ldots\ldots, u_{l+i-1})$, $P_5 = (u_{l-1}, u_0, u_1, u_2)$ is a minimum $2-$acyclic simple graph-oidal cover of $G$ so that $\eta_{2as}(G) = 5$.

**Case 2.** Suppose $l = 2$ and let $P$ denote $(u_{l+i-1}, w)$ section of $C_m$ such that it has atleast one internal vertex say $u_i$ and $w \in C_m$. Let $P_1$ and $P_2$ denote the $(u_{l+i-1}, u_i)$ and $(u_i, w)$ section of $P$ respectively. Then there are two subcases.

**Subcase 2.1.** When $deg(u_{l-1}) \geq 4$ and $(deg(u_{l+i-1}) = 3$ (Or) $deg(u_{l+i-1}) \geq 5)$

Take $G_1 = G - \{P\}$ is a unicyclic graph with $(n+1)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l = 1$. Hence by the theorem 2.9, $\eta_{2as}(G_1) = ((n+1)+2) - m = (n+3) - m$.

Let $\psi_1$ is a minimum $2-$acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P_1\} \cup \{P_2\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+5)-m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and four vertices in $D(l:m:i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+4), t_2(\psi) \leq m$. Hence $t \geq (n+4), t_2 \leq m$ so that $\eta_{2as}(G) = q-p-t_2+t = (p+1)-p-m+(n+4) \geq (n+5)-m$. Thus $\eta_{2as}(G) = (n+5)-m$.

**Subcase 2.2.** When $deg(u_{l-1}) \geq 4$ and $deg(u_{l+i-1}) = 4$

Take $G_1 = G - \{P\}$ is a unicyclic graph with $(n+1)$ pendent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l = 1$. Hence by the theorem 2.9, $\eta_{2as}(G_1) = ((n+1)+2)-(m-1) = (n+4)-m$. Let $\psi_1$ is a minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $u_{l+i-1}$ is an external vertex. Take $\psi = (\psi_1 - P_3) \cup \{P_3P_1\} \cup \{P_2\}$ is a $2-$acyclic simple graphoidal cover of $G$ and $\eta_{2as}(G) \leq ((n+4)-m)-1+2 = (n+5)-m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and four vertices in $D(l:m:i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+4), t_2(\psi) \leq m$. Hence $t \geq (n+4), t_2 \leq m$ and hence $\eta_{2as}(G) = q-p-t_2+t = (p+1)-p-m+(n+4) \geq (n+5)-m$. Thus $\eta_{2as}(G) = (n+5)-m$.

**Case 3.** When $l = 3$ and let $u$ be the only vertex is of degree greater than 2 other than $u_{l-1}, u_{l+i-1}$. Let $P$ denote $(u_{l+i-1}, w)$ section of $C_m$ such that it has atleast one internal vertex say $u_i$ and $w \in C_m$. Let $P_1$ and $P_2$ denote the $(u_{l+i-1}, u_i)$ and $(u_i, w)$ section of $P$ respectively. Suppose $deg(u_{l-1}) \geq 3, deg(u) \geq 3, u \in C_m$, then there are two subcases.

**Subcase 3.1.** When $deg(u_{l+i-1}) = 3$ (Or) $deg(u_{l+i-1}) \geq 5$

Take $G_1 = G - \{P\}$ is a unicyclic graph with $(n+1)$ pendent vertices and $m$ vertices is of degree $\geq 4$ with $l = 2$. Hence by the theorem 2.9, $\eta_{2as}(G_1) = ((n+1)+1)-m = (n+2)-m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Then $\psi = \psi_1 \cup \{P_1\} \cup \{P_2\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+4)-m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and three vertices in $D(l:m:i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+3), t_2(\psi) \leq m$. Hence $t \geq (n+3), t_2 \leq m$ and hence $\eta_{2as}(G) = q-p-t_2+t = (p+1)-p-m+(n+3) \geq (n+4)-m$.

Thus $\eta_{2as}(G) = (n+4) - m$.

**Subcase 3.2.** When $deg(u_{l+i-1}) = 4$

Take $G_1 = G - \{P\}$ is a unicyclic graph with $(n+1)$ pendent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l = 2$. Hence by the theorem 2.9, $\eta_{2as}(G_1) = ((n+1)+1) - (m-1) = (n+3) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $u_{l+i-1}$ is external vertex. Take $\psi = (\psi_1 - P_3) \cup \{P_3 P_1\} \cup \{P_2\}$ is a $2-$acyclic simple graphoidal cover of $G$ and $\eta_{2as}(G) \leq ((n+3) - m) - 1 + 2 = (n+4) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and three vertices in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+3), t_2(\psi) \leq m$. Hence $t \geq (n+3), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+3) \geq (n+4) - m$. Thus $\eta_{2as}(G) = (n+4) - m$.

**Case 4.** When $l = 4$ and $u, v$ be the only vertices is of degree greater than 2 other than $u_{l-1}, u_{l+i-1}$. Then there are four sub cases.

**Subcase 4.1.** When $deg(u_{l-1}) \geq 3, deg(u_{l+i-1}) = 3$, then there are two subcases.

**Subcase 4.1.1.** When $deg(u) \geq 3, deg(v) \geq 3$ and $u, v \in C_l$

Take $G_1 = G - P$, where $P$ denotes $(u_{l+i-1}, w)$ section of $C_m$ such that it has atleast one internal vertex say $u_i$ and $w \in C_m$. Let $P_1$ and $P_2$ denote the $(u_{l+i-1}, u_i)$ and $(u_i, w)$ section of $P$ respectively. It is clear that $G_1$ is a unicyclic graph with $(n+1)$ pendnent vertices and $m$ vertices is of degree $\geq 4$ with $l = 2$. Therefore by therorem 2.9, $\eta_{2as}(G_1) = (n+1) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{P_1\} \cup \{P_2\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+1) - m + 2 = (n+3) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and two vertices in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+2) \geq (n+3) - m$. Thus $\eta_{2as}(G) = (n+3) - m$.

**Subcase 4.1.2.** When $deg(u) \geq 3$ and $deg(v) \geq 3$, $u \in C_l, v \in C_m$ and let $P$ denotes $(u_{l+i-1}, v)$ section of $C_m$ such that it has atleast one internal vertex say $u_i$. Let $P_1$ and $P_2$ denote the $(u_{l+i-1}, u_i)$ and $(u_i, v)$ section of $P$ respectively.Then there are two subcases.

**Subcase 4.1.2.1.** When $deg(v) = 3$ (Or) $deg(v) \geq 5$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $m$ vertices is of degree $\geq 4$ with $l = 2$. Therefore by therorem 2.9, $\eta_{2as}(G_1) = (n+1) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{P_1\} \cup \{P_2\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+1) - m + 2 = (n+3) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and two vertices in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+2) \geq (n+3) - m$. Thus $\eta_{2as}(G) = (n+3) - m$.

**Subcase 4.1.2.2.** When $deg(v) = 4$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l = 2$. Hence by therorem 2.9, $\eta_{2as}(G_1) = (n+1) - (m-1) = (n+2) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $v$ is an external vertex. Take $\psi = \psi_1 \cup \{P_1\} \cup \{P_2 P_3\}$ is a $2-$acyclic simple graphoidal cover of $G$ and $\eta_{2as}(G) \leq (n+2) - m + 1 + 2 = (n+3) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and two vertices in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+2) \geq (n+3) - m$. Thus $\eta_{2as}(G) = (n+3) - m$.

**Subcase 4.2.** When $deg(u_{l-1}) \geq 4$ and $deg(u_{l+i-1}) \geq 4$, then there are three subcases.

**Subcase 4.2.1.** When $deg(u) \geq 3, deg(v) \geq 3$, $u, v \in C_l$

Take $G_1 = G - P$, where $P$ denotes $(x, y)$ section of $C_m$ and $x, y \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n+2)$ pendnent vertices and $m$ vertices is of degree $\geq 4$ with $l = 3$. Hence by therorem 2.9, $\eta_{2as}(G_1) = (n+2) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{P\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+2) - m + 1 = (n+3) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and two vertices in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+2) \geq (n+4) - m$. Thus $\eta_{2as}(G) = (n+3) - m$.

**Subcase 4.2.2.** When $deg(u_{l+i-1}) = 4$, $deg(u) \geq 3, deg(v) \geq 3$, $u \in C_l, v \in C_m$ and let $P$ denotes $(u_{l+i-1}, v)$ section of $C_m$ such that it has atleast one internal vertex say $u_i$. Let $P_1$ and $P_2$ denote the $(u_{l+i-1}, u_i)$ and $(u_i, v)$ section of $P$ respectively. Then there are two subcases.

**Subcase 4.2.2.1.** When $deg(v) = 3$ (Or) $deg(v) \geq 5$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l = 2$. Hence by therorem 2.9, $\eta_{2as}(G_1) = (n+1) - (m-1) = (n+2) - m$. Let $\psi_1$ be the minimum 2−acyclic simple graphoidal cover of $G_1$. Let $P_4$ be a path in $\psi_1$ in which $u_{l+i-1}$ is an external vertex. Take $\psi = (\psi_1 - P_4) \cup \{P_4P_1\} \cup \{P_2\}$ is a 2−acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+2) - m - 1 + 2 = (n+3) - m$. For any 2−acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast two vertices in $D(l:m:i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m(n+2) \geq (n+3) - m$. Thus $\eta_{2as}(G) = (n+3) - m$.

**Subcase 4.2.2.2.** When $deg(v) = 4$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $(m-2)$ vertices is of degree $\geq 4$ with $l = 2$. Hence by therorem 2.9, $\eta_{2as}(G_1) = (n-1) - (m-2) = (n+3) - m$. Let $\psi_1$ be the minimum 2−acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $v$ is an external vertex and let $P_4$ be a path in $\psi_1$ in which $u_{l+i-1}$ as the external vertex. Take $\psi = (\psi_1 - P_3 - P_4) \cup \{P_4P_1\} \cup \{P_2P_3\}$ is a 2−acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+3) - m - 2 + 2 = (n+3) - m$. For any 2−acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast two vertices in $D(l:m:i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m(n+2) \geq (n+3) - m$. Thus $\eta_{2as}(G) = (n+3) - m$.

**Subcase 4.3.** When $deg(u_{l+i-1}) \geq 5$ and $deg(u) \geq 3$, $deg(v) \geq 3$, $u \in C_l, v \in C_m$ and let $P$ denotes $(u_{l+i-1}, v)$ section of $C_m$ such that it has atleast one internal vertex say $u_i$. Let $P_1$ and $P_2$ denote the $(u_{l+i-1}, u_i)$ and $(u_i, v)$ section of $P$ respectively. Then there are two subcases.

**Subcase 4.3.1.** When $deg(v) = 3$ Or $deg(v) \geq 5$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $m$ vertices is of degree $\geq 4$ with $l = 2$. Hence by therorem 2.9, $\eta_{2as}(G_1) = (n+1) - m$. Let $\psi_1$ be the minimum 2−acyclic simple graphoidal cover of $G_1$. Let $P_4$ be a path in $\psi_1$ in which $u_{l+i-1}$ is an external vertex. Take $\psi = \psi_1 \cup \{P_1\} \cup \{P_2\}$ is a 2−acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq$

$(n+3)-m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast two vertices in $D(l:m:i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ and hence $\eta_{2as}(G) = q-p-t_2+t = (p+1)-p-m(n+2) \geq (n+3)-m$. Thus $\eta_{2as}(G) = (n+3)-m$.

**Subcase 4.3.2.** When $deg(v) = 4$

Take $G_1 = G-P$ is a unicyclic graph with $n$ pendnent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l = 2$. Hence by therorem 2.9, $\eta_{2as}(G_1) = (n+1)-(m-1) = (n+2)-m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $v$ is an external vertex. Take $\psi = (\psi_1 - P_3) \cup \{P_1\} \cup \{P_2P_3\}$ is a $2-$acyclic simple graphoidal cover of $G$ and $\eta_{2as}(G) \leq (n+2)-m-1+2 = (n+3)-m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast two vertices in $D(l:m:i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ and hence $\eta_{2as}(G) = q-p-t_2+t = (p+1)-p-m(n+2) \geq (n+3)-m$. Thus $\eta_{2as}(G) = (n+3)-m$.

**Case 5.** When $l = 5$ and $u, v, w$ be the only vertices is of degree greater than 2 other than $u_{l-1}, u_{l+i-1}$. Then there are three subcases.

**Subcase 5.1.** Suppose $deg(u_{l-1}) \geq 3, deg(u_{l+i-1}) = 3$, then there are two subcases.

**Subcase 5.1.1.** When $deg(u) \geq 3, deg(v) \geq 3, deg(w) \geq 3$, $u, v, w \in C_l$

Take $G_1 = G-P$, where Let $P = (x, y)$ be a path in $C_m$ and $x, y \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n+2)$ pendnent vertices and $m$ vertices is of degree $\geq 4$ with $l > 3$. Hence by therorem 2.9, $\eta_{2as}(G_1) = (n+2)-m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{\cup P\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq ((n+2)-m)+1 = (n+3)-m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and two vertices in $D(l:m:i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ and hence $\eta_{2as}(G) = q-p-t_2+t = (p+1)-p-m+(n+2) \geq (n+3)-m$. Thus $\eta_{2as}(G) = (n+3)-m$.

**Subcase 5.1.2.** When $deg(u) \geq 3$, $deg(v) \geq 3$, $deg(w) \geq 3$ and $u, v \in C_l, w \in C_m$. Let $P$ denotes $(u_{l+i-1}, w)$ section of $C_m$ such that it has atleast one internal vertex say $u_i$. Let $P_1$ and $P_2$ denote the $(u_{l+i-1}, u_i)$ and $(u_i, w)$ section of $P$ respectively. Then there are two subcases.

**Subcase 5.1.2.1.** When $deg(w) = 3$ (Or) $deg(w) \geq 5$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $m$ vertices is of degree $\geq 4$ with $l = 3$. Hence by therorem 2.9, $\eta_{2as}(G_1) = n - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{P_1\} \cup \{P_2\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+2) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast one vertex in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+1), t_2(\psi) \leq m$. Hence $t \geq (n+1), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+1) \geq (n+2) - m$. Thus $\eta_{2as}(G) = (n+2) - m$.

**Subcase 5.1.2.2.** When $deg(w) = 4$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l = 3$. Hence by therorem 2.9, $\eta_{2as}(G_1) = n - (m-1) = (n+1) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi$ in which $w$ as the external vertex. Take $\psi = (\psi_1 - P_3) \cup \{P_1\} \cup \{P_2 P_3\}$ is a $2-$acyclic simple graphoidal cover of $G$ and $\eta_{2as}(G) \leq (n+1) - m - 1 + 2 = (n+2) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast one vertex in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+1), t_2(\psi) \leq m$. Hence $t \geq (n+1), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+1) \geq (n+2) - m$. Thus $\eta_{2as}(G) = (n+2) - m$.

**Subcase 5.2.** When $deg(u_{l-1}) \geq 4$ and $deg(u_{l+i-1}) \geq 4$, then there are three subcases.

**Subcase 5.2.1.** Suppose $deg(u_{l+i-1}) \geq 4$, $deg(u) \geq 3$, $deg(v) \geq 3$, $deg(w) \geq 3$, $u, v, w \in C_l$

Take $G_1 = G - P$, where Let $P = (y, z)$ be a path in $C_m$ and $y, z \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n+2)$ pendnent vertices and $m$ vertices is of degree $\geq 4$ with $l > 3$. Hence by therorem 2.9, $\eta_{2as}(G_1) = (n+2) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{P\}$ is a $2-$acyclic simple graphoidal cover of $G$ so that $\eta_{2as}(G) \leq ((n+2) - m) + 1 = (n+3) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and two vertices in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+2) \geq (n+3) - m$. Thus $\eta_{2as}(G) = (n+3) - m$.

**Subcase 5.2.2.** Suppose $deg(u_{l+i-1}) = 4$, $deg(u) \geq 3$, $deg(v) \geq 3$, $deg(w) \geq 3$, $u, v \in C_l, w \in C_m$ and let $P$ denotes $(u_{l+i-1}, w)$ section of $C_m$ such that it has atleast one internal vertex say $u_i$. Let $P_1$ and $P_2$ denote the $(u_{l+i-1}, u_i)$ and $(u_i, w)$ section of $P$ respectively. Then there are two subcases.

**Subcase 5.2.2.1.** When $deg(w) = 3$ (Or) $deg(w) \geq 5$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l = 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n+1) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_4$ be a path in $\psi_1$ in which $u_{l+i-1}$ is an external vertex. Take $\psi = (\psi_1 - P_4) \cup \{P_4P_1\} \cup \{P_2\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+1) - m - 1 + 2 = (n+2) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast one vertex in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+1), t_2(\psi) \leq m$. Hence $t \geq (n+1), t_2 \leq m$ and so $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+1) \geq (n+2) - m$. Thus $\eta_{2as}(G) = (n+2) - m$.

**Subcase 5.2.2.2.** When $deg(w) = 4$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $(m-2)$ vertices is of degree $\geq 4$ with $l = 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n+2) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $w$ is an external vertex and let $P_4$ be a path in $\psi_1$ in which $u_{l+i-1}$ is an external vertex. Take $\psi = (\psi_1 - P_3 - P_4) \cup \{P_4P_1\} \cup \{P_2P_3\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+2) - m - 2 + 2 = (n+2) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast one vertex in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+1), t_2(\psi) \leq m$. Hence $t \geq (n+1), t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+1) \geq (n+2) - m$. Thus $\eta_{2as}(G) = (n+2) - m$.

**Subcase 5.2.3.** When $deg(u_{l+i-1}) \geq 5$, $deg(u) \geq 3$, $deg(v) \geq 3$, $deg(w) \geq 3$ and $u, v, \in C_l, w \in C_m$ and let $P$ denotes $(u_{l+i-1}, w)$ section of $C_m$ such that it has atleast one internal vertex say $u_i$. Let $P_1$ and $P_2$ denote the $(u_{l+i-1}, u_i)$ and $(u_i, w)$ section of $P$ respectively. then there are two subcases.

**Subcase 5.2.3.1.** When $deg(w) = 3$ Or $deg(w) \geq 5$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $m$ vertices is of degree $\geq 4$ with $l = 3$. Hence by therorem 2.9, $\eta_{2as}(G_1) = n - m$. Let $\psi_1$ be the minimum 2$-$acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{P_1\} \cup \{P_2\}$ is a 2$-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+2) - m$. For any 2$-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast one vertex in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+1), t_2(\psi) \leq m$. Hence $t \geq (n+1), t_2 \leq m$ and so $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m(n+1) \geq (n+2) - m$. Thus $\eta_{2as}(G) = (n+2) - m$.

**Subcase 5.2.3.2.** When $deg(w) = 4$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l > 3$. Hence by therorem 2.9, $\eta_{2as}(G_1) = (n+1) - m$. Let $\psi_1$ be the minimum 2$-$acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $w$ is an external vertex. Take $\psi = (\psi_1 - P_3) \cup \{P_1\} \cup \{P_2P_3\}$ is a 2$-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+1) - m - 1 + 2 = (n+2) - m$. For any 2$-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast one vertex in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+1), t_2(\psi) \leq m$. Hence $t \geq (n+1), t_2 \leq m$ and so $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+1) \geq (n+2) - m$. Thus $\eta_{2as}(G) = (n+2) - m$.

**Case 6.** When $l > 5$ and $u, v, w, x$ be the vertices is of degree greater than 2 on $D(l : m : i)$ other than $u_{l-1}, u_{l+i-1}$. Then there are three subcases.

**Subcase 6.1.** Suppose $deg(u_{l-1}) \geq 3$ and $deg(u_{l+i-1}) = 3$, then there are three sub cases.

**Subcase 6.1.1.** When $deg(u) \geq 3$, $deg(v) \geq 3$, $deg(w) \geq 3$ and $deg(x) \geq 3$, $u, v, w, x \in C_l$

Take $G_1 = G - P$ where $P = (y, z)$ be a path in $C_m$ and $y, z \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n+2)$ pendnent vertices and $m$ vertices is of degree $\geq 4$ with $l > 3$. Hence by therorem 2.9, $\eta_{2as}(G_1) = (n+2) - m$. Let $\psi_1$ be the minimum 2$-$acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{\cup P\}$ is a 2$-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq ((n+2) - m) + 1 = (n+3) - m$. For any 2$-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and two vertices in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+2) \geq (n+3) - m$. Thus $\eta_{2as}(G) = (n+3) - m$.

**Subcase 6.1.2.** When $deg(u) \geq 3$, $deg(v) \geq 3$, $deg(w) \geq 3$ and $deg(x) \geq 3$, $u, v, w \in C_l$, $x \in C_m$

and let $P$ denotes $(u_{l+i-1}, x)$ section of $C_m$ such that it has atleast one internal vertex say $u_i$. Let $P_1$ and $P_2$ denote the $(u_{l+i-1}, u_i)$ and $(u_i, x)$ section of $P$ respectively. Then there are two subcases.

**Subcase 6.1.2.1.** When $deg(x) = 3$ (Or) $deg(x) \geq 5$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $m$ vertices is of degree $\geq 4$ with $l > 3$. Hence by therorem 2.9, $\eta_{2as}(G_1) = n - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{P_1\} \cup \{P_2\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+2) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast one vertex in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+1), t_2(\psi) \leq m$. Hence $t \geq (n+1), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+1) \geq (n+2) - m$. Thus $\eta_{2as}(G) = (n+2) - m$.

**Subcase 6.1.2.2.** When $deg(x) = 4$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l = 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n+1) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $w$ is an external vertex. Take $\psi = (\psi_1 - P_3) \cup \{P_1\} \cup \{P_2 \cup P_3\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+1) - m - 1 + 2 = (n+2) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast one vertex in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+1), t_2(\psi) \leq m$. Hence $t \geq (n+1), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+1) \geq (n+2) - m$. Thus $\eta_{2as}(G) = (n+2) - m$.

**Subcase 6.1.3.** When $deg(u) \geq 3$, $deg(v) \geq 3$, $deg(w) \geq 3$ and $deg(x) \geq 3$, $u, v, \in C_l, w, x \in C_m$ and let $P$ denotes $(w, x)$ section of $C_m$. Then there are three subcases.

**Subcase 6.1.3.1.** When $(deg(w) = deg(x) = 3)$ (Or) $(deg(w) \geq 5, deg(x) \geq 5)$ (or) $(deg(w) = 3, deg(x) \geq 5)$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $m$ vertices is of degree $\geq 4$ with $l > 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = n - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{P\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+1) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq n, t_2(\psi) \leq m$.

Hence $t \geq n, t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + n \geq (n+1) - m$. Thus $\eta_{2as}(G) = (n+1) - m$.

**Subcase 6.1.3.2.** When $deg(w) = deg(x) = 4$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $(m-2)$ vertices is of degree $\geq 4$ with $l \geq 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n+2) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $w$ is an external vertex and let $P_4$ be a path in $\psi$ in which $x$ is an external vertex. Take $\psi = (\psi_1 - P_3 - P_4) \cup \{P_3PP_4\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+2) - m - 2 + 1 = (n+1) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq n, t_2(\psi) \leq m$. Hence $t \geq n, t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + n \geq (n+1) - m$. Thus $\eta_{2as}(G) = (n+1) - m$.

**Subcase 6.1.3.3.** When $(deg(w) = 3, deg(x) = 4)$ Or $(deg(w) \geq 5, deg(x) = 4)$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l \geq 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = n - (m-1) = (n+1) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $w$ is an external vertex. Take $\psi = (\psi_1 - P_3) \cup \{PP_3\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+1) - m - 1 + 1 = (n+1) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq n, t_2(\psi) \leq m$. Hence $t \geq n, t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + n \geq (n+1) - m$. Thus $\eta_{2as}(G) = (n+1) - m$.

**Subcase 6.2.** When $deg(u_{l-1}) \geq 4$ and $deg(u_{l+i-1}) \geq 4$, then there are three sub cases.

**Subcase 6.2.1.** When $deg(u_{l+i-1}) \geq 4$, $deg(u) \geq 3$, $deg(v) \geq 3$, $deg(w) \geq 3$ and $deg(x) \geq 3, u, v, w, x \in C_l$

Take $G_1 = G - P$ where $P = (y, z)$ be a path in $C_m$ and $y, z \in C_m$. It is clear that $G_1$ is a unicyclic graph with $(n+2)$ pendnent vertices and $m$ vertices is of degree $\geq 4$ with $l > 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n+2) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{P\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq ((n+2) - m) + 1 = (n+3) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and two vertices in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+2), t_2(\psi) \leq m$. Hence $t \geq (n+2), t_2 \leq m$ so that $\eta_{2as}(G) = q-p-t_2+t = (p+1)-p-m(n+2) \geq (n+3)-m$. Thus $\eta_{2as}(G) = (n+3)-m$.

**Subcase 6.2.2.** Suppose $deg(u_{l+i-1}) = 4$ and $deg(u) \geq 3, deg(v) \geq 3, deg(w) \geq 3$ and $deg(x) \geq 3, u,v,w \in C_l, x \in C_m$ and let $P$ denotes $(u_{l+i-1},x)$ section of $C_m$ such that it has atleast one internal vertex say $u_i$. Let $P_1$ and $P_2$ denote the $(u_{l+i-1},u_i)$ and $(u_i,x)$ section of $P$ respectively. Then there are two subcases.

**Subcase 6.2.2.1.** When $deg(x) = 3$ Or $deg(x) \geq 5$

Take $G_1 = G-P$ is a unicyclic graph with $n$ pendnent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l > 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n+1)-m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_4$ be a path in $\psi_1$ in which $u_{l+i-1}$ is an external vertex. Take $\psi = (\psi_1 - P_4) \cup \{P_4 \cup P_1\} \cup \{P_2\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+1)-m-1+2 = (n+2)-m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast one vertex in $D(l:m:i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+1), t_2(\psi) \leq m$. Hence $t \geq (n+1), t_2 \leq m$ and hence $\eta_{2as}(G) = q-p-t_2+t = (p+1)-p-m+(n+1) \geq (n+2)-m$. Thus $\eta_{2as}(G) = (n+2)-m$.

**Subcase 6.2.2.2.** When $deg(x) = 4$

Take $G_1 = G-P$ is a unicyclic graph with $n$ pendnent vertices and $(m-2)$ vertices is of degree $\geq 4$ with $l > 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n+2)-m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $x$ is an external vertex and let $P_4$ be a path in $\psi_1$ in which $u_{l+i-1}$ is an external vertex. Take $\psi = (\psi_1 - P_3 - P_4) \cup \{P_4P_1\} \cup \{P_2P_3\}$ is a $2-$acyclic simple graphoidal cover of $G$ and $\eta_{2as}(G) \leq (n+2)-m-2+2 = (n+2)-m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast one vertex in $D(l:m:i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+1), t_2(\psi) \leq m$. Hence $t \geq (n+1), t_2 \leq m$ and hence $\eta_{2as}(G) = q-p-t_2+t = (p+1)-p-m+(n+1) \geq (n+2)-m$. Thus $\eta_{2as}(G) = (n+2)-m$.

**Subcase 6.2.3.** When $deg(u_{l+i-1}) \geq 5$ and $deg(u) \geq 3$, $deg(v) \geq 3$, $deg(w) \geq 3$ and $deg(x) \geq 3$, $u,v,w \in C_l, x \in C_m$ and let $P$ denotes $(u_{l+i-1},x)$ section of $C_m$ such that it has atleast one internal vertex say $u_i$. Let $P_1$ and $P_2$ denote the $(u_{l+i-1},u_i)$ and $(u_i,x)$ section of $P$ respectively. Then there are two subcases.

**Subcase 6.2.3.1.** When $deg(x) = 3$ (Or) $deg(x) \geq 5$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $m$ vertices is of degree $\geq 4$ with $l > 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n+1) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_4$ be a path in $\psi_1$ in which $u_{l+i-1}$ is an external vertex. Take $\psi = \psi_1 \cup \{P_1\} \cup \{P_2\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+2) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast one vertex in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+1), t_2(\psi) \leq m$. Hence $t \geq (n+1), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m(n+1) \geq (n+2) - m$. Thus $\eta_{2as}(G) = (n+2) - m$.

**Subcase 6.2.3.2.** When $deg(x) = 4$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l > 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n+1) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $x$ is an external vertex. Take $\psi = (\psi_1 - P_3) \cup \{P_1\} \cup \{P_2 P_3\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+1) - m - 1 + 2 = (n+2) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices and atleast one vertex in $D(l : m : i)$ are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq (n+1), t_2(\psi) \leq m$. Hence $t \geq (n+1), t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + (n+1) \geq (n+2) - m$. Thus $\eta_{2as}(G) = (n+2) - m$.

**Subcase 6.2.4.** When $deg(u_{l+i-1}) \geq 4, deg(u) \geq 3, deg(v) \geq 3, deg(w) \geq 3$ and $deg(x) \geq 3, u, v \in C_l, w, x \in C_m$ and let $P$ denotes $(w, x)$ section of $C_m$. Then there are three subcases.

**Subcase 6.2.4.1.** When $(deg(w) = deg(x) = 3)$ Or $(deg(w) = 3$ and $deg(x) \geq 5)$ (Or) $(deg(w) \geq 5$ and $deg(x) \geq 5)$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $m$ vertices is of degree $\geq 4$ with $l > 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = n - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Take $\psi = \psi_1 \cup \{P\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+1) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq n, t_2(\psi) \leq m$. Hence $t \geq n, t_2 \leq m$ and hence $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + n \geq (n+1) - m$.

Thus $\eta_{2as}(G) = (n+1) - m$.

**Subcase 6.2.4.2.** When $(deg(w) = 4$ and $deg(x) = 3)$ Or $(deg(w) = 4$ and $deg(x) \geq 5)$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $(m-1)$ vertices is of degree $\geq 4$ with $l > 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n+1) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_4$ be a path in $\psi_1$ in which $w$ is an external vertex. Take $\psi = (\psi_1 - P_4) \cup \{P_4 P\}$ is a $2-$acyclic simple graphoidal cover of $G$ and $\eta_{2as}(G) \leq (n+1) - m - 1 + 1 = (n+1) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq n, t_2(\psi) \leq m$. Hence $t \geq n, t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + n \geq (n+1) - m$. Thus $\eta_{2as}(G) = (n+1) - m$.

**Subcase 6.2.4.3.** When $deg(w) = deg(x) = 4$

Take $G_1 = G - P$ is a unicyclic graph with $n$ pendnent vertices and $(m-2)$ vertices is of degree $\geq 4$ with $l \geq 3$. Hence by theorem 2.9, $\eta_{2as}(G_1) = (n+2) - m$. Let $\psi_1$ be the minimum $2-$acyclic simple graphoidal cover of $G_1$. Let $P_3$ be a path in $\psi_1$ in which $w$ is an external vertex and let $P_4$ be a path in $\psi$ in which $x$ as the external vertex. Take $\psi = (\psi_1 - P_3 - P_4) \cup \{P_3 P P_4\}$ is a $2-$acyclic simple graphoidal cover of $G$, hence $\eta_{2as}(G) \leq (n+2) - m - 2 + 1 = (n+1) - m$. For any $2-$acyclic simple graphoidal cover of $G$ atleast $n$ pendent vertices are external and atmost $m$ vertices are internal twice. Therefore $t_\psi \geq n, t_2(\psi) \leq m$. Hence $t \geq n, t_2 \leq m$ so that $\eta_{2as}(G) = q - p - t_2 + t = (p+1) - p - m + n \geq (n+1) - m$. Thus $\eta_{2as}(G) = (n+1) - m$. $\square$

**Example 3.4.** Consider the Bicyclic graph $D(l:m:i)$ shown in Figure 2

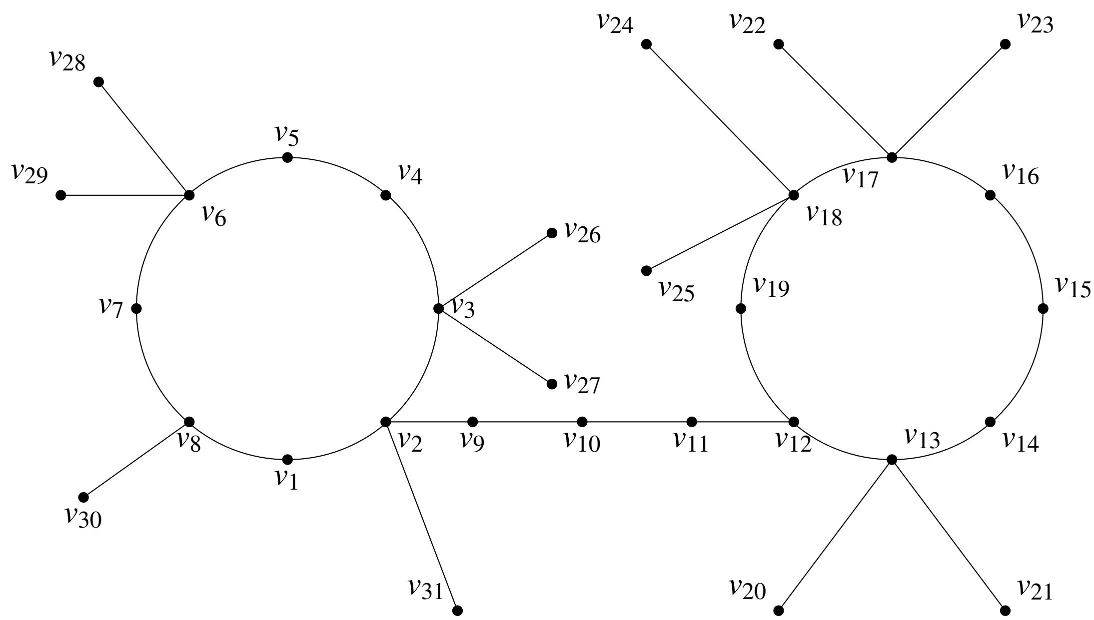


FIGURE 2. Bicyclic Graphs $D(l:m:i)$

Here $p=31$, $q=32$, $n=12$, $l=8$ and $m=6$, then $\eta_{2as}(G)=\{(v_{28},v_6,v_5,v_4,v_3,v_{26}),(v_{27},v_3,v_2,v_{31}),(v_{29},v_6,v_7,v_8,v_{30}),(v_8,v_1,v_2,v_9,v_{10},v_{11},v_{12},v_{13},v_{20}),(v_{21},v_{13},v_{14},v_{15},v_{16},v_{17},v_{23}),(v_{22},v_{17},v_{18},v_{24}),(v_{25},v_{18},v_{19},v_{12})\}=7=(n+1)-m$.

## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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