Available online at http://scik.org

J. Math. Comput. Sci. 10 (2020), No. 3, 507-521

https://doi.org/10.28919/jmcs/4459

ISSN: 1927-5307

NEW FIXED POINT THEOREMS FOR MODIFIED GENERALIZED RATIONAL $\alpha-\psi-$GERAGHTY CONTRACTION TYPE MAPS

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Abstract: In this paper, we introduce the notion of modified generalized rational α-ψ-Geraghty contraction type maps in the context of metric space and establish some fixed point theorems for such maps. This new contraction map is motivated by the different Geraghty contraction type maps introduced by many authors over the years. Examples are also given to illustrate the validity of our results.

Keywords: Metric space, fixed point, α-orbital admissible mapping with respect to η, triangular α-orbital admissible mapping with respect to η, modified generalized rational α-ψ-Geraghty contraction type map.

2010 AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

The celebrated Banach Contraction Principle is one of the most important and most used fixed point results in all analysis. Therefore this result has been generalized in different directions by various researchers ever since. In 1973, Geraghty [4] generalized the Banach contraction principle in the setting of a complete metric space by considering an auxiliary function. This remarkable result of Geraghty was further generalized and improved upon by the works of many authors namely Amini-Harandi & Emami [1], Caballero et al.[2] and Gordji et al.

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Received January 6, 2020

[5] etc. In 2012, Samet et al.[16] defined the notion of α-ψ-contractive mappings and obtained remarkable fixed point results. Then, Karapinar & Samet [8] introduced the concept of generalized α-ψ-contractive mappings and obtained fixed point results for such mappings. Salimi et al. [15] again modified the notions of α-ψ-contractive and α-admissible mappings and established fixed point results for such mappings. In 2013, Cho et al. [3] defined the concept of α-Geraghty contraction type maps in the setting of a metric space and proved the existence and uniqueness of a fixed point of such maps. Erdal Karapinar [9] introduced the concept of α-ψ-Geraghty contraction type maps and proved fixed point results generalizing the results obtained by Cho et al.[3]. In 2014, Popescu [14] generalized the results of Cho et al. [3] and gave other conditions for the existence and uniqueness of a fixed point of α-Geraghty contraction type maps. Then, K. Anthony Singh [6] introduced extended generalized $\alpha-\psi-$Geraghty contraction type maps and proved some fixed point results generalizing the results of Popescu [14]. In 2017, Muhammad Arshad & Aftab Hussain [13] defined generalized rational $\alpha-$Geraghty contraction type maps and proved some fixed point results. Again very recently, K. Anthony Singh et al. [7] introduced the notion of generalized rational $\alpha-\psi-$Geraghty contraction type maps and proved some fixed point results.

In this paper, motivated by the different Geraghty contraction type maps introduced by many authors and the works of Popescu [14], Salimi et al. [15], Muhammad Arshad & Aftab Hussain [13], K. Anthony Singh et al.[7], we define modified generalized rational $\alpha-\psi-$Geraghty contraction type maps in the setting of metric space and obtain the existence and uniqueness of a fixed point of such maps. We also give examples to illustrate the validity of our results.

2. PRELIMINARIES

In this section, we recall some basic definitions and related results on the topic in the literature.

Let $\mathcal{F}$ be the family of all functions $\beta:[0,\infty)\to[0,1)$ which satisfy the condition

$$\lim_{n\to\infty}\beta(t_n)=1 \quad \text{implies} \quad \lim_{n\to\infty}t_n=0.$$

By using such a map, Geraghty proved the following interesting result.

Theorem 2.1. [4] Let (X,d) be a complete metric space and let T be a mapping on X. Suppose there exists $\beta \in \mathcal{F}$ such that for all $x,y \in X$,

$$d(Tx,Ty) \leq \beta(d(x,y))d(x,y).$$

Then T has a unique fixed point $x_* \in X$ and $\{T^n x\}$ converges to x_* for each $x \in X$.

Popescu [14] introduced the following two new concepts.

Definition 2.2. [14] Let $T: X \to X$ be a map and $\alpha: X \times X \to R$ be a function. Then T is said to be α-orbital admissible if $\alpha(x,Tx) \geq 1$ implies $\alpha(Tx,T^2x) \geq 1$.

Definition 2.3. [14] A map $T: X \to X$ is said to be triangular α-orbital admissible if

(T1) T is α-orbital admissible,

(T2) $\alpha(x,y) \geq 1$ and $\alpha(y,Ty) \geq 1$ imply $\alpha(x,Ty) \geq 1$.

Lemma 2.4. [14] Let $T: X \to X$ be a triangular α-orbital admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n,x_m) \geq 1$ for all $m,n \in \mathbb{N}$ with $n < m$.

Cho et al. [3] introduced the following contraction and proved some interesting fixed point results generalising many results in the existing literature.

Definition 2.5. [3] Let (X,d) be a metric space and $\alpha: X \times X \to \mathbb{R}$ be a function. A map $T: X \to X$ is called a generalized α-Geraghty contraction type map if there exists $\beta \in \mathcal{F}$ such that for all $x,y \in X$,

$$\alpha(x,y)d(Tx,Ty) \leq \beta(M(x,y))M(x,y),$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$.

Erdal Karapinar [9] defined the following class of auxiliary functions.

Let Ψ denote the class of functions $\psi: [0,\infty) \to [0,\infty)$ which satisfy the following conditions:

(a) ψ is nondecreasing;

(b) ψ is subadditive, that is, $\psi(s+t) \leq \psi(s) + \psi(t)$;

(c) ψ is continuous;

(d) $\psi(t) = 0 \Leftrightarrow t = 0$.

Erdal Karapinar [9] also introduced the following contraction and proved some interesting fixed point results generalising the results of Cho et al. [3].

Definition 2.6. [9] Let (X,d) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called a generalized α-ψ-Geraghty contraction type mapping if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,

$$\alpha(x,y)\psi(d(Tx,Ty)) \le \beta(\psi(M(x,y)))\psi(M(x,y)),$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$ and $\psi \in \Psi$.

Popescu [14] extended the notion of generalized α-Geraghty contraction type map and gave the following definition.

Definition 2.7. [14] Let (X,d) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called a generalized α-Geraghty contraction type map if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,

$$\alpha(x,y)d(Tx,Ty) \le \beta(M_T(x,y))M_T(x,y),$$

where $M_T(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \dfrac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

K. Anthony Singh [6] further introduced the following contraction and proved some fixed point results generalising the results of Popescu [14].

Definition 2.8. [6] Let (X,d) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called an extended generalized α-ψ-Geraghty contraction type map if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,

$$\alpha(x,y)\psi(d(Tx,Ty)) \le \beta(\psi(M_T(x,y)))\psi(M_T(x,y)),$$

where $M_T(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \dfrac{d(x,Ty)+d(y,Tx)}{2}\right\}$ and $\psi \in \Psi$.

Let Ω be the family of all functions $\theta : [0,\infty) \to [0,1]$ which satisfy the following conditions

(1) $\theta(t) < 1$ for $t > 0$, and

(2) $\lim_{n\to\infty} \theta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$.

Remark 2.9. Here instead of the family $\mathcal{F}$ we are introducing a slightly extended family Ω.

K. Anthony Singh et al.[7] further introduced the following contraction and proved some fixed point results.

Definition 2.10. [7] Let (X,d) be a metric space and let $\alpha: X\times X\to\mathbb{R}$ be a function. Then the mapping $T: X\to X$ is called a generalized rational α-ψ-Geraghty contraction type mapping if there exists $\theta\in\Omega$ such that for all $x,y\in X$,

$$\alpha(x,y)\psi(d(Tx,Ty))\le\theta(\psi(N(x,y)))\psi(N(x,y))$$

where $N(x,y)=\max\left\{d(x,y),\dfrac{d(x,Tx)d(y,Ty)}{1+d(x,y)},\dfrac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}\right\}$ and $\psi\in\Psi$.

If we take $\psi(t)=t$ in Definition 2.10., then T can be called generalized rational α-Geraghty contraction type mapping.

3. MAIN RESULTS

We now state and prove our main results.

First we introduce some new definitions and concepts and then define modified generalized rational α-ψ-Geraghty contraction type map. These are motivated by the works of Popescu [14], Salimi et al. [15], Muhammad Arshad & Aftab Hussain [13], K. Anthony Singh et al. [7] and the different types of Geraghty contraction type maps introduced by various authors over the years.

Definition 3.1. Let $T: X\to X$ be a map and $\alpha,\eta: X\times X\to\mathbb{R}$ be two functions. Then T is said to be α-orbital admissible with respect to η if $\alpha(x,Tx)\ge\eta(x,Tx)$ implies

$$\alpha(Tx,T^2x)\ge\eta(Tx,T^2x).$$

Note that if $\eta(x,y)=1$, then T becomes an α-orbital admissible mapping and if $\alpha(x,y)=1$, then T is called an η-orbital subadmissible mapping.

Definition 3.2. Let $T: X\to X$ be a map and $\alpha,\eta: X\times X\to\mathbb{R}$ be two functions. Then T is said to be triangular α-orbital admissible with respect to η if T is α-orbital admissible with respect to η and $\alpha(x,y)\ge\eta(x,y)$ and $\alpha(y,Ty)\ge\eta(y,Ty)$ imply $\alpha(x,Ty)\ge\eta(x,Ty)$.

Note that if $\eta(x,y)=1$, then T becomes a triangular α-orbital admissible mapping and if $\alpha(x,y)=1$, then T is called a triangular η-orbital subadmissible mapping.

Lemma 3.3. Let $T: X \to X$ be a triangular *α*-orbital admissible mapping with respect to η. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ for all $m, n \in \mathbb{N}$ with $n < m$.

Proof: Since T is *α*-orbital admissible mapping with respect to η and $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$, we have $\alpha(Tx_1, T^2x_1) \geq \eta(Tx_1, T^2x_1)$ i.e. $\alpha(x_2, x_3) \geq \eta(x_2, x_3)$. Continuing in this way, we get $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \geq 1$. Let us suppose that $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ and prove that $\alpha(x_n, x_{m+1}) \geq \eta(x_n, x_{m+1})$, where $n < m$. Since T is triangular *α*-orbital admissible mapping with respect to η and $\alpha(x_m, x_{m+1}) \geq \eta(x_m, x_{m+1})$, we get $\alpha(x_n, x_{m+1}) \geq \eta(x_n, x_{m+1})$. Thus we have proved that $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ for all $m, n \in \mathbb{N}$ with $n < m$.

Now we give our Theorem below. The contraction T defined in the Theorem can be called modified generalized rational α-ψ-Geraghty contraction type map.

Theorem 3.4. Let (X, d) be a complete metric space, $\alpha, \eta : X \times X \to \mathbb{R}$ be two functions and let $T: X \to X$ be a map. Assume that

$$x, y \in X,\ \ \alpha(x, y) \geq \eta(x, y) \Rightarrow\ \psi(d(Tx, Ty)) \leq \theta(\psi(N_T(x, y)))\psi(N_T(x, y)),$$

where $N_T(x, y) = \max\left\{d(x, y), \dfrac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}, \dfrac{d(x, Tx)d(y, Ty)}{1 + d(Tx, Ty)}\right\}$ and $\psi \in \Psi$, $\theta \in \Omega$.

Suppose that the following conditions are satisfied

(1) T is a triangular α - orbital admissible mapping with respect to η,

(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$,

(3) T is continuous.

Then T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.

Proof : Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. We construct a sequence of points $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is clearly a fixed point of T and the proof is complete. Hence, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By hypothesis,

$\alpha(x_1,x_2)\geq\eta(x_1,x_2)$ and T is triangular α - orbital admissible mapping with respect to η. Therefore by Lemma 3.3., we have

$$\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1}) \text{ for all } n\in\mathbb{N}.$$

Then, we have

$$\psi\big(d(x_{n+1},x_{n+2})\big)=\psi\big(d(Tx_n,Tx_{n+1})\big) \leq\theta\Big(\psi\big(N_T(x_n,x_{n+1})\big)\Big)\psi\big(N_T(x_n,x_{n+1})\big) \text{ for all } n\in\mathbb{N}. \quad (1)$$

Here we have

$$N_T(x_n,x_{n+1})=\max\left\{d(x_n,x_{n+1}),\frac{d(x_n,Tx_n)d(x_{n+1},Tx_{n+1})}{1+d(x_n,x_{n+1})},\frac{d(x_n,Tx_n)d(x_{n+1},Tx_{n+1})}{1+d(Tx_n,Tx_{n+1})}\right\}$$

$$=\max\left\{d(x_n,x_{n+1}),\frac{d(x_n,x_{n+1})d(x_{n+1},x_{n+2})}{1+d(x_n,x_{n+1})},\frac{d(x_n,x_{n+1})d(x_{n+1},x_{n+2})}{1+d(x_{n+1},x_{n+2})}\right\}$$

$$\leq\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\}.$$

If $\max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\}=d(x_{n+1},x_{n+2})$ i.e. $d(x_{n+1},x_{n+2})\geq d(x_n,x_{n+1})$, then from (1) and the definition of θ, we have $\psi\big(d(x_{n+1},x_{n+2})\big)<\psi\big(d(x_{n+1},x_{n+2})\big)$, which is a contradiction.

Therefore, we have

$$d(x_{n+1},x_{n+2})<d(x_n,x_{n+1}) \text{ for all } n\in\mathbb{N}.$$

Thus the sequence $\{d(x_n,x_{n+1})\}$ is nonnegative and nonincreasing and also we have $N_T(x_n,x_{n+1})=d(x_n,x_{n+1})$.

Now, we prove that $d(x_n,x_{n+1})\to 0$ as $n\to\infty$.

It is clear that $\{d(x_n,x_{n+1})\}$ is a decreasing sequence which is bounded from below. Therefore there exists $r\geq 0$ such that $\lim_{n\to\infty}d(x_n,x_{n+1})=r$. We show that $r = 0$. And we suppose on the contrary that $r > 0$.

Then, we have $\dfrac{\psi\big(d(x_{n+1},x_{n+2})\big)}{\psi\big(d(x_n,x_{n+1})\big)}\leq\theta\Big(\psi\big(d(x_n,x_{n+1})\big)\Big)<1.$

Now by taking limit $n\to\infty$, we have

$$\lim_{n\to\infty}\theta\Big(\psi\big(d(x_n,x_{n+1})\big)\Big)=1.$$

By the property of θ, we have $\lim_{n\to\infty}\psi\left(d\left(x_n,x_{n+1}\right)\right)=0 \Rightarrow \lim_{n\to\infty}d\left(x_n,x_{n+1}\right)=0,$ which is a contradiction. Hence $r=0$ i.e.

$$\lim_{n\to\infty}d\left(x_n,x_{n+1}\right)=0. \tag{2}$$

Now we show that the sequence $\{x_n\}$ is a Cauchy sequence. Let us suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon>0$ such that, for all positive integers k, there exist $m_k>n_k>k$ with

$$d\left(x_{m_k},x_{n_k}\right)\geq\varepsilon. \tag{3}$$

Let m_k be the smallest number satisfying the conditions above. Then we have

$$d\left(x_{m_k-1},x_{n_k}\right)<\varepsilon. \tag{4}$$

By (3) and (4), we have

$$\begin{aligned}\varepsilon &\leq d\left(x_{m_k},x_{n_k}\right)\\ &\leq d\left(x_{m_k},x_{m_k-1}\right)+d\left(x_{m_k-1},x_{n_k}\right)\\ &< d\left(x_{m_k-1},x_{m_k}\right)+\varepsilon\end{aligned}$$

that is,

$$\varepsilon\leq d\left(x_{m_k},x_{n_k}\right)<\varepsilon+d\left(x_{m_k-1},x_{m_k}\right) \text{ for all } k\in\mathbb{N}. \tag{5}$$

Then in view of (2) and (5), we have

$$\lim_{k\to\infty}d\left(x_{m_k},x_{n_k}\right)=\varepsilon. \tag{6}$$

Again, we have

$$\begin{aligned}d\left(x_{m_k},x_{n_k}\right)&\leq d\left(x_{m_k},x_{m_k-1}\right)+d\left(x_{n_k},x_{m_k-1}\right)\\ &\leq d\left(x_{m_k},x_{m_k-1}\right)+d\left(x_{n_k},x_{n_k-1}\right)+d\left(x_{m_k-1},x_{n_k-1}\right)\end{aligned}$$

and $\quad d\left(x_{m_k-1},x_{n_k-1}\right)\leq d\left(x_{m_k-1},x_{m_k}\right)+d\left(x_{n_k-1},x_{n_k}\right)+d\left(x_{m_k},x_{n_k}\right).$

Taking limit as $k\to\infty$ and using (2) and (6), we obtain

$$\lim_{k\to\infty}d\left(x_{m_k-1},x_{n_k-1}\right)=\varepsilon. \tag{7}$$

By Lemma 3.3., we get $\alpha\left(x_{n_k-1}, x_{m_k-1}\right) \geq \eta\left(x_{n_k-1}, x_{m_k-1}\right)$. Therefore, we have

$$\psi\left(d\left(x_{m_k}, x_{n_k}\right)\right)=\psi\left(d\left(Tx_{m_k-1}, Tx_{n_k-1}\right)\right) \leq \theta\left(\psi\left(N_T\left(x_{n_k-1}, x_{m_k-1}\right)\right)\right)\psi\left(N_T\left(x_{n_k-1}, x_{m_k-1}\right)\right).$$

Here we have

$$N_T\left(x_{n_k-1}, x_{m_k-1}\right)=\max\left\{d\left(x_{n_k-1}, x_{m_k-1}\right), \frac{d\left(x_{n_k-1}, Tx_{n_k-1}\right)d\left(x_{m_k-1}, Tx_{m_k-1}\right)}{1+d\left(x_{n_k-1}, x_{m_k-1}\right)}, \frac{d\left(x_{n_k-1}, Tx_{n_k-1}\right)d\left(x_{m_k-1}, Tx_{m_k-1}\right)}{1+d\left(Tx_{n_k-1}, Tx_{m_k-1}\right)}\right\}$$

$$=\max\left\{d\left(x_{n_k-1}, x_{m_k-1}\right), \frac{d\left(x_{n_k-1}, x_{n_k}\right)d\left(x_{m_k-1}, x_{m_k}\right)}{1+d\left(x_{n_k-1}, x_{m_k-1}\right)}, \frac{d\left(x_{n_k-1}, x_{n_k}\right)d\left(x_{m_k-1}, x_{m_k}\right)}{1+d\left(x_{n_k}, x_{m_k}\right)}\right\}$$

And we see that

$$\lim_{k\to\infty} N_T\left(x_{n_k-1}, x_{m_k-1}\right)=\varepsilon.$$

Now we have

$$\frac{\psi\left(d\left(x_{n_k}, x_{m_k}\right)\right)}{\psi\left(N_T\left(x_{n_k-1}, x_{m_k-1}\right)\right)} \leq \theta\left(\psi\left(N_T\left(x_{n_k-1}, x_{m_k-1}\right)\right)\right)<1.$$

By using (6) and taking limit as $k\to\infty$ in the above inequality, we obtain

$$\lim_{k\to\infty}\theta\left(\psi\left(N_T\left(x_{n_k-1}, x_{m_k-1}\right)\right)\right)=1.$$

So, $\lim_{k\to\infty}\psi\left(N_T\left(x_{n_k-1}, x_{m_k-1}\right)\right)=0 \Rightarrow \lim_{k\to\infty} N_T\left(x_{n_k-1}, x_{m_k-1}\right)=0=\varepsilon,$ which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x^*\in X$ such that $x_n \to x^*$. As T is continuous, we have $Tx_n \to Tx^*$ i.e. $\lim_{n\to\infty} x_{n+1}=Tx^*$ and so $x^*=Tx^*$. Hence x^* is a fixed point of T.

For the uniqueness of a fixed point of the mapping T, we consider the following hypothesis:

(G) For any two fixed points x and y of T, there exists $z\in X$ such that $\alpha(x,z)\geq\eta(x,z)$,

$$\alpha(y,z)\geq\eta(y,z) \text{ and } \alpha(z,Tz)\geq\eta(z,Tz).$$

Theorem 3.5. Adding condition (G) to the hypotheses of Theorem 3.4., we obtain that x^* is the unique fixed point of T.

Proof: Due to Theorem 3.4., we obtain that $x^*\in X$ is a fixed point of *T*. Let $y^*\in X$ be another fixed point of *T*. Then by hypothesis (G), there exists $z\in X$ such that $\alpha\left(x^*,z\right)\geq\eta\left(x^*,z\right)$,

$$\alpha\left(y^*,z\right)\geq\eta\left(y^*,z\right) \text{ and } \alpha\left(z,Tz\right)\geq\eta\left(z,Tz\right).$$

Since *T* is triangular α - orbital admissible mapping with respect to η, we get

$$\alpha\left(x^*,T^nz\right)\geq\eta\left(x^*,T^nz\right) \text{ and } \alpha\left(y^*,T^nz\right)\geq\eta\left(y^*,T^nz\right) \text{ for all } n\in\mathbb{N}.$$

Then we have

$$\begin{aligned}\psi\left(d\left(x^*,T^{n+1}z\right)\right)&=\psi\left(d\left(Tx^*,TT^nz\right)\right)\\&\leq\theta\left(\psi\left(N_T\left(x^*,T^nz\right)\right)\right)\psi\left(N_T\left(x^*,T^nz\right)\right),\ \forall n\in\mathbb{N}.\end{aligned}$$

Here we have

$$\begin{aligned}N_T\left(x^*,T^nz\right)&=\max\left\{d\left(x^*,T^nz\right),\frac{d\left(x^*,Tx^*\right)d\left(T^nz,TT^nz\right)}{1+d\left(x^*,T^nz\right)},\frac{d\left(x^*,Tx^*\right)d\left(T^nz,TT^nz\right)}{1+d\left(Tx^*,TT^nz\right)}\right\}\\&=d\left(x^*,T^nz\right)\end{aligned}$$

By Theorem 3.4., we deduce that the sequence $\left\{T^nz\right\}$ converges to a fixed point $z^*\in X$. Then taking limit $n\to\infty$ in the above equality, we get $\lim\limits_{n\to\infty}N_T\left(x^*,T^nz\right)=d\left(x^*,z^*\right)$. And let us suppose that $z^*\neq x^*$. Then we have

$$\frac{\psi\left(d\left(x^*,T^{n+1}z\right)\right)}{\psi\left(N_T\left(x^*,T^nz\right)\right)}\leq\theta\left(\psi\left(N_T\left(x^*,T^nz\right)\right)\right)<1.$$

And taking limit $n\to\infty$, we get $\lim\limits_{n\to\infty}\theta\left(\psi\left(N_T\left(x^*,T^nz\right)\right)\right)=1$. Therefore we have $\lim\limits_{n\to\infty}\psi\left(N_T\left(x^*,T^nz\right)\right)=0$. This implies $\lim\limits_{n\to\infty}N_T\left(x^*,T^nz\right)=0$ i.e. $d\left(x^*,z^*\right)=0$, which is a contradiction. Therefore we must have $z^*=x^*$. Similarly, we get $z^*=y^*$. Thus we have $y^*=x^*$. Hence x^* is the unique fixed point of *T*.

By taking $\eta\left(x,y\right)=1$ in Theorem 3.4., we get the following result.

Theorem 3.6. Let $\left(X,d\right)$ be a complete metric space, $\alpha:X\times X\to\mathbb{R}$ be a function and let $T:X\to X$ be a map. Assume that

$$x, y \in X,\ \ \alpha(x, y) \geq 1 \Rightarrow \psi(d(Tx, Ty)) \leq \theta(\psi(N_T(x, y)))\psi(N_T(x, y)),$$

where $N_T(x, y) = \max\left\{d(x, y), \dfrac{d(x, Tx)d(y, Ty)}{1+d(x, y)}, \dfrac{d(x, Tx)d(y, Ty)}{1+d(Tx, Ty)}\right\}$ and $\psi \in \Psi$, $\theta \in \Omega$.

Suppose that the following conditions are satisfied

(1) T is a triangular α - orbital admissible mapping ,

(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$,

(3) T is continuous.

Then T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.

Again by taking $\alpha(x, y) = 1$ in Theorem 3.4., we get the following result.

Theorem 3.7. Let (X, d) be a complete metric space, $\eta : X \times X \to \mathbb{R}$ be a function and let $T : X \to X$ be a map. Assume that

$$x, y \in X,\ \ \eta(x, y) \leq 1 \Rightarrow \psi(d(Tx, Ty)) \leq \theta(\psi(N_T(x, y)))\psi(N_T(x, y)),$$

where $N_T(x, y) = \max\left\{d(x, y), \dfrac{d(x, Tx)d(y, Ty)}{1+d(x, y)}, \dfrac{d(x, Tx)d(y, Ty)}{1+d(Tx, Ty)}\right\}$ and $\psi \in \Psi$, $\theta \in \Omega$.

Suppose that the following conditions are satisfied

(1) T is a triangular η - orbital subadmissible mapping,

(2) there exists $x_1 \in X$ such that $\eta(x_1, Tx_1) \leq 1$,

(3) T is continuous.

Then T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.

Theorem 3.8. Adding condition (G) to the hypotheses of Theorem 3.6. (or Theorem 3.7.), we obtain that x^* is the unique fixed point of T.

Clearly Theorem 3.6. implies the following result.

Theorem 3.9. [7] Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and let $T : X \to X$ be a mapping. Suppose that the following conditions hold:

(i) T is a generalized rational α -ψ -Geraghty contraction type mapping,

(ii) T is triangular α - admissible,

(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$,

(iv) T is continuous.

Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.

Note that condition (ii) can be replaced by the weaker condition 'T is triangular α-orbital admissible'.

Theorem 3.10. [7] Adding condition (G) to the hypotheses of Theorem 3.9., we obtain that x^* is the unique fixed point of *T*.

Here we give two examples to illustrate Theorem 3.6. and Theorem 3.7.

Example 3.11. Let $X = \mathbb{R}$ with the metric d defined by $d(x,y) = |x-y|,\ \forall x,y \in X$. Then (X,d) is a complete metric space. And let $\theta(t) = \frac{1}{3}$ for all $t \geq 0$. Then $\theta \in \Omega$. Also let the function $\psi : [0,\infty) \to [0,\infty)$ be defined as $\psi(t) = \frac{t}{3}$. Then we have $\psi \in \Psi$.

Let a mapping $T : X \to X$ be defined by $T(x) = \begin{cases} \dfrac{x}{5} & if\ \ x \geq 0, \\ -x & if\ \ x < 0. \end{cases}$

Also let a function $\alpha : X \times X \to \mathbb{R}$ be defined by $\alpha(x,y) = \begin{cases} 1 & if\ x,y \geq 0, \\ -1 & \text{otherwise}. \end{cases}$

Then we show that *T* is triangular α-orbital admissible mapping.

Let $\alpha(x,Tx) \geq 1$. Then $Tx \geq 0$ and so $T^2 x \geq 0$. Therefore $\alpha(Tx, T^2 x) \geq 1$. Also if $\alpha(x,y) \geq 1$ and $\alpha(y,Ty) \geq 1$, then $x \geq 0$ and $Ty \geq 0$. Therefore $\alpha(x,Ty) \geq 1$.

Thus *T* is triangular α-orbital admissible mapping, which is condition (1) of Theorem 3.6. Also, condition (3) of Theorem 3.6. is satisfied because *T* is continuous.

And $\alpha(1,T1) = \alpha\left(1, \frac{1}{5}\right) = 1$. Therefore condition (2) of Theorem 3.6. is satisfied with $x_1 = 1$.

Let $\alpha(x,y) \geq 1$. Then $x,y \geq 0$.

Now $d(Tx,Ty) = \frac{1}{5}|x-y| \leq \frac{1}{3}|x-y| \leq \frac{1}{3} N_T(x,y)$

$$\Rightarrow \frac{d(Tx,Ty)}{3} \leq \frac{1}{3} \frac{N_T(x,y)}{3}$$

$$\Rightarrow \psi(d(Tx,Ty)) \leq \theta(\psi(N_T(x,y)))\psi(N_T(x,y)).$$

Thus all the conditions of Theorem 3.6. are satisfied and T has a unique fixed point $x^*=0$.

Example 3.12. Let $X=\{1,2,3\}$ with the metric d defined as $d(1,1)=d(2,2)=d(3,3)=0$, $d(1,2)=d(2,1)=1$, and $d(1,3)=d(3,1)=d(2,3)=d(3,2)=\frac{1}{2}$. Then (X,d) is a complete metric space. And let $\theta(t)=\frac{1}{1+t}$ for all $t\geq 0$. Then $\theta\in\Omega$. Also let the function $\psi:[0,\infty)\to[0,\infty)$ be defined as $\psi(t)=\frac{t}{3}$. Then we have $\psi\in\Psi$.

Let a mapping $T:X\to X$ be defined by $T(1)=1,\ T(2)=3,\ T(3)=2.$

And let a function $\eta:X\times X\to\mathbb{R}$ be defined by

$$\eta(x,y)=\begin{cases}1 & if\ \ x=y,\\ \frac{1}{2} & if\ \ (x,y)=(1,2)\,or\,(2,1),\\ 2 & otherwise.\end{cases}$$

Then we show that T is triangular η-orbital subadmissible mapping.

Let $\eta(x,Tx)\leq 1$. Then $x=1$ and $Tx=1$ and so $T^2x=1$. Therefore $\eta(Tx,T^2x)\leq 1$. Also if $\eta(x,y)\leq 1$ and $\eta(y,Ty)\leq 1$, then the possible cases are:

(i) $(x,y)=(1,1)$. Then $\eta(x,Ty)=\eta(1,1)=1\leq 1$.

(ii) $(x,y)=(2,1)$. Then $\eta(x,Ty)=\eta(2,1)=\frac{1}{2}\leq 1$.

Thus, T is triangular η-orbital subadmissible mapping, which is condition (1) of Theorem 3.7. Condition (2) of Theorem 3.7. is satisfied with $x_1=1$. And condition (3) of Theorem 3.7. is satisfied because T is continuous.

Let $\eta(x,y)\leq 1$. Then we have the following two possible cases:

(i) $(x,y)=(1,1)\,or\,(2,2)\,or\,(3,3)$.

Then $d(Tx,Ty)=0$ and $\psi(d(Tx,Ty))=0$.

Therefore, obviously we have $\psi(d(Tx,Ty))\leq\theta(\psi(N_T(x,y)))\psi(N_T(x,y))$.

(ii) $(x,y)=(1,2)\,or\,(2,1)$.

Then $d\left(Tx,Ty\right)=\frac{1}{2}$ and $\psi\left(d\left(Tx,Ty\right)\right)=\frac{1}{6}$.

And $N_T\left(x,y\right)=N_T\left(1,2\right)=\max\left\{d\left(1,2\right),\frac{d\left(1,T1\right)d\left(2,T2\right)}{1+d\left(1,2\right)},\frac{d\left(1,T1\right)d\left(2,T2\right)}{1+d\left(T1,T2\right)}\right\}$

$$=d\left(1,2\right)=1.$$

Similarly $N_T\left(x,y\right)=N_T\left(2,1\right)=1$.

Therefore $\theta\left(\psi\left(N_T\left(x,y\right)\right)\right)\psi\left(N_T\left(x,y\right)\right)=\frac{\psi\left(N_T\left(x,y\right)\right)}{1+\psi\left(N_T\left(x,y\right)\right)}=\frac{N_T(x,y)/3}{1+N_T(x,y)/3}=\frac{1/3}{1+1/3}=\frac{1}{4}$.

Thus we have $\psi\left(d\left(Tx,Ty\right)\right)\le\theta\left(\psi\left(N_T\left(x,y\right)\right)\right)\psi\left(N_T\left(x,y\right)\right)$.

Hence all the conditions of Theorem 3.7. are satisfied and T has a unique fixed point $x^*=1$.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] Amini-Harandi A., Emami H., A fixed point theorem for contraction type maps in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Anal., Theory Methods Appl. 72 (2010), 2238-2242.

[2] Caballero J., Harjani J., Sadarangani K., A best proximity point theorem for Geraghty- contractions, Fixed Point Theory Appl. 2012 (2012), Article ID 231.

[3] Cho SH., Bae JS., Karapinar E., Fixed point theorems for α-Geraghty contraction type maps in metric spaces, Fixed Point Theory Appl. 2013 (2013), Article ID 329.

[4] Geraghty M., On contractive mappings, Proc. Amer. Math. Soc. 40 (1973), 604-608.

[5] Gordji ME., Ramezami M., Cho YJ., Pirbavafa S., A generalization of Geraghty's Theorem in partially ordered metric spaces and applications to ordinary differential equations, Fixed Point Theory Appl. 2012 (2012), Article ID 74.

[6] K. Anthony Singh, Fixed point theorems for extended generalized α-ψ-Geraghty contraction type maps in metric space, South East Asian J. Math. Math. Sci. 15 (3) (2019), 159-170.

[7] K. Anthony Singh, M.R. Singh, Th. Chhatrajit Singh, Fixed point theorems for generalized rational α-ψ-Geraghty contraction type mappings in metric space, Electron. J. Math. Anal. Appl. Communicated.

[8] Karapinar E., Samet B., Generalized α-ψ-contractive type mappings and related fixed point theorems with applications, Abstr. Appl. Anal. 2012 (2012), Article ID 793486.

[9] Karapinar E., α-ψ-Geraghty contraction type mappings and some related fixed point results, Filomat 28 (2014), 37-48.

[10] L. Shanjit, Y. Rohen, Best proximity point theorems in *b*–metric space satisfying rational contraction, J. Nonlinear Anal. Appl. 2019 (2) (2019), 12-22.

[11] M. Bina, N. Priyobarta, Y. Rohen, N. Mlaiki, Coupled coincidence results in *A*– metric space satisfying Geraghty type contraction, J. Math. Anal. 10 (1) (2019), 62-85.

[12] M. S. Khan, N. Priyobarta, Y. Rohen, Fixed points of generalized rational Geraghty contraction for multivalued mappings, J. Adv. Math. Stud. 12 (2) (2019), 156-169.

[13] Muhammad Arshad, Aftab Hussain, Fixed point results for generalized rational α-Geraghty contraction, Miskolc Math. Notes, 18 (2) (2017), 611-621.

[14] Popescu, Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces, Fixed Point Theory Appl. 2014 (2014), Article ID 190.

[15] P. Salimi, A. Latif, N. Hussain, Modified α-ψ-contractive mappings with applications, Fixed Point Theory Appl. 2013 (2013), Article ID 151.

[16] Samet B., Vetro C., Vetro P., Fixed point theorems for α-ψ-contractive mappings, Nonlinear Anal. Theory Methods Appl. 75 (2012), 2154-2165.