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REMARK ON STABILITY OF FRACTIONAL ORDER PARTIAL DIFFERENTIAL EQUATION

GOVIND P. KAMBLE[1,*], MOHAMMED MAZHAR-UL-HAQUE[2]

[1]Department of Mathematics, P. E. S. College of Engineering, Nagsenvana, Aurangabad (M.S.) 431002, India

[2]Department of Mathematics, GRACE, Swami Ramanand Teerth Marathwada University, Nanded (M.S.) 431606, India

Abstract. In this paper, using a fractional order partial derivative with non-singular kernel we investigate, the stability and its generalization on semi-closed and semi-open interval for the solution of a fractional order partial differential equation with the help of an inequality.

In this paper, we will consider the following fractional order partial differential equation

$$\frac{\partial^{3\alpha}_{\beta,\psi} u}{\partial_{\beta,\psi} x^{\alpha}\,\partial_{\beta,\psi} y^{\alpha}\,\partial_{\beta,\psi} z^{\alpha}} = f\left(x,y,z,u(x,y,z),\frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi} x^{\alpha}}u(x,y,z),\frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi} y^{\alpha}}u(x,y,z),\frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi} x^{\alpha}}u(x,y,z)\right) \tag{0.1}$$

where, $\frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi} x^{\alpha}}u(x,y,z)$ is the ψ - Hilfer fractional partial derivative [1], with parameter $0<\alpha<1$ and $0\le\beta\le1$, $0\le x\le a,\ \ 0\le y\le b,\ 0\le x\le c$ and $f\in C([0,a)\times[0,b)\times[0,c)\times\mathbb{B}^4,\mathbb{B})$ and $(\mathbb{B},|\ .\ |)$ a real or complex Banach space.

Keywords: stability; non-singular kernel; Banach space; Hilfer fractional partial differential equation.

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*Corresponding author

E-mail address: kamblegp.14@gmail.com

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1. Introduction

In recent years, the fractional calculus plays an significant role in numerous fields, such as a pure and applied Mathematics, Science and Engineering Technology.

Recently, the class of fractional integro - differential equations allows the stability of Ulam-Hyers, Ulam- Hyers-Rassias and semi-Ulam -Hyers in the interval $[a,b]$ and $[a,\infty)[1]$.

There has been a considerable development in fractional ordinary differential equations and partial differential equations. For more details on fractional calculus theory, one see the monographs of Kilbas et. al.[2], Hermann [3], Podlumny [4], Oldhan K., Spanier [5], Samko, Kilbas [6] and the papers of Sousa $[7,8]$, Abbas and Benchohra $[9,10]$.

Furthermore, this is concerned with existence of mild solution of evolution with Hilfer fractional derivative generalized the well-known Riemann -Liouville fractional derivative by non-compact measure method and acquire some sufficient conditions to make certain the existence of mild solution [11], An initial value problem for a class of non-linear fractional differential equations concerning Hilfer fractional derivative and prove the existence and uniqueness of universal solutions in the space of weighted continuous functions. Also analyze the stability of the solution for a weighted Cauchy - type problem [12].

They found the existence and the uniqueness of a positive solution in the space of weighted continuous functions and boundary performance of such solution [13], Particularly, sufficient conditions for the existence of solution for a class of initial value problems for impulsive fractional differential equations connecting the Caputo fractional derivative [14].

Additionally, the existence and uniqueness of a solution of a class of initial boundary value problems for implicit fractional differential equations with fractional derivative and the outcome are based upon technique of measures of compactness and the fixed point theorems of Darbo and Monch [15], in addition the existence and uniqueness results for implicit differential equations of Hilfer type fractional order via Schaeder's fixed point theorem and Banach contraction principle [16], Ulam stability and data dependence for fractional differential equations with Caputo fractional derivative of order α and presents four types of Ulam stability results for the fractional differential equation [17], Ulam ψ -Hilfer fractional derivative and present the

Hyers -Ulam -Rassias stability and the Hyers-Ulam stability of the fractional Voltera integral -differential equation by means of fixed point method [18].

In continuation of this are the generalization of semi- group property [19], Mittag-Lefller function [20], the existence of non-local fractional differential equations and its approximations [21-24], bounds for these solutions [25-27] and monotone nature of the solution [28].

We organize this paper as : In the second section, we define some basic definitions and notations, in the third section, we investigate the stability and its generalization on semi-closed and semi-open interval for the solution of a fractional order partial differential equation with the help of inequality.

2. Technical Background

In this section, we use some definitions and notations which are given in [1] with details and present technical preparation needed for further discussion.

Definition 2.1. [1] *Let* $\theta = (\theta_1, \theta_2, ..., \theta_N)$ *and* $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$*,where* $0 < \alpha_1, \alpha_2, ..., \alpha_N < 1$*,* $N \in \mathbb{N}$*. Also put* $\tilde{I} = I_1 \times I_2 \times ..... \times I_N = [\theta_1, a_1) \times [\theta_2, a_2) \times ... \times [\theta_N, a_N)$*, where* $a_1, a_2...., a_N$ *and* $\theta_1, \theta_2...., \theta_N$ *are positive constants. Also let* $\psi(.)$ *be an increasing and positive monotone function on* $[\theta_1, a_1), [\theta_2, a_2), ..., [\theta_N, a_N)$ *having a continuous derivative* $\psi'(.)$ *on* $[\theta_1, a_1), [\theta_2, a_2), ..., [\theta_N, a_N)$*. The* ψ *- Riemann - Liouville partial integral of N variables.*

$u = (u_1, u_2, ..., u_N) \in L'(\tilde{I})$ *is defined by*

$$I^{\alpha,\varphi}_{\theta,x} u(x) = \frac{1}{\Gamma(\alpha_j)} \int\int\int\int_{\tilde{I}} \psi'(s_j)(\psi(x_j) - \psi(s_j))^{\alpha_j - 1} u(s_j) ds_j \tag{2.1}$$

with

$$\begin{aligned}
&\psi'(s_j)(\psi(x_j) - \psi(s_j))^{\alpha_j - 1} \\
&\qquad = \psi'(s_1)(\psi(x_1) - \psi(s_1))^{\alpha_1 - 1} \quad \psi'(s_2)(\psi(x_2) - \psi(s_2))^{\alpha_2 - 1} .... \\
&\qquad .... \psi'(s_N)(\psi(x_N) - \psi(s_N))^{\alpha_N - 1}
\end{aligned}$$

and using the notation

$\Gamma(\alpha_i) = \Gamma(\alpha_1), \Gamma(\alpha_2), ..., \Gamma(\alpha_N)$

and $u(s_j) = u(s_1, s_2, ..., s_N)$

$ds_j = ds_1, ds_2, ..., ds_N,\ j \in \{1, 2, ..., N\}$ *with* $N \in \mathbb{N}$.

In particular, take $N = 3, \theta_1 = \theta_2 = \theta_3 = 0$

$$I_{\theta}^{\alpha,\psi} u(x_1, x_2, x_3) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \int_0^{x_1}\int_0^{x_2}\int_0^{x_3} \psi'(s_1)\psi'(s_2)\psi'(s_3)\times [(\psi(x_1)-\psi(s_1))^{\alpha_1-1}(\psi(x_2)-\psi(s_2))^{\alpha_2-1}(\psi(x_3)-\psi(s_3))^{\alpha_3-1} ds_1 ds_2 ds_3]$$

with $0 < \alpha_1, \alpha_2, \alpha_3 \leq 1$

Also, we have

$$I_{0^+,x_1}^{\alpha_1,\psi} u(x_1, x_2, x_3) = \frac{1}{\Gamma(\alpha_1)} \int_0^{x_1} \psi'(s_1)(\psi(x_1)-\psi(s_1))^{\alpha_1-1} u(s_1, s_2, s_3) ds_1$$

$$I_{0^+,x_2}^{\alpha_2,\psi} u(x_1, x_2, x_3) = \frac{1}{\Gamma(\alpha_2)} \int_0^{x_2} \psi'(s_2)(\psi(x_2)-\psi(s_2))^{\alpha_2-1} u(s_1, s_2, s_3) ds_2$$

$$I_{0^+,x_3}^{\alpha_3,\psi} u(x_1, x_2, x_3) = \frac{1}{\Gamma(\alpha_3)} \int_0^{x_3} \psi'(s_3)(\psi(x_3)-\psi(s_3))^{\alpha_3-1} u(s_1, s_2, s_3) ds_3$$

with $0 < \alpha_1, \alpha_2, \alpha_3 \leq 1$

Definition 2.2. [1] *Let* $\theta = (\theta_1, \theta_2, ..., \theta_N)$ *and* $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$, *where* $\theta < \alpha_1, \alpha_2, \alpha_3, ..., \alpha_N < 1, N \in \mathbb{N},$. *Also put* $\tilde{I} = I\alpha_1 \times I\alpha_2 \times ..... \times I\alpha_N = [\theta_1, a_1) \times [\theta_2, a_2) \times ... \times [\theta_N, a_N)$, *where* $a_1, a_2 ...., a_N$ *and* $\theta_1, \theta_2 ...., \theta_N$ *are positive constants. Also let* $u, \psi \in C^n(\tilde{I}, \mathbb{R})$ *are two functions such that* ψ *is increasing and* $\psi'(x_i) \neq 0, i \in \{1, 2, ..., N\}$, $x_i \in \tilde{I}, N \in \mathbb{N}$. *The* ψ *is Hilfer fractional partial derivative of N variables denoted by* ${}^H\mathbb{D}_{\theta,x}^{\alpha,\beta,\psi}(.)$ *of a derivative of order* α *and* $0 \leq \beta_1, \beta_2, ..., \beta_N \leq 1$ *is defined by*

$${}^H\mathbb{D}_{\theta,x}^{\alpha,\beta,\psi} u(x) = I_{\theta,x_j}^{\beta(1-\alpha),\psi}\left(\frac{1}{\psi'(x_j)}\frac{\partial^N}{\partial x_j}\right) I_{\theta,x_j}^{(1-\beta)(1-\alpha),\psi} u(x_j)$$

with $\partial x_j = \partial x_1, \partial x_2, ..., \partial x_N$ *and* $\psi'(x_j) = \psi'(x_1), \psi'(x_2), ..., \psi'(x_N)$,

$j \in \{1, 2, ..., N\}, N \in \mathbb{N}$.

taking N = 3,

$$^{H}\mathbb{D}_{\theta}^{\alpha,\beta,\psi}u(x_1,x_2,x_3) = I_{\theta}^{\beta(1-\alpha),\psi}\Big(\frac{1}{\psi'(x_1)\psi'(x_2)\psi'(x_3)}\frac{\partial^3}{\partial x_1\partial x_2\partial x_3}\Big)I_{\theta}^{(1-\beta)(1-\alpha),\psi}u(x_1,x_2,x_3)$$

We use notation

$$^{H}\mathbb{D}_{\theta}^{\alpha,\beta,\psi}u(x_1,x_2,x_3) = \frac{\partial_{\beta,\psi}^{3\alpha}u(x_1,x_2,x_3)}{\partial_{\beta,\psi}x_1^{\alpha}\,\partial_{\beta,\psi}x_2^{\alpha}\,\partial_{\beta,\psi}x_3^{\alpha}}$$

Lemma 2.1. [2] *If*

(i) $u,v,h \in C([a,b];\mathbb{R}_+)$

(ii) for any $t \geq a$ *and* $\psi(t)$ *is increasing and* $\psi'(t)\ \forall t \in [a,b]$

$u(t) \leq v(t) + h(t)\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}u(s)ds$

(iii) h(t) is non - negative and non decreasing then,

$u(t) \leq v(t)E_{\alpha}[h(t)\Gamma(\alpha)(\psi(t)-\psi(a))^{\alpha}]$

for any $t \geq \alpha$ *and being* $E_{\alpha}(.)$ *the one-parameter Mittag-Leffler function.*

Definition 2.3. [1]. *Let* $(a,b)\,(-\infty \leq a < b \leq \infty)$ *be a finite interval (or infinite) of the real line* $\mathbb{R}$ *and let* $\alpha > 0$ *. Also let* $\psi(x)$ *be an increasing and positive monotone function on* $(a,b]$*, having a continuous derivative* $\psi'(x)$ *(we denote first derivative as* $\frac{d}{dx}\psi(x) = \psi'(x)$ *on (a,b)). The left-sided fractional integral of a function f with respect to a function on* $[a,b]$ *is defined by*

$$I_{a^+}^{\alpha,\psi}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \psi'(s)(\psi(x)-\psi(s))^{\alpha-1}f(s)ds.$$

The right-sided fractional integral is defined in an analogous form.

Definition 2.4. [1]. *Let* $n-1 < \alpha < n$ *with* $n \in \mathbb{N}$*, let* $I = [a,b]$ *be an interval such that* $-\infty \leq a < b \leq \infty$ *and let* $f,\psi \in C^n[a,b]$ *be two functions such that* ψ *is increasing and* $\psi'(x) \neq 0$*, for all* $x \in I$*. The left-sided* ψ *- Hilfer fractional derivative* $^{H}\mathbb{D}_{a^+}^{\alpha,\beta,\psi}(.)$ *of a function of order* α *and type* $0 \leq \beta \leq 1$*, is defined by*

$$^{H}\mathbb{D}_{a^+}^{\alpha,\beta,\psi}f(x) = I_{a^+}^{\beta(n-\alpha),\psi}\Big(\frac{1}{\psi'(x)}\frac{d}{dx}\Big)^n I_{a^+}^{(1-\beta)(n-\alpha),\psi}f(x)$$

The right-sided ψ*-Hilfer fractional derivative is defined in an analogous form.*

Theorem 2.2. [1] *If $f \in C^1[a,b]$, $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, then*

$$I_{a^+}^{\alpha,\psi} \, {}^H\mathbb{D}_{a^+}^{\alpha,\beta,\psi} f(x) = f(x) - \frac{(\psi(x)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} I_{a^+}^{(1-\beta)(1-\alpha),\psi} f(a)$$

Theorem 2.3. [1] *If $f \in C^1[a,b]$, $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, then*

$${}^H\mathbb{D}_{a^+}^{\alpha,\beta,\psi} \, I_{a^+}^{\alpha,\psi} f(x) = f(x)$$

Definition 2.5. [1] *For each function y satisfying*

$$|{}^H\,\mathbb{D}_{a^+}^{\alpha,\beta,\psi} y(x) - f(x,y(x), \int_a^x K(x,\tau,y(\tau),y(\delta(\tau)))d\tau) \,| \leq \theta$$

$x \in [a,b]$, where $\theta \geq 0$, there is a solution y_0 of the fractional integro-differential equation and a constant $C > 0$ independent of y and y_0 such that

$$|\, y(x) - y_0(x) \,| \leq C\theta$$

for all $x \in [a,b]$, then we say that the integro-differential equation has the Ulam-Hyers stability.

Definition 2.6. [1] *If for each function y satisfying*

$$|{}^H\,\mathbb{D}_{a^+}^{\alpha,\beta,\psi} y(x) - f(x,y(x), \int_a^x K(x,\tau,y(\tau),y(\delta(\tau)))d\tau) \,| \leq \theta$$

$x \in [a,b]$, where $\theta \geq 0$, there is a solution y_0 of the fractional integro-differential equation and a constant $C > 0$ independent of y and y_0 such that

$$|\, y(x) - y_0(x) \,| \leq C\sigma(x)$$

$x \in [a,b]$, for some non-negative function σ defined on $[a,b]$, then we say that the fractional integro-differential equation has the so-called semi-Ulam-Hyers-Rassias stability.

Definition 2.7. [1] *We say that $d : X \times X \to [0,\infty]$ is a generalized metric on X if:*

(i) d(x,y) = 0 if and only if x = y,

(ii) $d(x,y) = d(y,x), for all x,y \in X,$

(iii) $d(x,z) \leq d(x,y) + d(y,z) for all x,y,z \in X.$

Theorem 2.4. [1] *(Banach) Let (X, d) be a generalized complete metric space and $T : X \to X$ a strictly contractive operator with Lipschtiz constant $L > 1$. If there exist a non-negative integer k such that $d(T^{k+1}x, T^k x) < \infty$ for some $x \in X$, then the following three propositions hold true:*

(i) The sequence $(T^n x)_{n\in\mathbb{N}}$ converges to a fixed point x^* of T;

(ii) x^* is the unique fixed point of T in $X^* = \{y \in X; d(T^k x, y) < \infty\}$;

(iii) If $y \in X^*$, then

$$d(y,x^*) \leq \frac{1}{1-L} d(Ty,y)$$

3. Main Result

In this paper, our aim is to investigate the stability and its generalization on semi-closed and semi-open interval for the solution of a fractional order partial differential equation with the help of inequality.

For our convenience in the calculations, we consider the following set of considerations and notations

(i) $\mathscr{M}^{p,v}_{v_1,v_2,v_3} f(x,y,z) = \int_0^p f(x,y,z)v(x,y,z)v_1(x,y,z)v_2(x,y,z)v_3(x,y,z)dp$

(ii) $\mathscr{M}^{p,v}_{u_1,u_2,u_3} f(x,y,z) = \int_0^p f(x,y,z)u(x,y,z)u_1(x,y,z)u_2(x,y,z)u_3(x,y,z)dp$

(iii) $\bar{\mathscr{M}}^{p}_{u,v} f(x,y,z)$

$= \int_0^p \mid f(x,y,z)v(x,y,z)v_1(x,y,z)v_2(x,y,z)v_3(x,y,z)$

$- f(x,y,z)u(x,y,z)u_1(x,y,z)u_2(x,y,z)u_3(x,y,z) \mid dp$

(iv) $\mathscr{M}^p \phi(x,y,z) = \int_0^p \phi(x,y,z)dp$

(v) $\Psi^\gamma(x,0,0) = \frac{(\psi(x)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}$

(vi) $\Psi^\gamma(0,y,0) = \frac{(\psi(y)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}$

(vii) $\Psi^\gamma(0,0,z) = \frac{(\psi(z)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}$

(viii) Let $a,b,c \in (0,\infty], \in > 0, \phi(x,y,z) \in C([0,a) \times [0,b) \times [0,c), \mathbb{R}_+)$ and $(\mathbb{B}, |.|)$ be a real or complex Banach space.

Consider,

$$|\frac{\partial^{3\alpha} v(x,y,z)}{\partial_{\beta,\psi}x^\alpha \, \partial_{\beta,\psi}y^\alpha \, \partial_{\beta,\psi}z^\alpha} - f(x,y,z,v(x,y,z), \frac{\partial^\alpha_{\beta,\psi} v}{\partial_{\beta,\psi}x^\alpha}, \frac{\partial^\alpha_{\beta,\psi} v}{\partial_{\beta,\psi}y^\alpha}, \frac{\partial^\alpha_{\beta,\psi} v}{\partial_{\beta,\psi}z^\alpha})| \leq \in \tag{3.1}$$

$x \in [0,a), y \in [0,b), z \in [0,c)$

$$|\frac{\partial^{3\alpha} v(x,y,z)}{\partial_{\beta,\psi} x^{\alpha}\, \partial_{\beta,\psi} y^{\alpha}\, \partial_{\beta,\psi} z^{\alpha}} - f(x,y,z,v(x,y,z), \frac{\partial^{\alpha}_{\beta,\psi} v}{\partial_{\beta,\psi} x^{\alpha}}, \frac{\partial^{\alpha}_{\beta,\psi} v}{\partial_{\beta,\psi} y^{\alpha}}, \frac{\partial^{\alpha}_{\beta,\psi} v}{\partial_{\beta,\psi} z^{\alpha}})| \leq \phi(x,y,z) \tag{3.2}$$

$x \in [0,a),\, y \in [0,b),\, z \in [0,c),\, 0 < \alpha \leq 1,\, 0 \leq \beta \leq 1.$

(ix) If function $u : [0,a) \times [0,b) \times [0,c) \longrightarrow \mathbb{B}$ is a solution of equation (0.1) ,
if $u \in C([0,a) \times [0,b) \times [0,c)) \bigcap C'([0,a) \times [0,b) \times [0,c))$

$$\frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi} x^{\alpha}} \in C([0,a) \times [0,b) \times [0,c))$$

$$\frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi} y^{\alpha}} \in C([0,a) \times [0,b) \times [0,c))$$

$$\frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi} z^{\alpha}} \in C([0,a) \times [0,b) \times [0,c))$$

and u satisfies (0.1)

Definition 3.1. *The solution of equation (0.1) is U-H stability , if there exist a real number* C^1_f, C^2_f, C^3_f *and* $C^n_f \geq 0$ *such that, for any* $\in > 0$ *and for any solution v to the inequality (3.1), with*

$$| \, v(x,y,z) - u(x,y,z) \, | \leq C^1_f \in,$$

$$| \, \frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi} x^{\alpha}} v(x,y,z) - \frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi} x^{\alpha}} u(x,y,z) \, | \leq C^2_f \in,$$

$$| \, \frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi} y^{\alpha}} v(x,y,z) - \frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi} y^{\alpha}} u(x,y,z) \, | \leq C^3_f \in,$$

and

$$| \, \frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi} z^{\alpha}} v(x,y,z) - \frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi} z^{\alpha}} u(x,y,z) \, | \leq C^4_f \in$$

$x \in [0,a),\, y \in [0,b),\, z \in [0,c)$ *with* , $0 < \alpha \leq 1,\, 0 \leq \beta \leq 1.$

Definition 3.2. *The solution of equation (0.1)admit generalised U-H-R stability ,if there exist a real number* $C^1_{f,\phi}, C^2_{f,\phi}, C^3_{f,\phi}$ *and* $C^n_{f,\phi} > 0$ *such that, for any* $\in > 0$ *and for any solution v to the inequality (3.2), with*

$$| v(x,y,z) - u(x,y,z) | \leq C^1_{f,\phi}\phi(x,y,z)$$

$$| \frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi}x^{\alpha}} v(x,y,z) - \frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi}x^{\alpha}} u(x,y,z) | \leq C^2_{f,\phi}\phi(x,y,z)$$

$$| \frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi}y^{\alpha}} v(x,y,z) - \frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi}y^{\alpha}} u(x,y,z) | \leq C^3_{f,\phi}\phi(x,y,z)$$

$$| \frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi}z^{\alpha}} v(x,y,z) - \frac{\partial^{\alpha}_{\beta,\psi}}{\partial_{\beta,\psi}z^{\alpha}} u(x,y,z) | \leq C^4_{f,\phi}\phi(x,y,z)$$

$x \in [0,a), y \in [0,b), z \in [0,c)$ *with* $0 < \alpha \leq 1, 0 \leq \beta \leq 1$.

Remark 1: If function v is a solution to the inequality (3.1), if and only if, there exist a function $g \in C([0,a) \times [0,b) \times [0,c), \mathbb{B})$, which depends on v, such that a) for all $\in > 0, | g(x,y,z) | \leq \in, \forall x \in [0,a), y \in [0,b), z \in [0,c)$
b) $\forall x \in [0,a), y \in [0,b), z \in [0,c), 0 < \alpha < 1, 0 \leq \beta \leq 1$.

$$\frac{\partial^{3\alpha} v(x,y,z)}{\partial_{\beta,\psi}x^{\alpha}\,\partial_{\beta,\psi}y^{\alpha}\,\partial_{\beta,\psi}z^{\alpha}} = f(x,y,z,v(x,y,z), \frac{\partial^{\alpha}_{\beta,\psi}v}{\partial_{\beta,\psi}x^{\alpha}}, \frac{\partial^{\alpha}_{\beta,\psi}v}{\partial_{\beta,\psi}y^{\alpha}}, \frac{\partial^{\alpha}_{\beta,\psi}v}{\partial_{\beta,\psi}z^{\alpha}}) + g(x,y,z) \tag{3.3}$$

Theorem 3.1. *If v is a solution to the inequality (3.1), then* (v, v_1, v_2, v_3) *in a solution of the following system of integral inequality*

$$\begin{aligned} &| v - \Psi^{\gamma}(0,0,z)v(x,0,0) - \Psi^{\gamma}(0,y,0)v(x,0,0) \\ &- \Psi^{\gamma}(x,0,0)v(0,y,0) - \Psi^{\gamma}(0,0,z)v(0,y,0) - \Psi^{\gamma}(0,y,0)v(0,0,z) \\ &- \Psi^{\gamma}(x,0,0)v(0,0,z) - I^{\alpha,\psi}_{\theta}(\mathscr{M}^{p,v}_{v_1,v_2,v_3} f(x,y,z)) | \\ &\leq \in x \frac{(\psi(x)-\psi(0))^{\alpha_1}(\psi(y)-\psi(0))^{\alpha_2}(\psi(z)-\psi(0))^{\alpha_3}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(\alpha_3+1)} \end{aligned}$$

$$| v_1(x,y,z) - \Psi^{\gamma}(x,0,0)v_1(0,0,z) - \Psi^{\gamma}(0,y,0)v_1(0,0,z) - I^{\alpha_1,\psi}_{\theta^+,x}(\mathscr{M}^{p,v}_{v_1,v_2,v_3} f(x,y,z)) |$$

$$\leq \in x\frac{(\psi(x)-\psi(0))^{\alpha_1}}{\Gamma(\alpha_1+1)}$$

$$| v_2(x,y,z)-\Psi^{\gamma}(0,0,z)v_2(x,0,0)-\Psi^{\gamma}(0,y,0)v_2(x,0,0)-I^{\alpha_2,\psi}_{\theta^+,y}(\mathscr{M}^{p,v}_{v_1,v_2,v_3}f(x,y,z)) |$$
$$\leq \in \frac{(\psi(y)-\psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}$$

$$| v_3(x,y,z)-\Psi^{\gamma}(x,0,0)v_3(0,y,0)-\Psi^{\gamma}(0,0,z)v_3(0,y,0)-I^{\alpha_3,\psi}_{\theta^+,z}(\mathscr{M}^{p,v}_{v_1,v_2,v_3}f(x,y,z)) |$$
$$\leq \in \frac{(\psi(z)-\psi(0))^{\alpha_3}}{\Gamma(\alpha_3+1)}$$

$x\in[0,a)$, $y\in[0,b)$, $z\in[0,c)$, $0<\alpha\leq 1$, $0\leq\gamma\leq 1$.

$$v_1=\frac{\partial^{\alpha}_{\beta,\psi}v}{\partial_{\beta,\psi}z^{\alpha}}, v_2=\frac{\partial^{\alpha}_{\beta,\psi}v}{\partial_{\beta,\psi}x^{\alpha}}, v_3=\frac{\partial^{\alpha}_{\beta,\psi}v}{\partial_{\beta,\psi}y^{\alpha}}$$

Proof: From equation (3.2), we have

$| \ v-\Psi^{\gamma}(0,0,z)v(x,0,0)-\Psi^{\gamma}(0,y,0)v(x,0,0)-\Psi^{\gamma}(x,0,0)v(0,y,0)-\Psi^{\gamma}(0,0,z)v(0,y,0)-\Psi^{\gamma}(x,0,0)v(0,0,z)-\Psi^{\gamma}(0,y,0)v(0,0,z)-I^{\alpha,\psi}_{\theta}(\mathscr{M}^{p,v}_{v_1,v_2,v_3}f(x,y,z)) |$

$$\leq | I^{\alpha,\psi}_{\theta}(\mathscr{M}^{p}\phi(x,y,z)) |$$

$$\leq I^{\alpha,\psi}_{\theta}(\int_0^p \in dp) \leq \in x\frac{(\psi(x)-\psi(0))^{\alpha_1}(\psi(y)-\psi(0))^{\alpha_2}(\psi(z)-\psi(0))^{\alpha_3}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(\alpha_3+1)}$$

In the similar manner, we have the inequalities

$$| v_1(x,y,z)-\Psi^{\gamma}(x,0,0)v_1(0,0,z)-\Psi^{\gamma}(0,y,0)v_1(0,0,z)-I^{\alpha_1,\psi}_{\theta^+,x}(\mathscr{M}^{p,v}_{v_1,v_2,v_3}f(x,y,z)) |$$
$$\leq \in x\frac{(\psi(x)-\psi(0))^{\alpha_1}}{\Gamma(\alpha_1+1)}$$

$$| v_2(x,y,z)-\Psi^{\gamma}(0,0,z)v_2(x,0,0)-\Psi^{\gamma}(0,y,0)v_2(x,0,0)-I^{\alpha_2,\psi}_{\theta^+,y}(\mathscr{M}^{p,v}_{v_1,v_2,v_3}f(x,y,z)) |$$
$$\leq \in \frac{(\psi(y)-\psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}$$

and

$$| v_3(x,y,z)-\Psi^{\gamma}(x,0,0)v_3(0,y,0)-\Psi^{\gamma}(0,0,z)v_3(0,y,0)-I^{\alpha_3,\psi}_{\theta^+,z}(\mathscr{M}^{p,v}_{v_1,v_2,v_3}f(x,y,z)) |$$

$$\leq\in \frac{(\psi(z)-\psi(0))^{\alpha_3}}{\Gamma(\alpha_3+1)}$$

Remark 2: If function v is a solution to the inequality (3.1), if and only if there exist a function $g \in C([0,a)\times[0,b)\times[0,c),\mathbb{B})$, which depends on v, such that

(a) $\mid g(x,y,z) \mid \leq \phi(x,y,z), \forall x \in [0,a), y \in [0,b), z \in [0,c)$.

(b) $\forall x \in [0,a), y \in [0,b), z \in [0,c), 0 < \alpha \leq 1, 0 \leq \beta \leq 1$.

$$\frac{\partial^{3\alpha} v(x,y,z)}{\partial_{\beta,\psi}x^{\alpha}\,\partial_{\beta,\psi}y^{\alpha}\,\partial_{\beta,\psi}z^{\alpha}} = f(x,y,z,v(x,y,z),\frac{\partial^{\alpha}_{\beta,\psi}v}{\partial_{\beta,\psi}x^{\alpha}},\frac{\partial^{\alpha}_{\beta,\psi}v}{\partial_{\beta,\psi}y^{\alpha}},\frac{\partial^{\alpha}_{\beta,\psi}v}{\partial_{\beta,\psi}z^{\alpha}})+g(x,y,z)$$

Remark 3: If function v is a solution to the inequality (3.2), then (v,v_1,v_2,v_3) is a solution of the following system of integral inequalities.

$$\begin{aligned}&\mid v-\Psi^{\gamma}(0,0,z)v(x,0,0)-\Psi^{\gamma}(0,y,0)v(x,0,0)\\&-\Psi^{\gamma}(x,0,0)v(0,y,0)-\Psi^{\gamma}(0,0,z)v(0,y,0)-\Psi^{\gamma}(x,0,0)v(0,0,z)\\&-\Psi^{\gamma}(0,y,0)v(0,0,z)-I^{\alpha,\psi}_{\theta}(\mathscr{M}^{p,v}_{v_1,v_2,v_3}f(x,y,z))\mid\\&\leq I^{\alpha,\psi}_{0^+}(\mathscr{M}^{p}\phi(x,y,z))\end{aligned}$$

$$\begin{aligned}&\mid v_1(x,y,z)-\Psi^{\gamma}(x,0,0)v_1(0,0,z)-\Psi^{\gamma}(0,y,0)v_1(0,0,z)-I^{\alpha_1,\psi}_{\theta^+,x}(\mathscr{M}^{p,v}_{v_1,v_2,v_3}f(x,y,z))\mid\\&\leq I^{\alpha_1,\psi}_{0^+,x}(\mathscr{M}^{p}\phi(x,y,z))\end{aligned}$$

$$\begin{aligned}&\mid v_2(x,y,z)-\Psi^{\gamma}(0,0,z)v_2(x,0,0)-\Psi^{\gamma}(0,y,0)v_2(x,0,0)-I^{\alpha_2,\psi}_{\theta^+,y}(\mathscr{M}^{p,v}_{v_1,v_2,v_3}f(x,y,z))\mid\\&\leq I^{\alpha_2,\psi}_{0^+,y}(\mathscr{M}^{p}\phi(x,y,z))\end{aligned}$$

$$\begin{aligned}&\mid v_3(x,y,z)-\Psi^{\gamma}(x,0,0)v_3(0,y,0)-\Psi^{\gamma}(0,0,z)v_3(0,y,0)-I^{\alpha_3,\psi}_{\theta^+,z}(\mathscr{M}^{p,v}_{v_1,v_2,v_3}f(x,y,z))\mid\\&\leq I^{\alpha_3,\psi}_{0^+,z}(\mathscr{M}^{p}\phi(x,y,z))\end{aligned}$$

$x \in [0,a), y \in [0,b), z \in [0,c), 0 < \alpha \leq 1, 0 \leq \gamma \leq 1$.

$$v_1 = \frac{\partial^{\alpha}_{\beta,\psi}v}{\partial_{\beta,\psi}z^{\alpha}}, v_2 = \frac{\partial^{\alpha}_{\beta,\psi}v}{\partial_{\beta,\psi}x^{\alpha}}, v_3 = \frac{\partial^{\alpha}_{\beta,\psi}v}{\partial_{\beta,\psi}y^{\alpha}}$$

Theorem 3.2. *We suppose that,*

(1) $a < \infty, b < \infty, c < \infty$.

(2) $f \in C([0,a) \times [0,b) \times [0,c) \times \mathbb{B}^4, \mathbb{B})$.

(3) $\exists L_f > 0$*, such that*

$$| f(x,y,z,u_1,u_2,u_3,u_4) - f(x,y,z,v_1,v_2,v_3,v_4) | \leq L_f \max_{i \in \{1,2,3,4\}} | u_i - v_i |$$

$\forall x \in [0,a), y \in [0,b), z \in [0,c), v_i, u_i \in \mathbb{B}$

Then, we have

a) for $h \in C'([0,a),\mathbb{B})$, $g \in C'([0,b),\mathbb{B})$ *and* $k \in C'([0,c),\mathbb{B})$*, the equation (3.2) has a unique solution with*

$$\begin{aligned} I_\theta^{1-\gamma,\psi} u(x,0,0) &= h(x), x \in [0,a) \\ I_\theta^{1-\gamma,\psi} u(0,y,0) &= g(y), y \in [0,b) \\ I_\theta^{1-\gamma,\psi} u(0,0,z) &= k(z), z \in [0,c) \end{aligned} \tag{3.4}$$

b) The equation (3.2) is stable.

Proof a) : If u(x,y,z) is a solution to the problem (0.1) and (3.3), then

$$(u, \frac{\partial^\alpha_{\beta,\psi} u}{\partial_{\beta,\psi} x^\alpha}, \frac{\partial^\alpha_{\beta,\psi} u}{\partial_{\beta,\psi} y^\alpha}, \frac{\partial^\alpha_{\beta,\psi} u}{\partial_{\beta,\psi} z^\alpha})$$

is a solution to the system.

$$\begin{aligned} u(x,y,z) &= \Psi^\gamma(0,0,z)h(x) + \Psi^\gamma(0,y,0)h(x) + \Psi^\gamma(x,0,0)g(y) + \Psi^\gamma(0,0,z)g(y) \\ &\quad + \Psi^\gamma(x,0,0)k(z) + \Psi^\gamma(0,y,0)k(z) + I_\theta^{\alpha,\psi}(\mathscr{M}^{p,u}_{u_1,u_2,u_3} f(x,y,z)) \\ u_1(x,y,z) &= \Psi^\gamma(x,0,0)k_{z^\alpha}(z) + \Psi^\gamma(0,y,0)k_{z^\alpha}(z) + I_{0^+,x}^{\alpha_1,\psi}(\mathscr{M}^{p,u}_{u_1,u_2,u_3} f(x,y,z)) \\ u_2(x,y,z) &= \Psi^\gamma(0,0,z)h_{x^\alpha}(x) + \Psi^\gamma(0,y,0)h_{z^\alpha}(x) + I_{0^+,y}^{\alpha_2,\psi}(\mathscr{M}^{p,u}_{u_1,u_2,u_3} f(x,y,z)) \\ u_3(x,y,z) &= \Psi^\gamma(x,0,0)g_{y^\alpha}(y) + \Psi^\gamma(0,0,z)g_{z^\alpha}(y) + I_{0^+,z}^{\alpha_1,\psi}(\mathscr{M}^{p,u}_{u_1,u_2,u_3} f(x,y,z)) \end{aligned} \tag{3.5}$$

Here,

$$u_1(x,y,z) = \frac{\partial^\alpha_{\beta,\psi} u(x,y,z)}{\partial_{\beta,\psi} z^\alpha}$$

$$u_2(x,y,z) = \frac{\partial^{\alpha}_{\beta,\psi} u(x,y,z)}{\partial_{\beta,\psi} x^{\alpha}}$$

$$u_3(x,y,z) = \frac{\partial^{\alpha}_{\beta,\psi} u(x,y,z)}{\partial_{\beta,\psi} y^{\alpha}}$$

$$h_{x^\alpha}(x) = \frac{\partial^{\alpha}_{\beta,\psi} u(x)}{\partial_{\beta,\psi} x^{\alpha}}$$

$$g_{y^\alpha}(y) = \frac{\partial^{\alpha}_{\beta,\psi} u(y)}{\partial_{\beta,\psi} y^{\alpha}}$$

$$k_{z^\alpha}(z) = \frac{\partial^{\alpha}_{\beta,\psi} u(z)}{\partial_{\beta,\psi} z^{\alpha}}$$

Let us denote right hand side of the system (3.5) by operators, A_1, A_2, A_3 and A_4 resply.

The system (3.5), then becomes

$$u(x,y,z) = A_1(u,u_1,u_2,u_3)(x,y,z)$$

$$u_1(x,y,z) = A_2(u,u_1,u_2,u_3)(x,y,z)$$

$$u_2(x,y,z) = A_3(u,u_1,u_2,u_3)(x,y,z)$$

$$u_3(x,y,z) = A_4(u,u_1,u_2,u_3)(x,y,z)$$

$u_1, u_2, u_3 \in C([0,a)\times[0,b)\times[0,c))$

Let $X : C([0,a)\times[0,b)\times[0,c))\times C([0,a)\times[0,b)\times[0,c))\times C([0,a)\times[0,b)\times[0,c))\times C([0,a)\times[0,b)\times[0,c))$

and for any $\delta > 0$, consider the Bielecki norm on v

$$\| (u,u_1,u_2,u_3) \|_{\mathbb{B}} := max\{M_1, M_2, M_3, M_4\}$$

$$M_1 := \max_{(x,y,z)\in[0,a)\times[0,b)\times[0,c)} | u(x,y,z) | e^{-\delta(x+y+z)}$$

$$M_2 := \max_{(x,y,z)\in[0,a)\times[0,b)\times[0,c)} | u_1(x,y,z) | e^{-\delta(x+y+z)}$$

$$M_3 := \max_{(x,y,z)\in[0,a)\times[0,b)\times[0,c)} | u_2(x,y,z) | e^{-\delta(x+y+z)}$$

$$M_4 := \max_{(x,y,z)\in[0,a)\times[0,b)\times[0,c)} | u_3(x,y,z) | e^{-\delta(x+y+z)}$$

Then $(X, \|\ .\ \|_{\mathbb{B}})$ is an ordered L - space. We will define the operator $A : X \to X$ by
$(u,u_1,u_2,u_3) \to (A_1(u,u_1,u_2,u_3), A_2(u,u_1,u_2,u_3), A_3(u,u_1,u_2,u_3), A_4(u,u_1,u_2,u_3))$
Using the hypothesis 1-3, we have

$$\| A(\bar{u},\bar{u}_1,\bar{u}_2,\bar{u}_3) - A(\bar{\bar{u}},\bar{\bar{u}}_1,\bar{\bar{u}}_2,\bar{\bar{u}}_3) \|_{\mathbb{B}} \leq \frac{L_f}{\delta} \| A(\bar{u},\bar{u}_1,\bar{u}_2,\bar{u}_3) - A(\bar{\bar{u}},\bar{\bar{u}}_1,\bar{\bar{u}}_2,\bar{\bar{u}}_3) \|_{\mathbb{B}} \tag{3.6}$$

taking $\delta > 0$, such that $\frac{L_f}{\delta} < 1$ in relation (3.6), the operator A is a contradiction and hence equation (0.1) has unique solution.

(b)Let v be a solution to the inequality (3.2), let v be the unique solution of (0.1) satisfying

$$I_{\theta}^{1-\gamma,\psi} u(x,0,0) = v(x,0,0),\ x \in [0,a)$$
$$I_{\theta}^{1-\gamma,\psi} u(0,y,0) = v(0,y,0),\ y \in [0,b)$$
$$I_{\theta}^{1-\gamma,\psi} u(0,0,z) = v(0,0,z),\ z \in [0,c)$$

with $\gamma = \alpha + \beta(1-\gamma)$

from remark (2), the hypothesis 3 and lemma, we have

$$
\begin{aligned}
&| v(x,y,z) - u(x,y,z) | \\
&\leq | v(x,y,z) - \Psi^{\gamma}(0,0,z)v(x,0,0) - \Psi^{\gamma}(0,y,0)v(x,0,0) - \Psi^{\gamma}(0,0,z)v(0,y,0) - \\
&\Psi^{\gamma}(x,0,0)v(0,0,z) - \Psi^{\gamma}(0,y,0)v(0,0,z) | \\
&- I_{\theta}^{\alpha,\psi}(\mathscr{M}^{p,v}_{v_1,v_2,v_3} f(x,y,z)) + I_{\theta}^{\alpha,\psi}(\bar{\mathscr{M}}^{p}_{u,v} f(x,y,z)) \\
&\leq \in x \frac{(\psi(x)-\psi(0))^{\alpha_1}(\psi(y)-\psi(0))^{\alpha_2}(\psi(z)-\psi(0))^{\alpha_3}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(\alpha_3+1)} \\
&+ L_f I_{\theta}^{\alpha,\psi}(p \max_{i\in\{1,2,3,4\}} | u_1(x,y,z) - v_1(x,y,z) |)
\end{aligned}
$$

$$
\begin{aligned}
\leq \in a \frac{(\psi(c)-\psi(0))^{\alpha_3}(\psi(b)-\psi(0))^{\alpha_2}(\psi(a)-\psi(0))^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(\alpha_3+1)} \times [E_{\alpha}[L_f a \Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3) \\
(\psi(c)-\psi(0))^{\alpha_3}(\psi(b)-\psi(0))^{\alpha_2}(\psi(a)-\psi(0))^{\alpha_1}]] = C_f^1 \in
\end{aligned}
$$

where,

$$C_f^1 = a\frac{(\psi(c)-\psi(0))^{\alpha_3}(\psi(b)-\psi(0))^{\alpha_2}(\psi(a)-\psi(0))^{\alpha_1}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(\alpha_3+1)} \times [E_\alpha[L_f a\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3) (\psi(c)-\psi(0))^{\alpha_3}(\psi(b)-\psi(0))^{\alpha_2}(\psi(a)-\psi(0))^{\alpha_1}]]$$

By performing the same process as above, we obtain the following inequalities

$$\mid v_1(x,y,z)-u_1(x,y,z)\mid \leq \in a\frac{(\psi(a)-\psi(0))^{\alpha_1}}{\Gamma(\alpha_1+1)}E_\alpha[L_f a(\psi(a)-\psi(0))^{\alpha_1}\Gamma(\alpha_1)] = C_f^2$$

where,

$$C_f^2 = a\frac{(\psi(a)-\psi(0))^{\alpha_1}}{\Gamma(\alpha_1+1)}E_\alpha[L_f a(\psi(a)-\psi(0))^{\alpha_1}\Gamma(\alpha_1)]$$

and

$$\mid v_2(x,y,z)-u_2(x,y,z)\mid \leq \in \frac{(\psi(b)-\psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}E_\alpha[L_f(\psi(b)-\psi(0))^{\alpha_2}\Gamma(\alpha_2)] = C_f^3$$

where,

$$C_f^3 = \frac{(\psi(b)-\psi(0))^{\alpha_2}}{\Gamma(\alpha_2+1)}E_\alpha[L_f(\psi(b)-\psi(0))^{\alpha_2}\Gamma(\alpha_2)]$$

and

$$\mid v_3(x,y,z)-u_3(x,y,z)\mid \leq \in \frac{(\psi(c)-\psi(0))^{\alpha_3}}{\Gamma(\alpha_3+1)}E_\alpha[L_f(\psi(c)-\psi(0))^{\alpha_3}\Gamma(\alpha_3)] = C_f^3$$

where,

$$C_f^4 = \frac{(\psi(c)-\psi(0))^{\alpha_3}}{\Gamma(\alpha_3+1)}E_\alpha[L_f(\psi(c)-\psi(0))^{\alpha_3}\Gamma(\alpha_3)]$$

so the equation (0.1) is stable.

4. Conclusion

In this paper, we have proposed stability remarks of the solution of fractional partial differential equation which is growing and gains special attention in the solution of linear heat equations and diffusion equations.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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