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SOME COMMON BEST PROXIMITY POINTS THEOREMS FOR GENERALISED RATIONAL α-ϕ-GERAGHATY PROXIMAL CONTRACTIONS

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Abstract. In this paper, we establish some common best proximity point theorems for generalised rational α-ϕ-Geraghaty proximal contraction mappings in complete metric spaces. Examples are also given to illustrate our results.

Keywords: common best proximity point; α-admissible; generalized α-ϕ-Geraghaty proximal rational contraction.

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1. Introduction

Fixed point theory has wide application in various branches of science, engineering and other allied fields. It deals with solving an equation of the form $Tx = x$ where $T : A \to A$ defined on a subset A of a space X. If $T : A \to B$ be a non self mapping and $A, B \in X$ then the equation $Tx = x$ may not have a solution. In this case, it can be considered an element x for which the error $d(x, Tx)$ is the global minimum and x is said to be in close proximity with Tx. With this concept K. Fan [3] state the following theorems.

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Theorem 1.1. *Let K be a non-empty compact convex subset of a normal space X and $T : K \to X$ be a continuous non-self mapping. There exists $x \in K$ such that*

$$\|x - Tx\| = d(K, Tx) = \inf\{\|Tx - u\| : u \in K\}.$$

When a non-self mapping $T : A \to B$ has no fixed point, it is quite natural to find an element x^* such that $d(x^*, Tx^*)$ is minimum. The best proximity point theorem assure the existence of an element x^* such that

$$d(x^*, Tx^*) = d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

This element x^* is called the best proximity point of T. If the mapping T under discussion is a self-mapping, then the best proximity point theorem becomes to a fixed point results.

Geraghty [1] introduced an important generalisation of Banach contraction Principle.

Theorem 1.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping. Suppose that there exists $\beta \in \mathfrak{F}$ such that, for any $x, y \in X$,*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y).$$

Then T has a unique fixed point, where the class $\mathfrak{F}$ is the set of function $\beta : [0, \infty) \to [0, \infty)$ satisfying

$$\beta(t_n) \to 1 \Rightarrow t_n \to 0.$$

Since the constant function $f(t) = k$, where $k \in [0, 1)$ in $\mathfrak{F}$, Theorem 1.2 extent Theorem 1.1

In 2014, Karapinar [7] introduced a generalized α-ϕ-Geraghaty contraction type mapping and, in 2016, Hamzehnejadi and Lashkaripour [8] introduced a generalized α-ϕ-Geraghaty proximal contraction mapping. They also established some best proximity point theorems for this mapping.

The aim of this paper is to prove some common best proximity theorems for generalised rational α-ϕ-Geraghty proximal contraction mappings in complete metric space.

2. Preliminaries

Now, we recall some elementary results and basic definitions for our main results in this paper.

Let Φ be the class of all functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i): ϕ is nondecreasing

(ii): ϕ is continuous

(iii): $\phi(t) = 0$ if and only if $t = 0$.

Let A and B are non-empty subsets of a metric space (X,d).

$$\begin{aligned} A_0 &= \{a \in A : d(a,b) = d(A,B) \; for\ some\ b \in B\} \\ B_0 &= \{b \in B : d(a,b) = d(A,B) \; for\ some\ a \in A\}, \end{aligned}$$

$$d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}.$$

Definition 2.1. *Let (X,d) be a metric space and A, B two nonempty subsets of X. A point $u \in X$ is called a common best proximity point of non-self mappings $S,T : A \to B$ if*

$$d(u,Su) = d(u,Tu) = d(A,B).$$

It is clear that a common fixed point coincides with a common best proximity point if $d(A,B) = 0$.

Definition 2.2. [10] *Let A, B be nonempty subsets of a metric space (X,d) and $\alpha : X \times X \to [0,\infty)$ be a function. A pair (S,T) of non-self mappings $S,T : A \to B$ is said to be α-proximal admissible if*

$$\begin{cases} \alpha(x_1,x_2) \geq 1. \\ d(u_1,Sx_1) = d(A,B) \Rightarrow \min\{\alpha(u_1,u_2), \alpha(u_2,u_1)\} \geq 1. \\ d(u_2,Tx_2) = d(A,B) \end{cases}$$

for all $x_1,x_2,u_1,u_2 \in A$.

Konrawut et. al. [6] introduced a generalised $\alpha-\phi$-Geraghty proximal contraction mappings in metric spaces.

Definition 2.3. *Let A and B be nonempty subsets of a metric space (X,d). Let $\phi \in \Phi$, $\alpha : X\times X \to [0,\infty)$ be a function, $S,T : A \to B$ be a non-self mappings. (S,T) is called a generalised α-ϕ-Geraghaty proximal contraction pair if*

$$\begin{cases} \alpha(x,y) \geq 1. \\ d(u,Sx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{cases}$$

$$\Rightarrow \alpha(x,y)\phi(d(u,v)) \leq \beta(\phi(M(x,y)))\phi(M(x,y)-d(A,B))$$

for all $x,y,u,v \in A$ with $\alpha(x,y) \geq 1$, where $\beta \in \mathfrak{F}$ and $M(x,y) = \max\{d(x,y), d(x,u), d(y,u)\}$.

Now, we introduce a generalized rational α-ϕ-Geraghty proximal contraction mappings in metric spaces.

Definition 2.4. *Let A and B be nonempty subsets of a metric space (X,d). Let $\phi \in \Phi$, $\alpha : X\times X \to [0,\infty)$ be a function, $S,T : A \to B$ be non-self mappings. (S,T) is called a generalized rational α-ϕ-Geraghty proximal contraction pair if*

$$\begin{cases} \alpha(x,y) \geq 1. \\ d(u,Sx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{cases}$$

$$\Rightarrow \alpha(x,y)\phi(d(u,v)) \leq \beta(\phi(M(x,y)))\phi(M(x,y)-d(A,B))$$

for all $x,y,u,v \in A$ with $\alpha(x,y) \geq 1$, where $\beta \in \mathfrak{F}$ and

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,u)d(y,v)}{1+d(x,y)}, \frac{d(y,u)d(y,v)}{1+d(u,v)}\right\} \tag{1}$$

3. Main Results

Now, we prove some common best proximity point theorems for a generalized rational α-ϕ-Geraghty proximal contraction pair in metric spaces.

Theorem 3.1. *Let A and B be nonempty subsets of a complete metric space (X,d) such that $A_0 \neq \phi$ and A is closed. Let $S,T : A \to B$ be a generalized rational α-ϕ-Geraghty proximal contraction pair. Suppose that*

(i): $S(A_0) \subseteq B_0$ *and* $T(A_0) \subseteq B_0$

(ii): *there exists* $x_0, x_1 \in A_0$ *such that*

$$d(x_1, Sx_0) = d(A,B) \min\{\alpha(x_0,x_1), \alpha(x_1,x_0)\} \geq 1,$$

(iii): *the pair* (S,T) *is* α*-proximal admissible.*

(iv): S *and* T *are continuous.*

Then S and T have a common best proximity point, that is, there exists a point $x^* \in A$ such that $d(x^*, Sx^*) = d(x^*, Tx^*) = d(A,B)$.

Proof. By the condition (ii), there exists $x_0, x_1 \in A_0$ such that

$$d(x_1, Sx_0) = d(A,B), \min\{\alpha(x_0,x_1), \alpha(x_1,x_0)\} \geq 1 \tag{2}$$

From the condition (i), we have $Tx_1 \in B_0$ and so there exists $x_2 \in A_0$ such that

$$d(x_2, Tx_1) = d(A,B) \tag{3}$$

From (2) and (3), since the pair (S,T) is α-proximal admissible, we obtain

$$\min\{\alpha(x_1,x_2), \alpha(x_2,x_1)\} \geq 1 \tag{4}$$

Again, by the condition (i), we have $Sx_2 \in B_0$ and so there exists $x_3 \in A_0$ such that

$$d(x_3, Sx_2) = d(A,B) \tag{5}$$

By induction, we can find the sequence $\{x_n\}$ in A_0 such that

$$\min\{\alpha(x_n,x_{n+1}), \alpha(x_{n+1},x_n)\} \geq 1 \tag{6}$$

and

$$d(x_{n+1}, Sx_n) = d(x_{n+1}, Tx_{n+1}) = d(A,B) \tag{7}$$

for all $n \geq 0$. Since (S,T) is a generalized rational α-ϕ-Geraghty proximal contraction pair, using (6) and $\alpha(x_n, x_{n+1}) \geq 1$, it follows that, for all $n \geq 0$,

$$\begin{aligned} \phi(d(x_{n+1},x_{n+2})) &\leq \alpha(x_n,x_{n+1})\phi(d(x_{n+1},x_{n+2})) \\ &\leq \beta(\phi(M(x_n,x_{n+1})))\phi(M(x_n,x_{n+1})-d(A,B)) \\ &< \phi(M(x_n,x_{n+1}))-d(A,B) \qquad (8) \end{aligned}$$

where

$$\begin{aligned} M(x_n,x_{n+1}) &= \max\left\{d(x_n,x_{n+1}), \frac{d(x_n,x_{n+1})d(x_{n+1},x_{n+2})}{1+d(x_n,x_{n+1})}, \frac{d(x_n,x_{n+1})d(x_{n+1},x_{n+2})}{1+d(x_{n+1},x_{n+2})}\right\} \\ &\leq \max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\} \\ &\leq \max\{d(x_n,x_{n+1}),d(x_{n+1},x_{n+2})\}+d(A,B) \qquad (9) \end{aligned}$$

If $M(x_n,x_{n+1}) = d(x_{n+1},x_{n+2})$, then, from (8), it follows that

$$\begin{aligned} \phi(d(x_{n+1},x_{n+2})) &< \phi(M(x_n,x_{n+1})-d(A,B)) \\ &= \phi(d(x_{n+1},x_{n+2})-d(A,B)) \end{aligned}$$

which, from the properties of ϕ, implies that

$$\begin{aligned} d(x_{n+1},x_{n+2}) &< d(x_{n+1},x_{n+2})-d(A,B) \\ &< d(x_{n+1},x_{n+2}) \end{aligned}$$

which is a contradiction. Therefore, we have

$$M(x_n,x_{n+1}) = d(x_n,x_{n+1}) \qquad (10)$$

for all $n \geq 0$.

Next, from (8) and (10), we obtain

$$\phi(d(x_{n+1},x_{n+2})) < \phi(d(x_n,x_{n+1})-d(A,B))$$

for all $n \geq 0$, which, from the properties of ϕ, implies that

$$d(x_{n+1},x_{n+2}) < d(x_n,x_{n+1})$$

for all $n \geq 0$ and hence we deduce that the sequence $\{d(x_{n+1}, x_{n+2})\}$ is non-negative and decreasing. Consequently, there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_{n+2}) = r$. Assume that there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$. From (8) and the property of ϕ, it follows that

$$\begin{aligned} d(x_{n_0+1}, x_{n_0+2}) &< M(x_{n_0}, x_{n_0+1}) - d(A,B) \\ &\leq M(x_{n_0}, x_{n_0+1}) \\ &\leq d(x_{n_0}, x_{n_0+1}). \end{aligned}$$

Since $d(x_{n_0}, x_{n_0+1}) = 0$, we have $0 \leq d(x_{n_0+1}, x_{n_0+2}) \leq 0$ and so $d(x_{n_0+1}, x_{n_0+2}) = 0$. This implies that $x_{n_0} = x_{n_0+1} = x_{n_0+2}$. By (7), we obtain

$$d(x_{n_0}, Sx_{n_0}) = d(x_{n_0}, Tx_{n_0}) = d(A,B) \tag{11}$$

which is the desired result.

Now, let $r = d(x_n, x_{n+1}) \neq 0$ for all $n \geq 0$. In the sequel, we prove that $r = 0$. Suppose that $r > 0$. From (8) and (10), we have

$$0 < \frac{\phi(d(x_{n+1}, x_{n+2}))}{\phi(d(x_n, x_{n+1}) - d(A,B))} \leq \beta(\phi(d(x_n, x_{n+1}))).$$

Using the fact that ϕ is non-decreasing, we have

$$0 < \frac{\phi(d(x_{n+1}, x_{n+2}))}{\phi(d(x_n, x_{n+1}))} \leq \frac{\phi(d(x_{n+1}, x_{n+2}))}{\phi(d(x_n, x_{n+1}) - d(A,B))} \leq \beta(\phi(d(x_n, x_{n+1})))$$

which implies that $\lim_{n\to\infty} \beta(\phi(d(x_n, x_{n+1}))) = 1$. By the property of $\beta \in \mathfrak{F}$, we have

$$\lim_{n\to\infty} \phi(d(x_n, x_{n+1})) = 0.$$

Hence, we get $r = 0$, which is a contradiction. Therefore, $\lim_{n\to\infty} \phi(d(x_{n+1}, x_{n+2})) = 0$. Note that, for all $m, n \geq 0$,

$$\min\{\alpha(x_n, x_m), \alpha(x_m, x_n)\} \geq 1 \tag{12}$$

and

$$d(x_{n+1}, Sx_n) = d(x_{m+1}, Tx_m) = d(A,B). \tag{13}$$

Thus, for all $m,n \geq 0$, we obtain

$$\begin{aligned} \phi(d(x_{n+1},x_{m+1})) &\leq \alpha(x_n,x_m)\phi(d(x_{n+1},x_{m+1})) \\ &\leq \beta(\phi(M(x_{n+1},x_{m+1})))\phi(M(x_{n+1},x_{m+1})-d(A,B)) \end{aligned} \tag{14}$$

where

$$\begin{aligned} M(x_{n+1},x_{m+1}) &= \max\left\{d(x_n,x_m), \frac{d(x_n,x_{n+1})d(x_m,x_{m+1})}{1+d(x_n,x_m)}, \frac{d(x_n,x_{n+1})d(x_m,x_{m+1})}{1+d(x_{n+1},x_{m+1})}\right\} \\ &\leq \max\left\{d(x_n,x_m), \frac{d(x_n,x_{n+1})d(x_m,x_{m+1})}{1+d(x_n,x_m)}, \frac{d(x_n,x_{n+1})d(x_m,x_{m+1})}{1+d(x_{n+1},x_{m+1})}\right\} \\ &\quad +d(A,B) \end{aligned}$$

Since $\lim_{n\to\infty}\phi(d(x_n,x_{n+1}))=0$, we have

$$\lim_{m,n\to\infty}\sup M(x_{n+1},x_{m+1}) = \lim_{m,n\to\infty}\sup d(x_n,x_m) \leq \lim_{m,n\to\infty}\sup d(x_n,x_m)+d(A,B) \tag{15}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. In the contrary, we suppose that

$$\lim_{m,n\to\infty}\sup d(x_n,x_m) = r > 0.$$

Letting $m,n\to\infty$ and using the triangular inequality, we have

$$\begin{aligned} \lim_{m,n\to\infty}\sup d(x_n,x_m) &\leq \lim_{m,n\to\infty}\sup(d(x_n,x_{n+1})+d(x_{n+1},x_{m+1})+d(x_{m+1},x_m)) \\ &\leq \lim_{m,n\to\infty}\sup d(x_{n+1},x_{m+1}) \end{aligned} \tag{16}$$

Combining (14), (15) and (16) with the continuous property of ϕ, we obtain

$$\lim_{m,n\to\infty}\sup d(x_n,x_m) \leq \lim_{m,n\to\infty}\sup\beta(\phi(M(x_{n+1},x_{m+1}))) \lim_{m,n\to\infty}\sup\phi(d(x_n,x_m))$$

Since, $\lim_{m,n\to\infty}\sup\phi(d(x_n,x_m)) = r > 0$, it follows that

$$\lim_{m,n\to\infty}\sup\beta(\phi(M(x_{n+1},x_{m+1}))) = 1.$$

Since $\beta \in \mathfrak{F}$, we have

$$\lim_{m,n\to\infty}\sup d(x_n,x_m) = \lim_{m,n\to\infty}\sup M(x_{n+1},x_{m+1}) = 0$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence in A. Since A is a closed subset of a complete metric space (X,d), there exists $x^* \in A$ such that $x_n \to x^*$ as $n\to\infty$. Since S is continuous at x^*, $\lim_{n\to\infty} Sx_n = Sx^*$. Moreover, from the continuous property of the metric d, it follows that $\lim_{n\to\infty} d(x_{n+1}, Sx_n) = d(x^*, Sx^*)$. Thus, by (7), $d(x^*, Sx^*) = d(A,B)$. Similarly by the continuity of T, we have $d(x^*, Tx^*) = d(A,B)$. Therefore, x^* is a common best proximity point of S and T. This completes the proof. $\square$

In the next result we remove the continuity condition.

Theorem 3.2. *Let A and B be nonempty subsets of a complete metric space (X,d) such that $A_0 \neq \phi$ and A is closed. Let $S,T : A \to B$ be a generalized rational α-ϕ-Geraghty proximal contraction pair. Suppose that*

- **(i):** $S(A_0) \subseteq B_0$ *and* $T(A_0) \subseteq B_0$
- **(ii):** *there exists $x_0, x_1 \in A_0$ such that*

$$d(x_1, Sx_0) = d(A,B), \min\{\alpha(x_0,x_1), \alpha(x_1,x_0)\} \geq 1,$$

- **(iii):** *the pair (S,T) is α-proximal admissible.*
- **(iv):** *If $\{x_n\}$ is a sequence in A such that $\min\{\alpha(x_n,x_{n+1}), \alpha(x_{n+1},x_n)\} \geq 1$ for all $n \geq 1$ and $x_n \to x \in A$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of x_n such that $\min\{\alpha(x_{n_k},x), \alpha(x,x_{n_k})\} \geq 1$ for all $k \geq 1$.*

Then there exists $x^* \in A$ such that $d(x^*, Sx^*) = d(x^*, Tx^*) = d(A,B)$.

Proof. Following the proof of Theorem 3.1, we obtain a sequence $\{x_n\}$ in A_0 such that (6) and (7) hold. Aiso, we can show that $\{x_n\}$ is a Cauchy sequence in A. Since A is a closed subset of a complete metric space (X,d), there exists $x^* \in A$ such that $x_n \to x^*$ as $n \to \infty$. By the condition (iv), we have $\alpha(x^*, x_{2n(k)+1}) \geq 1$. Next, from (7), it follows that

$$d(x_{2n(k)+1}, Sx^*) = d(x_{2n(k)+2}, Tx_{n(k)+1}) = d(A,B). \tag{17}$$

On the other hand, by using the triangle inequality we obtain

$$\begin{aligned} d(x^*, Sx^*) &\leq d(x^*, x_{2n(k)+2}) + d(x_{2n(k)+2}, x_{2n(k)+1}) + d(x_{2n(k)+2}, Sx^*) \\ &\leq d(x^*, x_{2n(k)+2}) + d(x_{2n(k)+2}, x_{2n(k)+1}) + d(A,B) \end{aligned}$$

which implies that

$$d(x^*,Sx^*)-d(x^*,x_{2n(k)+2})-d(A,B)\leq d(x_{2n(k)+2},x_{2n(k)+1}).$$

Since (S,T) is a generalized rational α-ϕ-Geraghty proximal contraction pair, using the property of ϕ and $\alpha(x^*,x_{2n(k)+1})\geq 1$, we have

$$\begin{aligned}
&\phi(d(x^*,Sx^*)-d(x^*,x_{2n(k)+2})-d(A,B))\\
\leq\ &\phi(d(x_{2n(k)+2},x_{2n(k)+1}))\\
\leq\ &\alpha(x^*,x_{2n(k)+1})\phi(d(x_{2n(k)+2},x_{2n(k)+1}))\\
\leq\ &\beta(\phi(M(x^*,x_{2n(k)+1})))\phi(M(x^*,x_{2n(k)+1})-d(A,B))\\
\leq\ &\phi(M(x^*,x_{2n(k)+1}))-d(A,B) \qquad (18)
\end{aligned}$$

where

$$\begin{aligned}
M(x^*,x_{2n(k)+1}) &= \max\left\{d(x^*,x_{2n(k)+1}),\frac{d(x^*,Sx^*)d(x_{2n(k)+1},x_{2n(k)+2})}{1+d(x^*,x_{2n(k)+1})},\right.\\
&\qquad\left.\frac{d(x^*,Sx^*)d(x_{2n(k)+1},x_{2n(k)+2})}{1+d(Sx^*,x_{2n(k)+2})}\right\}\\
&\leq \max\left\{d(x^*,x_{2n(k)+1}),\frac{d(x^*,Sx^*)d(x_{2n(k)+1},x_{2n(k)+2})}{1+d(x^*,x_{2n(k)+1})},\right.\\
&\qquad\left.\frac{d(x^*,Sx^*)d(x_{2n(k)+1},x_{2n(k)+2})}{1+d(Sx^*,x_{2n(k)+2})}\right\}+d(A,B)
\end{aligned}$$

Observe that

$$\lim_{n\to\infty}d(x^*,x_{2n(k)+1})=\lim_{n\to\infty}d(x_{2n(k)+1},x_{2n(k)+2})=0.$$

Thus there exists $N\in\mathbb{N}$ such that for all $k\geq N$

$$M(x^*,x_{2n(k)+1})=0<d(A,B) \qquad (19)$$

From (18), (19) and the property of ϕ, it follows that, for all $k\geq N$,

$$\begin{aligned}
d(x^*,Sx^*)-d(x^*,x_{2n(k)+2})-d(A,B) &< M(x^*,x_{2n(k)+1})-d(A,B)\\
&\leq d(A,B)-d(A,B)\\
&\leq 0
\end{aligned}$$

and so

$$d(x^*, Sx^*) = d(x^*, x_{2n(k)+2}) + d(A,B).$$

Letting $k \to \infty$, we obtain $d(x^*, Sx^*) = d(A,B)$. Similarly, we have $d(x^*, Tx^*) = d(A,B)$. Therefore, x^* is a common best proximity point of S and T. This complete the proof. $\square$

Now, we prove the uniqueness of such a common best proximity point as in Theorems 3.1 and 3.2. Here, we need the following additional condition:

(v): For all $x, y \in CB(S,T)$, we have $\alpha(x,y) \geq 1$, where $CB(S,T)$ denotes the set of common best proximity points of S and T.

Theorem 3.3. *Adding the condition (v) to the hypothesis of Theorem 3.1(resp. Theorem 3.2), the point x^* is the unique best proximity point of S and T.*

Proof. Suppose that there exists $x^*, y^* \in A$ such that

$$d(x^*, Sx^*) = d(x^*, Tx^*) = d(y^*, Sy^*) = d(y^*, Ty^*) = d(A,B)$$

where $x^* \neq y^*$. By the condition (v), we have $\alpha(x^*, y^*) \geq 1$ and, since (S,T) is a generalized rational α-ϕ-Geraghty proximal contraction pair, we have

$$\begin{aligned} \phi(d(x^*, y^*)) &\leq \alpha(x^*, y^*)\phi(d(x^*, y^*)) \\ &\leq \beta(\phi(M(x^*, y^*)))\phi(M(x^*, y^*) - d(A,B)) \\ &< \phi(M(x^*, y^*)) - d(A,B) \end{aligned} \tag{20}$$

where

$$\begin{aligned} M(x^*, y^*) &= \max\left\{ d(x^*, y^*), \frac{d(x^*, Sx^*)(y^*, Ty^*)}{1 + d(x^*, y^*)}, \frac{d(x^*, Sx^*)d(y^*, Ty^*)}{1 + d(Sx^*, Ty^*)} \right\} \\ &= d(x^*, y^*) \end{aligned}$$

From (20), we obtain

$$\phi(d(x^*, y^*)) < \phi(d(x^*, y^*)) - d(A,B) < \phi(d(x^*, y^*))$$

which is a contradiction. Hence $x^* = y^*$. $\square$

Next, we give an example to illustrate Theorem 3.1.

Example 1. *Let $X = [0,\infty) \times [0,\infty)$ be endowed with the metric*

$$d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

Take $A = \{0\} \times Q \cap [0,5]$, and $B = \{1\} \times [0,5]$. We know that $d(A,B) = 1$, $A_0 = A$ and $B_0 = B$. Consider the mappings $S,T : A \to B$ defined by

$$\begin{aligned} S(0,x) &= (1, \frac{4}{5}\log(1+\frac{x}{2})) \\ T(0,x) &= (1, \frac{4}{5}\log(1+x)) \end{aligned}$$

respectively. Then we have $S(A_0) \subseteq B_0$ and $T(A_0) \subseteq B_0$. Also, we define a function $\alpha : X \times X \to [0,\infty)$ by

$$\begin{cases} \alpha((x,y),(x^{'},y^{'})) = 1, & \textit{if } (x,y),(x^{'},y^{'}) \in [[0,1]\times[0,1]; \\ \alpha((x,y),(x^{'},y^{'})) = 1, & \textit{otherwise.} \end{cases}$$

Let $(0,x_1)$, $(0,x_2)$, $(0,u_1)$ and $(0,u_2)$ in A such that

$$\begin{cases} \alpha(((0,x_1),(0,x_2))) \geq 1, \\ d((0,u_1),S(0,x_1)) = d(A,B), \\ d((0,u_2),T(0,x_2)) = d(A,B). \end{cases}$$

Then we have $(x_1,x_2) \in [0,1] \times [0,1]$. Also, we have $u_1 = \frac{4}{5}\log(1+\frac{x_1}{2})$ and $u_2 = \frac{4}{5}\log(1+x_2)$, which implies that

$$\min\{\alpha((0,u_1),(0,u_2)), \alpha((0,u_2),(0,u_1))\} \geq 1.$$

Thus the pair (S,T) is α-proximal admissible.

Now, we check that (S,T) is a generalized α-ϕ-Geraghty contraction pair. Define the functions $\phi : [0,\infty) \to [0,\infty)$ and $\phi : [0,\infty) \to [0,1)$ by

$$\phi(t) = \frac{t^3}{2},\ \beta(t) = \frac{tan^{-1}(t)}{t}$$

for all $t \geq 0$, respectively. Then $\beta \in \mathfrak{F}$ and $\phi \in \Phi$.

Let $(x,y) \in A$. Then $t = d(x,y) = [0,1]$. Also, it is easy to show that

$$\frac{1}{2}(\frac{4}{5}log(1+t))^3 \leq tan^{-1}(\frac{t^3}{2}) \tag{21}$$

for all $t \in [0,1]$. Moreover, we have

$$\begin{aligned}
\alpha((0,x_1),(0,x_2)),\phi(d(0,u_1),(0,u_2)) &= \frac{1}{2}(|u_1-u_2|)^3 \\
&= \frac{1}{2}(|\frac{4}{5}\log(1+\frac{x_1}{2})-\frac{4}{5}\log(1+x_2)|)^3 \\
&= \frac{1}{2}\left(\frac{4}{5}|\log\left(\frac{1+\frac{x_1}{2}}{1+x_2}\right)|\right)^3 \\
&\leq \frac{1}{2}\left(\frac{4}{5}|\log\left(\frac{1+\frac{x_1}{2}}{1+\frac{x_2}{2}}\right)|\right)^3 \qquad (22)
\end{aligned}$$

Now, we show that

$$|\log\left(\frac{1+\frac{x_1}{2}}{1+\frac{x_2}{2}}\right)| \leq \log\left(1+|\frac{x_1}{2}-\frac{x_2}{2}|\right) \leq \log(1+|x_1-x_2|).$$

Suppose that $x_1 \geq x_2$ or $x_2 \geq x_1$. Observe that

$$\begin{aligned}
\log\left(\frac{1+\frac{x_1}{2}}{1+\frac{x_2}{2}}\right) &\leq \log\left(\frac{1+\frac{x_2}{2}+\frac{x_1}{2}-\frac{x_2}{2}}{1+\frac{x_2}{2}}\right) \\
&\leq \log\left(\frac{1+\frac{x_1}{2}-\frac{x_2}{2}}{1+\frac{x_2}{2}}\right) \\
&\leq \log\left(1+\frac{x_1}{2}-\frac{x_2}{2}\right) \\
&\leq \log(1+|x_1-x_2|).
\end{aligned}$$

From (22)*, it follows that*

$$\begin{aligned}
\alpha((0,x_1),(0,x_2)),\phi(d(0,u_1),(0,u_2)) &\leq \frac{1}{2}\left(\frac{4}{5}\log(1+|x_1-x_2|)\right)^3 \\
&\leq \tan^{-1}\left(\frac{1}{2}(|x_1-x_2|)^3\right) \\
&\leq \tan^{-1}\left(\frac{1}{2}(d(0,u_1),(0,u_2))^3\right) \qquad (23)
\end{aligned}$$

Consider

$$\begin{aligned}
M(x_1,x_2) &= \max\{d(0,x_1),(0,x_2),d(0,x_1),(0,u_1),d(0,x_2),(0,u_2)\} \\
&= \max\{d(0,x_1),(0,x_2),d(0,x_1),(0,u_1),d(0,x_2),(0,u_2)\}+d(A,B).
\end{aligned}$$

Taking $M^*(x_1,x_2) = M(x_1,x_2) + d(A,B)$, it follows form (23) *that*

$$\begin{aligned}
\alpha((0,x_1),(0,x_2)), \phi(d(0,u_1),(0,u_2)) &\leq \tan^{-1}\left(\frac{1}{2}(d(0,u_1),(0,u_2))^3\right) \\
&\leq \tan^{-1}\left(\frac{1}{2}(M(x_1,x_2))^3\right) \\
&\leq \tan^{-1}\left(\frac{1}{2}(M(x_1,x_2))^3\right)\frac{\frac{1}{2}(d(0,x_1),(0,x_2))^3}{\frac{1}{2}(d(0,x_1),(0,x_2))^3} \\
&\leq \tan^{-1}\left(\frac{1}{2}(M(x_1,x_2))^3\right)\frac{\frac{1}{2}(M(x_1,x_2)+1-1)^3}{\frac{1}{2}(M(x_1,x_2))^3}.
\end{aligned}$$

Since $d(A,B) = 1$ and $M^*(x_1,x_2) = M(x_1,x_2) + d(A,B)$, we have

$$\begin{aligned}
\alpha((0,x_1),(0,x_2)), \phi(d(0,u_1),(0,u_2)) &\leq \frac{\tan^{-1}\left(\frac{1}{2}(M(x_1,x_2))^3\right)}{\frac{1}{2}(M(x_1,x_2))^3}\frac{1}{2}(M(x_1,x_2)+d(A,B)-d(A,B))^3 \\
&\leq \frac{\tan^{-1}\left(\frac{1}{2}(M(x_1,x_2))^3\right)}{\frac{1}{2}(M(x_1,x_2))^3}\frac{1}{2}(M^*(x_1,x_2)-d(A,B))^3.
\end{aligned}$$

Hence, we have

$$\alpha((0,x_1),(0,x_2)), \phi(d(0,u_1),(0,u_2)) \leq \beta(\phi(M(x_1,x_2)))\phi(M^*(x_1,x_2)-d(A,B))$$

and so (S,T) is a generalised α-ϕ-Geraghty proximal contraction pair. Furthermore, S and T are continuous. Moreover, the condition (ii) of Theorem 3.1 is verified. Indeed, for $x_0 = (0,2)$ and $x_1 = (0,0.5545)$, we get

$$d(x_1,Sx_0) = d((0,0.5545),(1,0.5545)) = 1 = d(A,B)$$

and

$$\min\{\alpha(x_0,x_1),\alpha(x_1,x_0)\} \geq 1.$$

Hence all the hypothesis of Theorem 3.1 are verified. So, the pair (S,T) admits a common best proximity point, which is $x^* = (0,0)$. It is easy to show that the common best proximity point $x^* = (0,0)$ is unique.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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