RULED SURFACES IN THREE DIMENSIONAL LIE GROUPS

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Abstract. Motivated by a number of recent investigations, we define and investigate the various properties of the ruled surfaces depend on three dimensional Lie groups with a bi-variant metric. We give useful results involving the characterizations of these ruled surfaces. Some special ruled surfaces such as normal surface, binormal surface, tangent developable surface, rectifying developable surface and Darboux developable surface are worked. From those applications, we make use of such a work to interpret the Gaussian, mean curvatures of these surfaces and geodesic, normal curvature and geodesic torsion of the base curves with respect to these surfaces depend on three dimensional Lie groups.

1. INTRODUCTION

In the surface theory of geometry, ruled surfaces were found by French mathematician Gaspard Monge who was a founder of constructive geometry. Recently, many mathematicians have studied the ruled surfaces on Euclidean space and Minkowski space for a long time. The information about these topic, see, e.g., [1, 9, 10, 16, 17, 18] for a systematic work. A *ruled surface*

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in $\mathbb{R}^3$ is surface which can be described as the set of points swept out by moving a straight line in surface. It therefore has a parametrization of the form

$$\Phi(s,v) = \alpha(s) + v\delta(s)$$

where α and δ are a curve lying on the surface called *base curve* and *director curve*, respectively. The straight lines are called rulings. By using the equation of ruled surface we assume that α' is never zero and δ is not identically zero. The rulings of ruled surface are asymptotic curves. Furthermore, the Gaussian curvature of ruled surface is everywhere non-positive. The ruled surface is developable if and only if the distribution parameter vanishes and it is minimal if and olny if its mean curvature vanishes [7]. A ruled surface is doubly ruled if through every one of its points there are two distinct lines that lie on the surface. Cylinder, cone, helicoid, Mobius strip, right conoid are some examples of ruled surfaces and hyperbolic paraboloid and hyperboloid of one sheet are doubly ruled surfaces. Recently, there are many works about geometry and curve theory in three dimensional Lie groups. Çöken and Çiftçi studied the degenerate semi- Riemannian geometry of Lie Gruops. They found reductive homogeneous semi-Riemannian space from the Lie group in a natural way [5]. Next, general helices in three dimensional Lie group with bi-invariant metric are defined by Çiftçi in [4]. He generalized the Lancret's theorem and obtained so-called spherical general helices, and also he gave a relation between the geodesics of the so-called cylinders and general helices. In [4], a cylinder which is a surface was defined in a three dimensional Lie group with a bi-variant metric in accordance with the definition of a ruled surface in Riemannian manifold. If G is a three dimensional Lie group and $\mathfrak{g}$ is its Lie algebra, then a cylinder is a surface $\varphi(t,\lambda)$ given by $\varphi : \mathbb{R} \times \mathbb{R} \longrightarrow G$, $\varphi(t,\lambda) = \alpha(t)\exp(\lambda X)$, where $\alpha : \mathbb{R} \longrightarrow G$ is a curve in G, $X \in \mathfrak{g}$ and

$$\exp : \mathfrak{g} \longrightarrow G$$

is the exponential mapping of G. Meeks and Pérez studied geometry of constant mean curvature $H \geq 0$ surfaces which are called H-surfaces in three dimensional simply-connected Lie group see[12].

Slant helices in three dimensional Lie groups were defined by Okuyucu et al. in [13]. They obtained a characterization of slant helices and gave some relations between slant helices and

their involutes, spherical images. They also defined Bertrand curves and Mannheim curves in three dimensional lie groups in [8, 14] and gave the harmonic curvature function for some special curves such as helix, slant curves, Mannheim curves and Bertrand curves. In the present paper, we define and investigate the ruled surface in three dimensional Lie groups with a bi-variant metric. We obtain the Gaussian and mean curvatures, distribution parameter of the ruled surface. Also we find the geodesic, normal curvatures and geodesic torsion of the base curve of ruled surface with respect to ruled surface in three dimensional Lie groups. In the final part of this paper, we give some characterizations of the ruled surface using the curvatures.

2. PRELIMINARIES

A *Lie group* is a nonempty subset G which satisfies the following conditions;

1) G is a group.

2) G is a smooth manifold.

3) G is a topological group, in particular, the group operation $\circ : G \times G \longrightarrow G$ and the inverse map $inv : G \longrightarrow G$ are smooth.

Let $\mathfrak{g}$ be the *Lie algebra* of G. $\mathfrak{g}$ is a vector space together with a bilinear map

$$[\,,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

called Lie bracket on $\mathfrak{g}$, such that the following two identities hold for all $a, b, c \in \mathfrak{g}$

$$[a, a] = 0$$

and the so-called Jacobi identity

$$[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0.$$

It is immediately verified that $[a, b] = -[b, a]$.

If G is a Lie group, a vector field X on G is *left-invariant*, if

$$d(L_a)_b(X(b)) = X(L_a)_b) = X(ab)$$

for all $a, b \in G$. Here $L_a : G \longrightarrow G$ and $d(L_a) : T_G \longrightarrow T_G$ where T_G is a tangent vector space. Similarly X is *right-invariant*, if

$$d(R_a)_b(X(b)) = X(R_a)_b) = X(ba).$$

A Riemannian metric on a Lie group G is called left-invariant if

$$\langle u, v \rangle = \langle d(L_a)(u), d(L_a)(v) \rangle$$

where $u, v \in T_G(a)$, $a \in G$. A metric on G that is both left-invariant and right-invariant is called *bi-invariant* (see [11]). Let G be a Lie group with bi-invariant metric $\langle , \rangle$ and let D be the corresponding Levi-Civita connection. If $\mathfrak{g}$ is the Lie algebra of G, then $\mathfrak{g}$ is isomorph to T_eG

(1)
$$\langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle$$

(2)
$$D_X Y = \frac{1}{2}[X, Y]$$

for all $X, Y, Z \in \mathfrak{g}$. Let $\alpha : I \subset \mathbb{R} \longrightarrow G$ be a parametrized curve and $\{X_1, X_2, ..., X_n\}$ be an orthonormal basis of $\mathfrak{g}$. We can write two vector fields W and Z as $W = \sum_{i=1}^{n} \omega_i X_i$ and $Z = \sum_{i=1}^{n} z_i X_i$ where $\omega_i : I \longrightarrow \mathbb{R}$ and $z_i : I \longrightarrow \mathbb{R}$ are smooth functions. The Lie bracket of W and Z is defined by $[W, Z] = \sum_{i,j=1}^{n} \omega_i z_j [X_i, X_j]$. If the directional derivative of W is $\dot{W} = \sum_{i=1}^{n} \dot{\omega}_i X_i$ for $\dot{\omega}_i = \frac{d\omega}{dt}$, then the following equation hold as;

(3)
$$D_{\alpha'} W = \dot{W} + \frac{1}{2}[T, W]$$

where $\alpha' = T$ is the tangent vector field of α. Note that if W is left-invariant vector field of α, then $\dot{W} = 0$ (see [3, 4])

Now, let α be a parametrized curve in three dimensional Lie group G and $\{T, N, B, \kappa, \tau\}$ be the Frenet apparatus of the curve α. Then Çiftçi [4] defined τ_G as;

(4)
$$\tau_G = \frac{1}{2} \langle [T, N], B \rangle$$

or

$$\tau_G = \frac{1}{2\kappa^2 \tau} \left\langle \ddot{T}, \left[T, \dot{T}\right] \right\rangle + \frac{1}{4\kappa^2 \tau} \left\| \left[T, \dot{T}\right] \right\|^2.$$

Also the following equalities were given in [13];

$$(5) \qquad [T,N] = \langle [T,N], B \rangle B = 2\tau_G B$$

$$[T,B] = \langle [T,B], N \rangle N = -2\tau_G N$$

Let $\alpha : I \subset \mathbb{R} \longrightarrow G$ be a curve with arc-length parameter s, in three dimensional Lie group G, then the Frenet formulae in G is given by [2]

$$(6) \qquad \begin{aligned} \frac{dT}{ds} &= \kappa N \\ \frac{dN}{ds} &= -\kappa T + (\tau - \tau_G)B \\ \frac{dB}{ds} &= -(\tau - \tau_G)N \end{aligned}$$

After some computation which we use equations (3) and (6), the curvature κ and torsion τ are found by

$$\kappa = \left\| \frac{dT}{ds} \right\| = \left\| \dot{T} \right\|$$

$$(7) \qquad \tau = \left\| \frac{dB}{ds} \right\| + \tau_G$$

(for curvature κ see [4]). It is known that cross product $\times$ in $\mathbb{R}^3$ is a Lie bracket. If the three dimensional special orthogonal group with the bi-variant metric is $SO(3)$, then by identifying $\mathfrak{so}(3)$ with $(\mathbb{R}^3, \times)$, we have $[X,Y] = X \times Y$ for all $X,Y \in \mathfrak{so}(3)$. So for a curve in $SO(3)$, it is shown that (see [4])

$$(8) \qquad \tau_G = \frac{1}{2} \langle T \times N, B \rangle = \frac{1}{2}.$$

Also if G is Abelian ,then $\tau_G = 0$ (see [4]).

3. RULED SURFACES IN THREE DIMENSIONAL LIE GROUPS

We will define ruled surfaces in three dimensional Lie groups .Then we will obtain the distribution parameter, Gaussian curvature and mean curvature of these ruled surfaces. Also we will identify the geodesic curvature, the normal curvature and geodesic torsion of the base curve of ruled surfaces.

Definition 1: Let G be the three dimensional Lie group with a bi-invariant metric $\langle,\rangle$. A *ruled surface* $\varphi(s,v)$ in G , $\varphi : \mathbb{R} \times \mathbb{R} \longrightarrow G$, is given by

$$(9) \qquad\qquad \varphi(s,v) = \alpha(s) + vX(s)$$

where $\alpha : \mathbb{R} \longrightarrow G$ is called *base curve* and $X \in \mathfrak{g}$ is a left-invariant unit vector field which is called *director*. The directors denote straight lines which are called *rulings* of the ruled surface.

The base curve α is given with the arc-length parameter s, the set $\{T, N, B, \kappa, \tau\}$ denote the Frenet apparatus of α , $\alpha' = T$, $\kappa \neq 0$ and $\tau_G = \frac{1}{2} \langle [T,N], B \rangle$.

Definition 2: If there exists a common perpendicular to two constructive rulings in the surface, then the foot of the common perpendicular on the main ruling is called *central point*. The locus of the central point is called *striction curve*. The striction curve of the ruled surface φ in three dimensional Lie group G is given by

$$(10) \qquad\qquad \overline{\alpha} = \alpha - \frac{\langle \alpha', D_T X \rangle}{\|D_T X\|^2} X.$$

Definition 3: The *distribution parameter* λ of the ruled surface φ in three dimensional Lie group G given by equation (9) is dedicated as;

$$(11) \qquad\qquad \lambda = \frac{\det(T, X, D_T X)}{\|D_T X\|^2}.$$

The standard unit normal vector field U on the ruled surface φ is defined by

$$(12) \qquad\qquad U = \frac{\varphi_s \times \varphi_v}{\|\varphi_s \times \varphi_v\|}.$$

where $\varphi_s = \frac{d\varphi}{ds}$ and $\varphi_v = \frac{d\varphi}{dv}$.

Definition 4: The *Gaussian curvature* and *mean curvature* of the ruled surface φ in three dimensional Lie group G are given respectively by

$$(13) \qquad\qquad K = \frac{eg - f^2}{EG - F^2}$$

and

$$(14) \qquad\qquad H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}$$

where $E = \langle \varphi_s, \varphi_s \rangle$, $F = \langle \varphi_s, \varphi_v \rangle$, $G = \langle \varphi_v, \varphi_v \rangle$, $e = \langle \varphi_{ss}, U \rangle$, $f = \langle \varphi_{sv}, U \rangle$ and $g = \langle \varphi_{vv}, U \rangle$.

Definition 5: For a surface Φ in three dimensional Lie group G,

1) Φ is *developable* if and only if the distribution parameter of Φ vanishes.

2) Φ is called *minimal* if and if only the mean curvature of Φ vanishes.

Definition 6: If the Gaussian curvature of a surface in in three dimensional Lie group G is K, then

1) If $K\langle 0$, then a point on the surface is hyperbolic.

2) If $K = 0$, then a point on the surface is parabolic.

3) If $K\rangle 0$, then a point on the surface is elliptic.

Definition 7: If the curve α is the base curve of the ruled surface φ in three dimensional Lie group G, then the geodesic curvature, normal curvature and geodesic torsion with respesct to the ruled surface φ are computed as follows;

$$(15) \qquad\qquad\qquad \kappa_{g_\varphi} = \langle U \times T, D_T T \rangle$$

$$(16) \qquad\qquad\qquad \kappa_{n_\varphi} = \langle D_T T, U \rangle$$

and

$$(17) \qquad\qquad\qquad \tau_{g_\varphi} = \langle U \times D_T U, D_T T \rangle .$$

(For the formulas of κ_g, κ_n and τ_g in Euclidean space see [1]).

Remark 1 : Note that the curvatures and torsion of the curve α in equations (15), (16) and (17) are computed with respect to ruled surface φ and the geodesic torsion τ_G in equation (4) of α is given with respect to three dimensional Lie group G.

Definition 8: For a curve β which is lying on a surface in three dimensional Lie group G, the following statements are satisfied;

1) β is a geodesic curve if and only if the geodesic curvature of the curve with respect to the surface vanishes.

2) β is a asymptotic line if and only if the normal curvature of the curve with respect to the surface vanishes.

3) β is a principal line if and only if the geodesic torsion of the curve with respect to the surface vanishes.

Theorem 1: Let $\varphi(s,v) = \alpha(s) + vX(s)$ be a ruled surface in three dimensional Lie group G with unit left-invariant vector field X, $\alpha : \mathbb{R} \longrightarrow G$ be the base curve and $\{T,N,B,\kappa,\tau\}$ be the Frenet apparatus of α. The base curve α is always the striction curve of the ruled surface φ.

Proof. If we use the equation (3.2) and make the appropriate calculations , we find the striction curve as;

$$
\begin{aligned}
\overline{\alpha} &= \alpha - \frac{\langle \alpha', D_T X \rangle}{\|D_T X\|^2} X \\
&= \alpha - \frac{\langle T, \frac{1}{2}[T,X] \rangle}{\|[T,X]\|^2} X \\
&= \alpha - \frac{1}{2} \frac{\langle [T,T], X \rangle}{\|[T,X]\|^2} X \\
&= \alpha.
\end{aligned}
$$

$\square$

Theorem 2:Let $\varphi(s,v) = \alpha(s) + vX(s)$ be a ruled surface in three dimensional Lie group G with unit left-invariant vector field X, $\alpha : \mathbb{R} \longrightarrow G$ be the base curve and $\{T,N,B,\kappa,\tau\}$ be the Frenet apparatus of α .The distribution parameter, the Gaussian curvature and the mean curvature of φ are given respesctively as;

$$
\lambda = 2 \frac{\langle T \times X, [T,X] \rangle}{\|[T,X]\|^2}
$$

$$
K = -\frac{\langle T \times X, [T,X] \rangle^2}{4A^2 (1 + \frac{v^2}{4} \|[T,X]\|^2 - \langle T,X \rangle^2)}
$$

and

$$
H = \frac{\frac{1}{A}\left(-\kappa \langle B,X \rangle - \frac{v\kappa}{2} \langle N \times X, [T,X] \rangle + \frac{v\kappa}{2} \langle [N,X], T \times X, \rangle\right.}{2(1 + \frac{v^2}{4} \|[T,X]\|^2 - \langle T,X \rangle^2)}
$$

$$
\begin{aligned}
&\quad + \frac{v^2\kappa}{4} \langle [N,X], [T,X] \times X, \rangle + \frac{1}{2} \langle [T,[T,X]], T \times X \rangle \\
&\quad + \frac{v}{4} \langle [T,[T,X]], [T,X] \times X \rangle) - \frac{1}{A} \langle T,X \rangle \langle T \times X, [T,X] \rangle
\end{aligned}
$$

where $A = \|\varphi_s \times \varphi_v\|$.

Proof. If $\varphi(s,v) = \alpha(s) + vX(s)$ is a ruled surface in three dimensional Lie group G, then we can compute

$$E = 1 + \frac{v^2}{4}\|[T,X]\|^2 \quad , \quad F = \langle T, X \rangle \quad , \quad G = 1$$

$$e = \frac{1}{A}\begin{pmatrix} -\kappa\langle B,X\rangle - \frac{v\kappa}{2}\langle N\times X,[T,X]\rangle + \frac{v\kappa}{2}\langle[N,X],T\times X,\rangle \\ +\frac{v^2\kappa}{4}\langle[N,X],[T,X]\times X,\rangle + \frac{1}{2}\langle[T,[T,X]],T\times X\rangle \\ +\frac{v}{4}\langle[T,[T,X]],[T,X]\times X\rangle \end{pmatrix}$$

$$f = \frac{1}{2A}\langle T\times X,[T,X]\rangle \quad , \quad g = 0.$$

where $A = \|\varphi_s \times \varphi_v\|$. By using the equations (13) and (14), we easily find Gaussian and mean curvatures .

Also with the equations (3) and (11), distribution parameter is obtained directly. $\square$

Corollary 1: The ruled surface $\varphi(s,v) = \alpha(s) + vX(s)$ in three dimensional Lie group G is developable if and only if the vector fields $T \times X$ and $[T,X]$ are orthogonal. The ruled surface φ is minimal if and only if the following equation is satisfied;

$$-\kappa\langle B,X\rangle - \frac{v\kappa}{2}\langle N\times X,[T,X]\rangle + \frac{v\kappa}{2}\langle[N,X],T\times X,\rangle$$
$$+\frac{v^2\kappa}{4}\langle[N,X],[T,X]\times X,\rangle + \frac{1}{2}\langle[T,[T,X]],T\times X\rangle$$
$$+\frac{v}{4}\langle[T,[T,X]],[T,X]\times X\rangle$$
$$= \langle T,X\rangle\langle T\times X,[T,X]\rangle$$

Proof. By using the definition (5) and the distribution parameter, the mean curvature which are found in above theorem the results are apparent. $\square$

Remark 2 : Notice that if Φ is a ruled surface in Euclidean space, then $K \leq 0$ where K is the Gaussian curvature of Φ. Altough $K \leq 0$ for the ruled surface Φ in Euclidean space, it is not always true for a ruled surface in three dimensional Lie group .

Theorem 3:Let $\varphi(s,v) = \alpha(s) + vX(s)$ be a ruled surface in three dimensional Lie group G with unit left-invariant vector field X, $\alpha : \mathbb{R} \longrightarrow G$ be the base curve and $\{T,N,B,\kappa,\tau\}$ be the Frenet apparatus of α. The geodesic curvature, normal curvature and geodesic torsion of α with respect to ruled surface φ are given respectively as;

$$\kappa_{g\varphi} = \frac{\kappa}{A}(\langle X,N\rangle + v\tau_G\langle X,B\rangle)$$

$$\kappa_{n_\varphi} = \frac{\kappa}{A}(-\langle X,B\rangle + \frac{v}{2}\langle [T,X],X \times N\rangle)$$

and

$$\tau_{g_\varphi} = \langle X,N\rangle \, (\frac{\kappa}{A^2}(\kappa\langle X,B\rangle + \frac{v}{2}\langle T,L \times X\rangle)$$

$$+ \frac{v\kappa}{2A^2}(\kappa\langle [T,X],N \times X\rangle + \frac{v}{2}\langle [T,X],L \times X\rangle)$$

$$+ \frac{1}{2A^2}(\frac{v\kappa}{2}\langle [[T,X],T],T \times X\rangle + \frac{v^2\kappa}{4}\langle [[T,X],T],[T,X] \times X\rangle))$$

$$- \frac{v\kappa\tau_G}{A^2}\langle [T,X],T \times X\rangle \langle X,B\rangle$$

where $A = \|\varphi_s \times \varphi_v\|$, $\tau_G = \frac{1}{2}\langle [T,N],B\rangle$ and $L = \kappa[X,N] + \frac{1}{2}[T,[T,X]]$.

Proof. If the equation of ruled surface is $\varphi(s,v) = \alpha(s) + vX(s)$, then the unit normal vector field of φ is found as;

$$U = \frac{1}{A}(T \times X + \frac{v}{2}[T,X] \times X).$$

By using the equation (3), we have

$$D_T U = \dot{U} + \frac{1}{2}[T,U]$$

$$= \left(\frac{1}{A}\right)'(T \times X + \frac{v}{2}[T,X] \times X) + \frac{1}{A}(\kappa(N \times X) + \frac{1}{2}(T \times [T,X])$$

$$+ \frac{v}{2}((\kappa[N,X] + \frac{1}{2}[T,[T,X]]) \times X))$$

$$+ \frac{1}{2A}([T,T \times X] + \frac{v}{2}[T,[T,X] \times X]).$$

If we use the equations (6), (15), (16) , (17) and make the appropriate calculations, the proof is completed. $\square$

Corollary 2:If the director vector field X and the binormal vector field B are ortogonal and $\langle [T,X],X \times N\rangle = 0$ then the base curve α of φ is a asymptotic line.

Proof. If we consider the definition (8) and the normal curvature κ_{n_φ} given in theorem the result is clear. $\square$

Corollary 3:If the director vector field X is orthogonal to both the principal normal vector field N and the binormal vector field B, then the base curve α of φ is geodesic curve and principal line.

Proof. If the director vector field X is orthogonal to both the principal normal vector field N and the binormal vector field B, then

$$\langle X, N \rangle = 0 \quad \text{and} \quad \langle X, B \rangle = 0.$$

By using geodesic curvature and geodesic torsion given in the theorem, we get

$$\kappa_{g_\varphi} = 0 \quad \text{and} \quad \tau_{g_\varphi} = 0.$$

These equations denotes that α of φ is geodesic curve and principal line, by the definition (8).   □

Example : Let a ruled surface which is a cylinder in three dimensional Lie group G, is given with the equation

$$\varphi(t, v) = (\cos t, \sin t, 0) + v(0, 0, 1).$$

The Frenet vector fields of the base curve $\alpha(t) = (\cos t, \sin t, 0)$ are $T = (-\sin t, \cos t, 0)$, $N = (-\cos t, -\sin t, 0)$ and $B = (0, 0, 1)$.

Since the curve $\alpha(t) = (\cos t, \sin t, 0)$ is also a circle in $\mathbb{R}^3$, we can compute $\tau_G = \frac{1}{2} \langle [T, N], B \rangle = \frac{1}{2} \langle T \times N, B \rangle = \frac{1}{2}$. By the equations in (7), curvature and torsion of α are found as $\kappa = \left\| \dot{T} \right\| = 1$ and

$$\tau = \left\| \frac{dB}{ds} \right\| + \tau_G$$

$$= \|(0, 0, 0)\| + \frac{1}{2}$$

$$= \frac{1}{2}.$$

which also means $D_T B = \frac{1}{2} [T, B]$ with $[T, B] = (\cos t, \sin t, 0)$.

For the curvatures we find the following expressions

$$\langle X, N \rangle = 0 \ , \ \langle X, B \rangle = 1 \ , \ \langle T, X \rangle = 0 \ , \quad A = 1$$

$$[T, X] = T \times X = (\cos t, \sin t, 0) \ , \quad [T, [T, X]] = (0, 0, 1)$$

$$[N, X] = (-\sin t, \cos t, 0) \ , \quad [T, X] \times X = (\sin t, -\cos t, 0).$$

Now by using the expressions above, the distribution parameter, Gaussian curvature and mean curvature are obtained as follows;

$$\lambda = 2$$

$$K = -\frac{1}{v^2 + 4}$$

$$H = -\frac{v^2 + 2}{v^2 + 4}.$$

Also the geodesic curvature, normal curvature and geodesic torsion of α with respect to the cylinder are

$$\kappa_{g_\varphi} = \frac{v}{2}$$

$$\kappa_{n_\varphi} = -1$$

$$\tau_{g_\varphi} = -\frac{v}{2}.$$

Remark 3 : Notice that a cylinder in in Euclidean space is developable but a cylinder in three dimensional Lie group G is not developable.

4. SOME SPECIAL RULED SURFACES IN THREE DIMENSIONAL LIE GROUPS

In this section, we will identify some special ruled surfaces which are existed in Euclidean space. For details of these surfaces see [7, 9, 10].

Definition 9: Let G be the three dimensional Lie group with bi-invariant metric and $\alpha : \mathbb{R} \longrightarrow G$ be a parametrized curve with the Frenet apparatus $\{T, N, B, \kappa, \tau\}$, the modified Darboux vector field $W = \frac{1}{\sqrt{\kappa^2 + \tau^2}}(\tau T + \kappa B)$, $\alpha' = T$, $\kappa \neq 0$ and $\tau_G = \frac{1}{2}\langle [T, N], B \rangle$. Some types of ruled surfaces in three dimensional Lie group G are defined and given with their equations as follows;

1) Tangent developable surface ; $\varphi(s, v) = \alpha(s) + vT(s)$

2) Normal surface; $\varphi(s, v) = \alpha(s) + vN(s)$

3) Binormal surface; $\varphi(s, v) = \alpha(s) + vB(s)$

4) Darboux developable surface; $\varphi(s, v) = B(s) + vT(s)$

5) Rectifying surface; $\varphi(s, v) = \alpha(s) + vW(s)$.

Remark 4 : If $\tau_G = \tau$, then the binormal vector field B is left-invariant. We will take $\tau_G \neq \tau$ in our calculations without loss of generality.

Theorem 4: Let $\varphi(s,v) = \alpha(s) + vT(s)$ be a tangent developable surface in three dimensional Lie group G. The distribution parameter, Gaussian curvature and mean curvature of the surface φ are given by

$$\lambda = 0$$

$$K = 0$$

$$H = -\frac{\tau - \tau_G}{2v^2 \kappa}$$

and the geodesic curvature, the normal curvature, the geodesic torsion of α with respect to tangent developable surface are

$$\kappa_{g_\varphi} = -\kappa$$

$$\kappa_{n_\varphi} = 0$$

$$\tau_{g_\varphi} = 0.$$

Proof. For the tangent developable surface , the following expressions are computed as; $E = 1 + v^2 \kappa^2$, $F = 1$, $G = 1$, $e = -\kappa(\tau - \tau_G)$, $f = 0$, $g = 0$ and the normal vector field of the surface by the equation (12) is $U = -B$. By using the equations (11), (13), (14), (15), (16) and (17) the results are obtained clearly.   □

Corollary 4: The tangent developable surface in three dimensional Lie group G is developable and it is not minimal. A point on this surface is parabolic. The base curve α on the surface is asymptotic and principal line but it is not geodesic curve.

Proof. Since the distribution parameter of tangent developable surface is zero, then it is developable. If we pay attention to the equations in (6), the mean curvature can not be zero because of $\tau_G \neq \tau$. Also by definition (6), a point on the surface is parabolic.

By thinking the definition (8) and since $\kappa \neq 0$, the base curve α is asymptotic and principal line and not geodesic curve.   □

Theorem 5:Let $\varphi(s,v) = \alpha(s) + vN(s)$ be a normal surface in three dimensional Lie group G. The distribution parameter, the Gaussian curvature and the mean curvature of the surface φ are given by

$$\lambda = \frac{\tau - \tau_G}{\kappa^2 + (\tau - \tau_G)^2}$$

$$K = -\left(\frac{\tau - \tau_G}{A^2}\right)^2$$

$$H = -\frac{v(\tau - \tau_G)(1 - v\kappa + v\kappa')}{2A^3}$$

and the geodesic curvature, the normal curvature, the geodesic torsion of α with respect to normal surface are

$$\kappa_{g_\varphi} = \frac{\kappa(1 - v\kappa)}{A}$$

$$\kappa_{n_\varphi} = 0$$

$$\tau_{g_\varphi} = \kappa\left(\frac{v(\tau - \tau_G)}{A} \cdot \left(\frac{1 - v\kappa}{A}\right)' - \frac{1 - v\kappa}{A} \cdot \left(\frac{v(\tau - \tau_G)}{A}\right)'\right)$$

where $A = \sqrt{v^2(\tau - \tau_G)^2 + (1 - v\kappa)^2}$.

Proof. For the normal surface , the following expressions are computed as; $E = v^2(\tau - \tau_G)^2 + (1 - v\kappa)^2$, $F = 0$, $G = 1$, $e = \frac{v(\tau - \tau_G)(1 - v\kappa + v\kappa')}{A}$, $f = \frac{\tau - \tau_G}{A}$, $g = 0$ and the normal vector field of the surface by the equation (12) is found as;

$$U = \frac{1}{A}(-v(\tau - \tau_G)T + (1 - v\kappa)B).$$

By using the equations (11), (13), (14), (15), (16) and (17) the results are obtained clearly. □

Corollary 5: The normal surface in three dimensional Lie group G is developable is not developable. It is minimal if and only if the equation $v\kappa - v\kappa' = 1$ is satisfied. A point on this surface is hyperbolic. The base curve α on the surface is asymptotic line. α is geodesic curve if and only if $v\kappa = 1$ and it is principal line if and only if

$$\frac{v(\tau - \tau_G)}{A} \cdot \left(\frac{1 - v\kappa}{A}\right)' = \frac{1 - v\kappa}{A} \cdot \left(\frac{v(\tau - \tau_G)}{A}\right)'.$$

Proof. Since $\tau_G \neq \tau$, then the distribution parameter can not be zero, so the normal surface is not developable. By the mean curvature found in theorem and the definition (5) and since $v \neq 0$, $\tau_G \neq \tau$, the surface is minimal with the satisfied equation $1 - v\kappa + v\kappa' = 0$. Also by definition (6), a point on the surface is hyperbolic.

By deciding the definition (8) and $\kappa \neq 0$, since $\kappa_{n_\varphi} = 0$, then α is asymptotic line. $v\kappa = 1$ if and only if $\kappa_{g_\varphi} = 0$,so α is geodesic curve. $\tau_{g_\varphi} = 0$ if and only if

$$\frac{v(\tau - \tau_G)}{A} \cdot \left(\frac{1 - v\kappa}{A} \right)' = \frac{1 - v\kappa}{A} \cdot \left(\frac{v(\tau - \tau_G)}{A} \right)'.$$

So α is principal line with the satisfied equation above.　　　□

Theorem 6: Let $\varphi(s, v) = \alpha(s) + vB(s)$ be a binormal surface in three dimensional Lie group G . The distribution parameter, the Gaussian curvature and the mean curvature of the surface φ are given by

$$\lambda = \frac{1}{\tau - \tau_G}$$

$$K = -\left(\frac{\tau - \tau_G}{A^2} \right)^2$$

$$H = -\frac{-v^2\kappa(\tau - \tau_G) + v\tau' - \kappa}{2A^3}$$

and the geodesic curvature, the normal curvature, the geodesic torsion of α with respect to binormal surface are

$$\kappa_{g_\varphi} = \frac{\kappa}{A}$$

$$\kappa_{n_\varphi} = -\frac{\kappa}{A}$$

$$\tau_{g_\varphi} = \frac{v\kappa(\tau - \tau_G)(A(\tau - \tau_G) + \tau_G)}{A^2}$$

where $A = \sqrt{1 + v^2(\tau - \tau_G)^2}$.

Proof. For the binormal surface given, the following expressions are computed as; $E = 1 + v^2(\tau - \tau_G)^2$, $F = 0$, $G = 1$, $e = \frac{-v^2\kappa(\tau - \tau_G) + v\tau' - \kappa}{A}$, $f = \frac{\tau - \tau_G}{A}$, $g = 0$ and the normal vector field of the surface by the equation (12) is found as;

$$U = -\frac{1}{A}(v(\tau - \tau_G)T + N).$$

By using the equations (11), (13), (14), (15), (16) and (17) the results are obtained clearly.   □

Corollary 6: The binormal surface in three dimensional Lie group G is not developable. It is minimal if and only if the equation $v^2\kappa(\tau - \tau_G) = v\tau' - \kappa$ is satisfied. A point on this surface is hyperbolic. The base curve α on the surface is not geodesic curve and asymptotic line. α is principal line if and only if

$$\frac{\tau_G^2 - v^2(\tau - \tau_G)^4}{(\tau - \tau_G)^2} = 1.$$

Proof. Since $\lambda \neq 0$, the binormal surface is not developable. By using the definition (5), the surface is minimal with the satisfied equation $v^2\kappa(\tau - \tau_G) = v\tau' - \kappa$. Also by definition (6), a point on the surface is hyperbolic.

Since $\kappa \neq 0$, then $\kappa_{g\varphi} \neq 0$ and $\kappa_{n\varphi} \neq 0$. Also since $v \neq 0$, $\kappa \neq 0$ and $\tau_G \neq \tau$, then $\tau_{g\varphi} = 0$ if and only if $A(\tau - \tau_G) = -\tau_G$. If we make necessary calculations in the equation $A(\tau - \tau_G) = -\tau_G$, then we get $\frac{\tau_G^2 - v^2(\tau - \tau_G)^4}{(\tau - \tau_G)^2} = 1$.   □

Theorem 7: Let $\varphi(s, v) = B(s) + vT(s)$ be a Darboux developable surface in three dimensional Lie group G. The distribution parameter, the Gaussian curvature and the mean curvature of the surface φ are given by

$$\lambda = 0$$

$$K = 0$$

$$H = \frac{1}{2(\tau - \tau_G - v\kappa)}$$

and the geodesic curvature, the normal curvature, the geodesic torsion of α with respect to Darboux developable surface are

$$\kappa_{g\varphi} = \kappa$$

$$\kappa_{n\varphi} = 0$$

$$\tau_{g\varphi} = 0$$

Proof. For the Darboux developable surface, the following expressions are computed as; $E = (v\kappa - (\tau - \tau_G))^2$, $F = 0$, $G = 1$, $e = v\kappa - (\tau - \tau_G)$, $f = 0$, $g = 0$ and the normal vector

field of the surface by the equation (12) is found as;

$$U = B.$$

By using the equations (11), (13), (14), (15), (16) and (17) the results are obtained clearly.   □

Corollary 7: The Darboux developable surface in three dimensional Lie group G is developable. It is not minimal . A point on this surface is parabolic. The base curve α on the surface is asymptotic line and principal line but it is not geodesic curve.

Proof. By the definition (5) and since the distribution parameter $\lambda = 0$, the Darboux developable surface is developable. Since the mean curvature cannot be zero, ,it is not minimal. Also by definition (6), a point on the surface is parabolic.

If we use the definition (8) and since $\kappa \neq 0$, $\kappa_{g_\varphi} \neq 0$, then the base curve α is not geodesic curve. Also α is asymptotic line and principal line because of $\kappa_{n_\varphi} = 0$ and $\tau_{g_\varphi} = 0$.   □

Theorem 8: Let $\varphi(s,v) = \alpha(s) + vW(s)$ be a rectifying surface in three dimensional Lie group G . The distribution parameter, the Gaussian curvature and the mean curvature of the surface φ are given by

$$\lambda = -\frac{c^2 \kappa^2 \tau_G}{((c\tau)')^2 + ((c\kappa)')^2 + c^2 \kappa^2 \tau_G^2}$$

$$K = -\frac{1}{A^2} \cdot \frac{\left(c^2 \kappa^2 \tau_G (1 + v((c\kappa)' - (c\tau)'))\right)^2}{(1 + v(c\kappa)')^2 + v^2(c\kappa)'^2 + (vc\kappa\tau_G)^2 - (1 + vc(c\kappa)'(\kappa + \tau)}$$

$$H = \frac{1}{A} \cdot \frac{\begin{matrix} v^2 c^2 \kappa \tau_G (c\kappa)''(\kappa + \tau) - v^2 c(\kappa - \tau)^2 (c\kappa)'^2 + 2v^2 c\tau_G (c\kappa)'^2 (\kappa + \tau) \\ + 2vc\kappa(c\kappa)'(\tau - \kappa + \tau_G) - c\kappa^2 + v^2 c^3 \kappa^2 \tau_G^2 (\tau^2 - \kappa^2 - \tau\tau_G) \\ -2(c\tau + vc(c\kappa)'(\kappa + \tau))(vc^2 \kappa^2 \tau_G(-(c\kappa)' + (c\tau)') - c^2 \kappa^2 \tau_G) \end{matrix}}{2((1 + v(c\kappa)')^2 + (v(c\kappa)')^2 + v^2 c^2 \kappa^2 \tau_G^2 - (c\tau + vc(c\kappa)'(\kappa + \tau))^2)}$$

and the geodesic curvature, the normal curvature, the geodesic torsion of α with respect to rectifying surface are

$$\kappa_{g\varphi} = -\frac{vc^2\kappa^2\tau\tau_G}{A}$$

$$\kappa_{n\varphi} = \frac{\kappa(vc(c\kappa)'(\tau-\kappa)-c\kappa)}{A}$$

$$\tau_{g\varphi} = \frac{\kappa}{A} \cdot \left(\begin{array}{c} vc^2\kappa\tau\tau_G\left(-\left(\frac{vc^2\kappa^2\tau_G}{A}\right)'+\frac{vc\kappa(c\kappa)'(\tau-\kappa)-c\kappa^2}{A}\right) \\ +vc^2\kappa^2\tau_G\left(\frac{(vc(c\kappa)'(\tau-\kappa)-c\kappa)(\tau-2\tau_G)}{A}-\left(\frac{vc^2\kappa\tau\tau_G}{A}\right)'\right) \end{array} \right)$$

where $c = \frac{1}{\sqrt{\kappa^2+\tau^2}}$ and $A = \sqrt{v^2c^4\kappa^2\tau_G^2(\kappa^2+\tau^2)+(vc(c\kappa)'(\tau-\kappa)-c\kappa)^2}$.

Proof. For the rectifying surface, the following expressions are computed as;

$$E = (1+v(c\kappa)')^2+(v(c\kappa)')^2+(vc\kappa\tau_G)^2$$

$$F = c\tau+vc(c\kappa)'(\kappa+\tau)$$

$$G = 1$$

and

$$e = \begin{array}{c} v^2c^2\kappa\tau_G(c\kappa)''(\kappa+\tau)-v^2c(\kappa-\tau)^2(c\kappa)'^2+2v^2c\tau_G(c\kappa)'^2(\kappa+\tau) \\ +2vc\kappa(c\kappa)'(\tau-\kappa+\tau_G)-c\kappa^2+v^2c^3\kappa^2\tau_G^2(\tau^2-\kappa^2-\tau\tau_G) \end{array}$$

$$f = -c^2\kappa^2\tau_G(1+v((c\kappa)'-(c\tau)'))$$

$$g = 0.$$

The normal vector field of the surface by the equation (12) is found as;

$$U = \frac{1}{A} \left(\begin{array}{c} vc^2\kappa^2\tau_G T-(vc(c\kappa)'(\tau-\kappa)-c\kappa)N \\ -vc^2\kappa\tau\tau_G B \end{array} \right).$$

By using the equations (11), (13), (14), (15), (16), (17) and making necessary calculations and simplifications ,the results are obtained clearly. $\square$

Corollary 8: If G is Abelian, then the rectifying surface in three dimensional Lie group G is developable and a point on this surface is parabolic. If G is Abelian or the base curve α on the

surface is plane curve , then α is geodesic curve. α is asymptotic line and principal line if and only if the following equations are satisfied respectively;

$$vc(c\kappa)'(\tau - \kappa) = c\kappa$$

and

$$vc^2\kappa\tau\tau_G\left(-\left(\frac{vc^2\kappa^2\tau_G}{A}\right)' + \frac{vc\kappa(c\kappa)'(\tau-\kappa)-c\kappa^2}{A}\right) \\ +vc^2\kappa^2\tau_G\left(\frac{(vc(c\kappa)'(\tau-\kappa)-c\kappa)(\tau-2\tau_G)}{A} - \left(\frac{vc^2\kappa\tau\tau_G}{A}\right)'\right) = 0.$$

Proof. If G is Abelian, then $\tau_G = 0$(see[4]). Since $\tau_G = 0$, $c \neq 0$ and $\kappa \neq 0$, the distribution parameter is zero. So the surface is developable. Also if $\tau_G = 0$, then the Gaussian curvature is zero, this means that a point on this surface is parabolic.

If G is Abelian or $\tau = 0$, and $v \neq 0, c \neq 0$ and $\kappa \neq 0$,then geodesic curvature is zero. The normal curvature and geodesic torsion are zero if and only if the equations in corollary are satisfied. $\square$

Remark 5: Altough a rectifying surface with the equation $\varphi(s,v) = \alpha(s) + vW(s)$ in three dimensional Lie group G is not developable, it is developable in Euclidean space.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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