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# PARAMETERS OF A CLASS OF COMPLEX CONTINUOUS WAVELET FAMILY

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**Abstract.** In this paper, we have established the relationship between the wavelet scale and equivalent Fourier wavelength for the family of the new complex continuous wavelet. The same relation is used to relate the maximum of the Scalogram of each coefficient obtained from the wavelet transform of the signal to its amplitude. The relationship was applied to a synthetic signal and the simulated results show the effectiveness of using the proper wavelet transform definition to this proposed family of wavelets.

**Keywords:** proposed complex continuous wavelet family; wavelet transform; scalogram; wavelet scale and fourier wavelength.

**2010 AMS Subject Classification:** 42C40, 65T60.

## INTRODUCTION

In signal processing, wavelet analysis is an effective tool for analyzing localized variations of power within a time series by breaking down it into time-frequency space [1, 5, 6, 9, 10]. Meyers et al. [7] derived the relationship between the equivalent Fourier wavelength and the wavelet scale for Morlet wavelets. In this paper, we have established the relation for the proposed family of complex continuous wavelets.

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In the first section, we discussed the new family of the complex continuous wavelet (NCW), which we proposed recently [8]. This new family of wavelets possess lesser time-width and also time-bandwidth product as compared with the existing wavelets [2, 8]. In the second section, the wavelet transformation of the proposed complex continuous wavelet family with different definitions of time-frequency atoms is expressed. In the next section, we have established the relationship between the wavelet scale and Fourier wavelength of the proposed family and their conversion of scale to the frequency. At last, we formulated the relationship between the amplitude of the corresponding periodic components of the signal and its scalogram computed using the members of the proposed family of complex continuous wavelets with a different form of wavelet transform definitions and simulated the results by using a synthetic signal.

## 1. PROPOSED COMPLEX CONTINUOUS WAVELET (NCW) FAMILY

This proposed family of wavelet was derived from the kernel of cauchy's distribution function $\phi(x) = e^{-ix} \cdot \frac{1}{1+x^2}$. The successive derivatives of $\phi(x)$ gives rise to the mother wavelet function $\xi^k(x)$, where $k$ is the $k^{th}$ derivative of $\phi(x)$. i.e.

(1) $$\xi^k(x) = \frac{(-1)^j}{\sqrt{C_k}} \frac{d^k}{dx^k} \phi(x)$$

where

$$j = \begin{cases} \frac{k}{2} & k \text{ is even} \\ \frac{k+1}{2} & k \text{ is odd} \end{cases}$$

Here $j$ is selected in order to equalize positive maxima and its absolute maximum value, where $C_k$ is the normalization constant such that $\left\|\xi^k(x)\right\|^2 = 1$. So that,the value of

(2) $$C_k = \int_{-\infty}^{\infty} \left| \frac{d^k}{dx^k} \phi(x) \right|^2 dx$$

Table 1 provides the normalization constants for $k = 1, 2, 3, 4, 5$.

Figure 1 and Figure 2 are real and imaginary parts of the fourth member of an NCW family respectively.

The Fourier transform of $\phi(x)$ is defined as

(3) $$\mathfrak{F}(\phi(x)) = \int_{-\infty}^{\infty} \phi(x) \mathrm{e}^{-i\omega x} dx = \pi \mathrm{e}^{-|\omega+1|}$$

TABLE 1. The normalization constant $C_k$

| $k$ | $C_k$ |
|---|---|
| 1 | $\frac{3\pi}{4}$ |
| 2 | $\frac{11\pi}{4}$ |
| 3 | $\frac{169\pi}{8}$ |
| 4 | $\frac{1185\pi}{4}$ |
| 5 | $\frac{53329\pi}{8}$ |

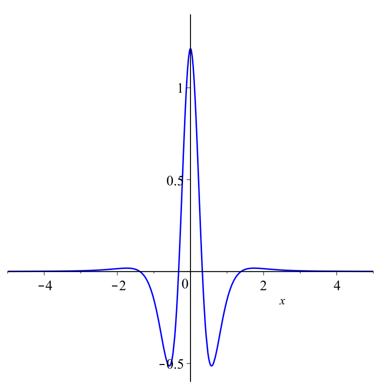


FIGURE 1. Real part of $\xi^4(x)$

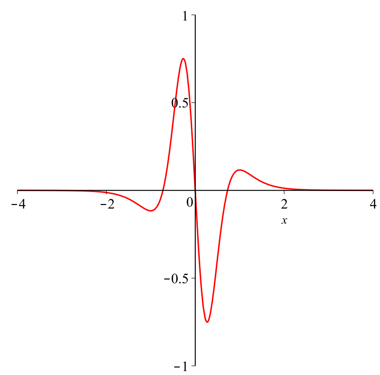


FIGURE 2. Imaginary part of $\xi^4(x)$

Hence we obtain, the fourier transform of the mother wavelet $\xi^k(x)$ as

$$\mathfrak{F}(\xi^k(x)) = \hat{\xi}^k(\omega) = \frac{(-1)^j\pi}{\sqrt{C_k}}(i\omega)^k e^{-|\omega+1|} \tag{4}$$

## 2. WAVELET TRANSFORMATION

The wavelet transform of a signal $s(x)$ using the time-frequency atoms $\xi^k_{a,b}(x)$ is defined as [1, 4, 6, 9]

$$W_s(a,b) = \int_{-\infty}^{\infty} s(x)\xi^{k*}_{a,b}(x)dx = \int_{-\infty}^{\infty} s(x)\frac{1}{\sqrt{a}}\xi^{k*}(\frac{x-b}{a})dx \tag{5}$$

The family of time-frequency atoms $\xi^k_{a,b}(x)$ are constructed by translating in time parameter $b$ and dilation with scale $a$. i.e.

$$\xi^k_{a,b}(x) = \frac{1}{\sqrt{a}}\xi^k(\frac{x-b}{a}) \tag{6}$$

where $a^{-1/2}$ is a normalization factor, which gives the mother wavelets at every scale same energy. Sometimes we use $a^{-1}$ instead of $a^{-1/2}$ [1, 9] in order to get accurate amplitude and frequency of a signal regardless of its sampling frequency, which we will discuss in the following sections. So the equation(6) is modified as

$$\xi^k_{a,b}(x) = a^{\alpha}\xi^k(\frac{x-b}{a}) \tag{7}$$

with $\alpha = -\frac{1}{2}$ or $-1$.
The fourier transform of time-frequency atoms $\xi^k_{a,b}(x)$ are given by

$$\mathfrak{F}(\xi^k_{a,b}(x)) = \hat{\xi}^k_{a,b}(\omega) = \int_{-\infty}^{\infty}\xi^k_{a,b}(x)\mathrm{e}^{-i\omega x}dx = a^{\alpha+1}\hat{\xi}^k(a\omega)\mathrm{e}^{-i\omega b} \tag{8}$$

By using Fourier-Parseval formula, the equation(5) converts in frequency domain as

$$W_s(a,b) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{s}(\omega)\hat{\xi}^{k*}_{a,b}(\omega)d\omega \tag{9}$$

Hence the equation(4) and equation(8), yields

$$W_s(a,b) = \frac{(-1)^j}{2\sqrt{C_k}}\int_{-\infty}^{\infty}\hat{s}(\omega)a^{\alpha+1}(ia\omega)^k\mathrm{e}^{-|a\omega+1|}\mathrm{e}^{i\omega b}d\omega \tag{10}$$

## 3. Relationship of Scale Parameter of NCW Family

**3.1. Relation between scale and Fourier wavelength.** The analytical relation between the wavelet scale and Fourier wavelength can be obtained by substituting a wave function of known frequency and calculating the scale at which the scalogram (or wavelet energy density function) reaches its maximum [4, 5, 7, 9, 10] . Let us take the cosine function ($s(x) = cos(\omega_0 x)$) of unit amplitude and angular frequency $\omega_0$, its Fourier transform is

$$\mathfrak{F}\{s(x)\} = \hat{s}(\omega) = \pi\{\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\} \tag{11}$$

Using equation(10), its wavelet transform is

$$W_s(a,b) = \frac{(-1)^j \pi}{2\sqrt{C_k}} a^{\alpha+1} (ia\omega_0)^k e^{-|a\omega_0+1|} e^{i\omega_0 b} \tag{12}$$

Its scalogram [6] or wavelet power spectrum [9] is given by

$$P_W s(a,b) = |W_s(a,b)|^2 = \frac{\pi^2}{4C_k} a^{2(\alpha+1)} (a\omega_0)^{2k} e^{-2|a\omega_0+1|} \tag{13}$$

The maximum of scalogram occurs at $\frac{\partial}{\partial a}|W_s(a,b)|^2 = 0$, which gives

$$a = \frac{2k+1}{2\omega_0} \quad or \quad \omega_0 = \frac{2k+1}{2a} \quad if \quad \alpha = -1/2 \tag{14}$$

and

$$a = \frac{k}{\omega_0} \quad or \quad \omega_0 = \frac{k}{a} \quad if \quad \alpha = -1. \tag{15}$$

(Note: Here $|a\omega_0 + 1| = a\omega_0 + 1$, because $a > 0$ and frequency $\omega_0$ is positive.)

On substituting $\omega_0 = \frac{2\pi}{\lambda_k}$ in equation(14) and equation(15), the Fourier wavelength

$$\lambda_k = \begin{cases} \frac{4\pi a}{2k+1} & \text{if} \quad \alpha = -\frac{1}{2} \\ \frac{2\pi a}{k} & \text{if} \quad \alpha = -1 \end{cases} \tag{16}$$

Hence we get a linear relationship between wavelet scale and equivalent Fourier wavelength for NCW family. Table 2 shows the wavelet scale and corresponding Fourier wavelength of an NCW Family.

TABLE 2. Wavelet scale and Fourier wavelength of an NCW Family.

| | $Scale\ parameter(a)$ | $Fourier\ Wavelength(\lambda_k)$ |
|---|---|---|
| $\alpha = -\frac{1}{2}$ | $\frac{2k+1}{2\omega_0}$ | $\frac{4\pi a}{2k+1}$ |
| $\alpha = -1$ | $\frac{k}{\omega_0}$ | $\frac{2\pi a}{k}$ |

**3.2. Conversion of scale to frequency.** The scale value $a$ can be converted into frequency (pseudo-frequency) $f$, the value of which depends on the central frequency $f_c$ of the applied mother wavelets and the scale value $a$ and is given by [1, 3]

$$f = \frac{f_c}{\delta t \cdot a} \tag{17}$$

where $\delta t$ is a sampling frequency of the analyzing signal. The central frequency $f_c$ of a mother wavelet is inverse of the Fourier wavelength $\lambda_k$ (the archetypal wavelet at scale $a = 1$ and location $b = 0$) of that mother wavelet. i.e.

$$f_c = \frac{1}{\lambda_k}\bigg|_{a=1} = \begin{cases} \frac{2k+1}{4\pi} & \text{if} \quad \alpha = -\frac{1}{2} \\ \frac{k}{2\pi} & \text{if} \quad \alpha = -1 \end{cases} \tag{18}$$

## 4. The Analysis of Scalogram of a Periodic Signal with an NCW Family

Most of the real-life signals can be expressed as a finite Fourier series, since the length of data and the sampling frequency are finite. Without loss of generality, we can consider the periodic items of a time series $\{s(x_i)\}_{i=1}^{T}$ with length $T$ as

$$s(x) = c_0 + \sum_{n=1}^{N} \{c_n cos(\omega_n x) + d_n sin(\omega_n x)\} \tag{19}$$

where $\omega_n = \frac{n \cdot 2\pi}{T}, n = 1,2,...,N$, and $N$ is determined by the sampling period. Applying the wavelet transform to $s(x)$ , we have

$$W_s(a,b) = \sum_{n=1}^{N} \{c_n \int_{-\infty}^{\infty} cos(\omega_n x) \xi_{a,b}^{k*}(x) dx + d_n \int_{-\infty}^{\infty} sin(\omega_n x) \xi_{a,b}^{k*}(x) dx\}.$$

Hence, there exists a relation between the periodic components and amplitude of the signal with the scalogram of coefficients obtained from the wavelet transform of sine and cosine functions.

The following theorem gives that relation between the amplitude of the sine and cosine functions with its scalogram.

**Theorem 4.1**. The sine and cosine function $s(x) = A \cdot cos(\omega_n x)$ and $r(x) = A \cdot sin(\omega_n x)$ produces the same scalogram and the maximum scalogram which occurs at $\omega = \omega_n$ and for any $b$ is

$$M_1 \cdot \frac{A^2}{\delta t \cdot \omega_n} \quad if \quad \alpha = -\frac{1}{2} \tag{20}$$

and

(21) $$M_2 \cdot A^2 \quad if \quad \alpha = -1$$

where

(22) $$M_1 = \frac{\pi^2}{4C_k}(\frac{2k+1}{2})^{2k+1} e^{-|2k+3|}, \quad M_2 = \frac{\pi^2}{4C_k}(k)^{2k} e^{-2|k+1|}$$

is constant for each member of the family.

**Proof.** **Case I**: If $\alpha = -\frac{1}{2}$. Without loss of generality, we assume that unit amplitude, i.e $A = 1$. Then according to equation(13), its scalogram are

$$P_W s(a,b) = |W_s(a,b)|^2 = \frac{\pi^2}{4C_k}\omega_n^{2k} a^{2k+1} \mathrm{e}^{-2|a\omega_n+1|}$$

and

$$P_W r(a,b) = |W_r(a,b)|^2 = \frac{\pi^2}{4C_k}\omega_n^{2k} a^{2k+1} \mathrm{e}^{-2|a\omega_n+1|}$$

Therefore

$$P_W s(a,b) = P_W r(a,b)$$

The maximum of $P_W s(a,b)$ obtained when $\frac{\partial}{\partial a} P_W s(a,b) = 0$, which gives

$$a = \frac{2k+1}{2\omega_n}$$

Then the maximum of $P_W s(a,b)$ and $P_W r(a,b)$ are $M_1 \cdot \frac{A^2}{\delta t \cdot \omega_n}$, where $M_1$ is defined in equation(22).

**Case II**: If $\alpha = -1$. By proceeding as above, we get a maximum of $P_W s(a,b)$ at $\frac{\partial}{\partial a} P_W s(a,b) = 0$, which gives

$$a = \frac{k}{\omega_n}$$

Then the maximum of $P_W s(a,b)$ and $P_W r(a,b)$ are $M_2 \cdot A^2$, where $M_2$ is defined in equation(22).

It can be observed from the above theorem, that the maximum scalogram depends on the sampling frequency and angular frequency of the signal, if $\alpha = -\frac{1}{2}$, and is independent in case of $\alpha = -1$.

This helps us in determining more accurately the amplitude of the analyzing signal with $\alpha = -1$, which we can see from the following example.

Let us generate a non-stationary cosine function in the interval $[0, 2.56]$ of length 1024 with sampling frequency $400Hz$ as shown in Figure 3 and is defined as follows

$$s(x) = \begin{cases} 0.4\cos(2\pi 30x); & 0 < x \leq 1, \\ 0.6\cos(2\pi 45x); & 1 < x \leq 2, \\ 0.8\cos(2\pi 60x); & otherwise. \end{cases}$$

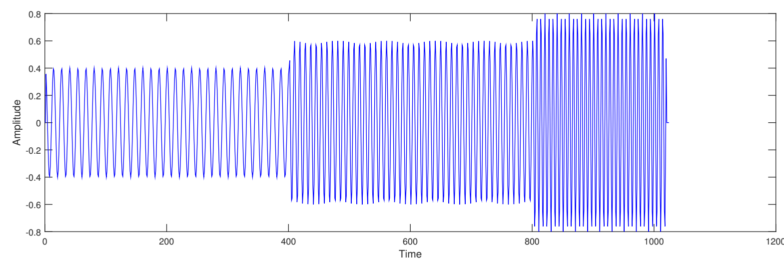


FIGURE 3. A Signal $s(x)$.

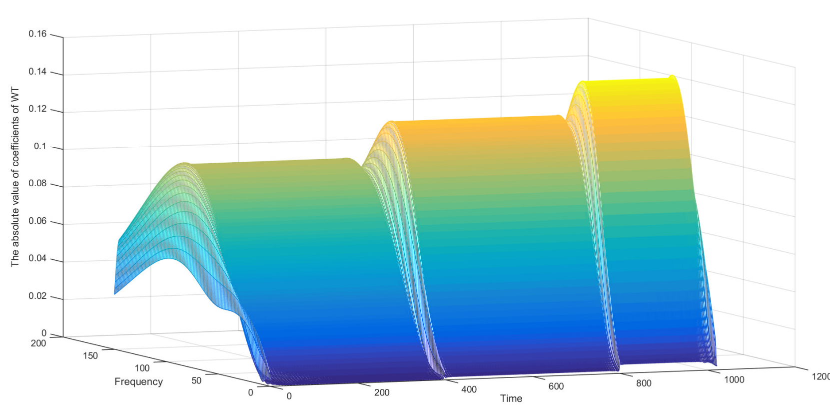


FIGURE 4. The absolute values of coefficients of NCW Family ($k = 4$) at $\alpha = -\frac{1}{2}$.

Figure 4 and Figure 5 shows the plot of absolute value of the coefficients obtain from the wavelet transform of $s(x)$ with $\alpha = -\frac{1}{2}$ and $\alpha = -1$ respectively using fourth member of the family($\xi^4_{a,b}(x)$).

It can be clearly seen from both the figures, the absolute values of the coefficients obtained using any member of the family are consistent according to the amplitude of the corresponding frequency components, which is in general not true (complex morlet wavelet [10]). These figures show three parts each of which is constant in absolute maxima within a particular interval. The point of maxima (scale) gives the corresponding frequency of the analyzing signal (see Table 5 and Table 6).

TABLE 3. Comparison of the original amplitude and the amplitude with respect to NCW family at $\alpha = -\frac{1}{2}$.

| Time | The Original Amplitude | *The Amplitude w.r.t. NCW Family* | | | |
|---|---|---|---|---|---|
| | | $k=1$ | $k=2$ | $k=3$ | $k=4$ |
| $0 < x \leq 1$ | 0.4 | 0.4025 | 0.4007 | 0.4011 | 0.4020 |
| $1 < x \leq 2$ | 0.6 | 0.6015 | 0.6014 | 0.6014 | 0.6017 |
| *otherwise* | 0.8 | 0.8027 | 0.8017 | 0.8011 | 0.8014 |

Table 3 and Table 4 shows the calculation of the amplitude of the synthetic signal from its wavelet transform computed using the first four members of the proposed family. i.e. $\xi^k_{a,b}(x)$ $(k = 1,2,3,4)$ and by using the relation obtained in Theorem 4.1. We are using FFT based algorithm for calculations [11, 13].

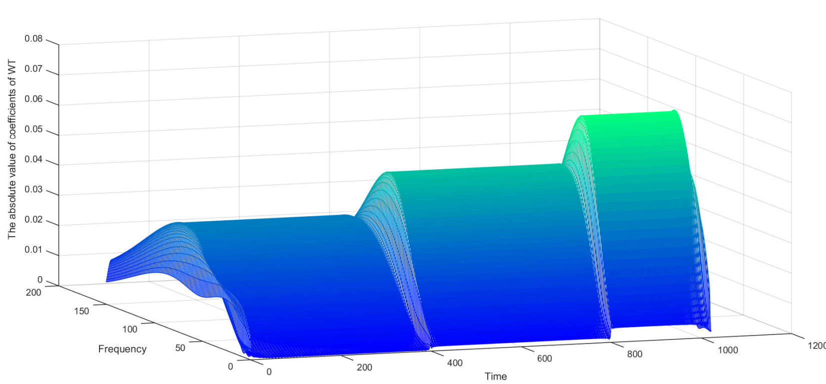


FIGURE 5. The absolute values of coefficients of NCW Family ($k = 4$) at $\alpha = -1$.

TABLE 4. Comparison of the original amplitude and the amplitude with respect to NCW family at $\alpha = -1$.

| Time | The Original Amplitude | *The Amplitude w.r.t. NCW Family* | | | |
|---|---|---|---|---|---|
| | | $k=1$ | $k=2$ | $k=3$ | $k=4$ |
| $0 < x \leq 1$ | 0.4 | 0.4014 | 0.4002 | 0.4005 | 0.4008 |
| $1 < x \leq 2$ | 0.6 | 0.6014 | 0.6003 | 0.6008 | 0.6001 |
| *otherwise* | 0.8 | 0.8028 | 0.8014 | 0.8001 | 0.8005 |

On comparing Table 3 and Table 4, it can be seen that the value of the amplitude obtained is close to the true value in case of $\alpha = -1$ and a little deviated in case of $\alpha = -\frac{1}{2}$. This is due to the fact that, for $\alpha = -1$, the scalogram is independent of sampling frequency and angular frequency of the analyzing signal. The above fact is also true in the case of frequency estimation (see Table 5 and Table 6).

TABLE 5. Comparison of the original frequency and the frequency with respect to NCW family at $\alpha = -\frac{1}{2}$.

| Time | The Original Frequency | *The Frequency w.r.t. NCW Family* | | | |
|---|---|---|---|---|---|
| | | $k=1$ | $k=2$ | $k=3$ | $k=4$ |
| $0 < x \leq 1$ | 30 | 29.84 | 29.91 | 29.98 | 29.89 |
| $1 < x \leq 2$ | 45 | 44.83 | 44.83 | 44.74 | 44.82 |
| *otherwise* | 60 | 60.05 | 59.83 | 59.89 | 59.91 |

TABLE 6. Comparison of the original frequency and the frequency with respect to NCW family at $\alpha = -1$.

| Time | The Original Frequency | *The Frequency w.r.t. NCW Family* | | | |
|---|---|---|---|---|---|
| | | $k=1$ | $k=2$ | $k=3$ | $k=4$ |
| $0 < x \leq 1$ | 30 | 29.88 | 29.95 | 29.98 | 29.92 |
| $1 < x \leq 2$ | 45 | 45.15 | 44.99 | 44.93 | 44.91 |
| *otherwise* | 60 | 60.05 | 60.05 | 60.05 | 59.91 |

## 5. CONCLUSION

The various forms of wavelet transform definitions are applied to the proposed family of complex continuous wavelets and discussed the suitable form applicable to these wavelets. Successfully established the relationship between the wavelet scale and Fourier wavelength of the wavelets and their conversion of scale to the frequency. The results were simulated by using a synthetic signal.


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## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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