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NON-STANDARD FITTED OPERATOR SCHEME FOR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM

G. SANGEETHA[1,*], P. THIRUPATHI[2], AND K. PHANEENDRA[2]

[1]Department of Mathematics, Bhoj Reddy Engineering College for Women, Hyderabad, India

[2]Department of Mathematics, University College of Engineering, Osmania University, Hyderabad, India

Abstract: A non-standard fitted operator finite difference technique is designed for the solution of singularly perturbed two-point boundary value problems. Using the solution of reduced problem of the singular perturbation problem, it is transformed to an equivalent equation of first order. This first order singularly perturbed problem is solved by using an exact non-standard fitted operator method with an appropriate condition. To demonstrate the method computationally, it is implemented on several examples. From the numerical results, it is noticed that the proposed scheme is in respectable agreement with the exact solution with minimal computational effort.

Keywords: singular perturbation problems; reduced problem; boundary layer; fitted operator.

2010 AMS Subject Classification: 65L10, 65L11, 65L12.

1. INTRODUCTION

Introduction The numerical direction of a class of singular perturbation problems (SPP) is a field where effective research is taking place. These problems are of widespread in fluid mechanics and other division of applied mathematics. Bender and Orszag [1], Nayfeh [11], O' Malley [7] will be

*Corresponding author
E-mail address: dsangeethaou@gmail.com
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used for the elaborate discussion on SPPs. The authors [2,3,4,5,6,8,9,10,14,15] demonstrate the idea of replacing SPP with an equivalent initial value problem.

Robert [12] introduced a boundary value technique to figure certain classes of SPP. Essam [2] derived a new initial value technique for the solution of a class of nonlinear SPPs. Kadalbajoo and Reddy [7] considered a class of nonlinear SPP replaced by an equivalent problem of first-order and solved by an initial value technique. Gasparo and Macconi [3] integrated a semilinear SPP, to obtain initial value problems of first order, and considered both the outer and inner solutions. Gasparo and Macconi [4, 5] have given a new matching idea of integrate the reduced problem and a WKB approximation for linear, semilinear and quasilinear problems. El-Zahar and Habib [6] considered a class of nonlinear SPP, replaced by an analogous first order IVP and integrated by locally exact integration.

2. NUMERICAL METHOD

Consider a singularly perturbed linear boundary value problem:

$$\varepsilon x''(t) + a(t)x'(t) + b(t)x(t) = f(t) \ , t \in [0, 1] \tag{1}$$

with boundary conditions

$$x(0) = \alpha, \ x(1) = \beta \tag{2}$$

Here ε (0< ε <<1) is a smaller positive parameter and α, β are identified constants. We arrogate that $a(t), b(t)$ and $f(t)$ are sufficiently differentiable functions in [0, 1]. If $a(t) \geq P > 0$ over the domain [0, 1], where P is positive constant, then the layer will be in the locality of left end $t = 0$. If $a(t) \leq N < 0$ over the domain [0, 1], where N is negative constant, then the layer will be in the area of right end $t = 1$.

2.1. Problem with boundary layer at Left-end

The reduced problem is obtained by setting $\varepsilon = 0$ in Eq. (1) and is solved with right boundary condition $x(1) = \beta$. Let $x_0(t)$ be the reduced problem solution and is achieved by solving the equation

$$a(t)x'(t) + b(t)x(t) = f(t), \ x(1) = \beta.$$

Now Eq. (1) is taken as:

$$\varepsilon x''(t) + a(t)x'(t) + b(t)x_o(t) = f(t) \tag{3}$$

Revise Eq. (3) in the form $\varepsilon x''(t) + a(t)x'(t) = R(t)$ (4)

where $R(t) = f(t) - b(t)x_o(t)$. Taking integration of Eq. (4), we get corresponding first-order problem

$$\varepsilon x'(t) + a(t)x(t) = Q(t) + K \tag{5}$$

where $Q(t) = \int (R(t) - a'(t)x_0(t))dt$ and K is the constant to be determined.

To find *K*, use a condition that the reduced Eq. (5) should meet the boundary condition $x(1) = \beta$, that is

$$a(1)x(1) = Q(1) + K. \quad \text{Hence,} \quad K = a(1)x(1) - Q(1) \tag{6}$$

Thus, the original problem Eq. (1) - (2) replaced with an analogous first-order problem Eq. (5) with condition $x(0) = \alpha$. The region [0, 1] is decomposed into *N* equal subregions with mesh length *h* such that $h = \frac{1}{N}$ and $t_i = ih, i = 1,2, \ldots N$. The To solve the first order problem in our numerical experimentation, an exact fitted operator technique [13] is used. The non-standard fitted operator method for the Eq. (5) given by

$$\varepsilon\left(\frac{x_{i+1} - x_i}{\varphi_i}\right) + a_0 x(t) = Q(t) + K, \text{ for } \quad i = 1,2,\ldots,N \tag{7}$$

where $\varphi_i = \left(\frac{1 - e^{-a_i h}}{a_i}\right)$ is a fitting operator which gives a uniform solution to Eq. (5). The truncation error in this scheme is $T(h) = \left(\frac{\varepsilon h}{2}\right) x_i'' + O(h^2)$. Using the condition $x(0) = \alpha$, the two-term relation Eq. (7) solved very easily.

2.2. Problem with Right-end boundary layer

In this section, the solution of the reduced problem be $x_0(t)$ and is achieved by solving the equation $a(t)x'(t) + b(t)x(t) = f(t)$ with $x(0) = \alpha$. With the help of the reduced problem solution, Eq. (1) can be taken as:

$$\varepsilon x''(t) + a(t)x'(t) + b(t)x_o(t) = f(t) \tag{8}$$

Eq. (8) is rewritten as: $\varepsilon x''(t) + a(t)x'(t) = H(t)$ (9)

where $H(t) = f(t) - b(t)x_o(t)$.

Taking integration on Eq. (9), we get $\varepsilon x'(t) + a(t)y(t) = Q(t) + K$ (10)

where $Q(t) = \int (H(t) + a'(t)x_0(t))dt$ and K is an integrating constant, to be determined.

To find *K*, substitute $\varepsilon = 0$ in Eq. (10) we get

$$K = a(t)x_0 - Q(t) \quad (11)$$

The non-standard fitted operator method for the Eq. (10) given by

$$\varepsilon(\frac{x_i - x_{i-1}}{\varphi_i}) + a_n x(t) = Q(t) + K, \text{ for } i = N, N-1, \ldots, 1 \quad (12)$$

where $\varphi_i = \left(\frac{e^{a_i h} - 1}{a_i}\right)$ is the fitted operator.

The truncation error in this scheme is $T(h) = \left(\frac{\varepsilon h}{2}\right)x_i'' + O(h^2)$. We solve Eq. (12) using the condition $x(1) = \beta$.

3. NUMERICAL ILLUSTRATIONS

To explain the method computationally, it is implemented on four linear and one nonlinear problem with layer behaviour. These examples have been broadly discussed in the literature and are obtainable for comparability because exact solutions are available.

Example 1. $\varepsilon x''(t) + x'(t) - x(t) = 0$ with $x(0) = 1$ and $x(1) = 1$.

The reduced problem is $x_0'(t) - x_0(t) = 0$ with $x_o(1) = 1$ and its solution is $x_o(t) = e^{t-1}$.

The initial value problem corresponding to the given problem is

$$\varepsilon x'(t) + x(t) = e^{(t-1)} \text{ with } x(0) = 1.$$

The exact solution of the problem is $x(t) = \frac{[(e^{m_2} - 1)e^{m_1 t} + (1 - e^{m_1})e^{m_2 t}]}{[e^{m_2} - e^{m_1}]}$

where $m_{1,2} = \frac{-1 \pm \sqrt{1 + 4\varepsilon}}{2\varepsilon}$. The results of computation are acknowledged in the Table 1 and layer structure in shown in Figure 1.

Example 2. $\varepsilon x''(t) + x'(t) = 1 + 2t$ with $x(0) = 0$ and $x(1) = 1$.

The reduced problem is $x_0'(t) = 1 + 2t$ with $x_o(1) = 1$ and its solution is

$x_0(t) = t^2 + t - 1$.

The exact solution is $x(t) = t(t + 1 - 2\varepsilon) + \frac{(2\varepsilon-1)(1-e^{-t/\varepsilon})}{(1-e^{-1/\varepsilon})}$.

The corresponding initial value problem is $\varepsilon x'(t) + x(t) = t^2 + t - 1$ with $x(0) = 0$.

The results are acknowledged in Table 2 and layer behaviour in shown in Figure 2.

Example 3. Consider the nonlinear SPP [1]

$$\varepsilon x''(t) + 2x'(t) + e^{x(t)} = 0 \quad \text{with } x(0) = 0 \text{ and } x(1) = 0.$$

The reduced problem is $2x_0'(x) + e^{x_0(t)} = 0$ with $x_0(1) = 0$ and its solution is $x_0(t) = \log_e\left(\frac{2}{t+1}\right)$.

The Exact solution is $x(t) = log_e\left(\frac{2}{t+1}\right) - (log_e 2)e^{-2t/\varepsilon}$.

The initial value problem corresponding to this problem is

$\varepsilon x'(t) + 2x(t) = -2\,log(t + 1) - 2\,log\left(\frac{1}{2}\right)$. The computational results are acknowledged in Table 3 and behaviour of the layer is shown in Figure 3.

Example 4. $\varepsilon x''(t) - x'(t) - (1 + \varepsilon)x(t) = 0$ with $x(0) = 1 + e^{\frac{-(1+\varepsilon)}{\varepsilon}}$, $x(1) = 1 + \frac{1}{\varepsilon}$.

The problem has a boundary layer at $t = 1$.

The reduced problem is $x_o'(t) + x_o(t) = 0$; $x_o(0) = 1$ and its solution is $x_o(t) = e^{-t}$.

The exact solution is given by $x(t) = e^{(1+\varepsilon)(t-1)/\varepsilon} + e^{-t}$. The in initial value problem corresponding to this problem is $\varepsilon x'(t) - x(t) = -(1 + \varepsilon)e^{-t}$ with $x(1) = 1 + \frac{1}{\varepsilon}$.

The computational results are shown in Table 4 and Figure 4 shows the layer behaviour.

Example 5. $-\varepsilon x''(t) + x'(t) = e^t$; with $x(0) = 0$ and $x(1) = 0$.

The reduced problem is $x_o'(t) - e^t = 0$; $x_o(0) = 0$ and its solution is $x_o(t) = e^t - 1$.

The exact solution is given by $x(t) = \frac{1}{1-\varepsilon}\left[e^t - \frac{1-e^{(1-\frac{1}{\varepsilon})}+(e^t-1)e^{(\frac{t-1}{\varepsilon})}}{1-e^{\frac{-1}{\varepsilon}}}\right]$.

Asymptotically equivalent initial value problem is $\varepsilon x'(t) - x(t) = 1 - e^t$ with $x(1) = 0$.

The computational results are tabulated in Table 5 and layer behaviour is pictured in Figure 5.

3. DISCUSSIONS AND CONCLUSION

A numerical solution of SPP is achieved by solving an analogous first order equation. The equivalent first-order equation of the original problem is obtained using the reduced problem solution. This first order SPP is solved by using an exact fitted operator method with an appropriate condition. We possess implemented the method on three linear examples, one non-linear example with boundary layer at left-end and two examples with boundary layer at right-end by taking various values of ε. Numerical results of the examples are compared with the upwind scheme and acknowledged in the Tables 1-5 to justify the scheme. The layer profile is pictured in th Figures 1-5. It is noticed from the results that the method presented here is simple and approximates the exact solution very well with lowest computational effort.

Table 1. Maximum absolute errors in the solution of Example 1

ε/h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
Present method						
2^{-5}	4.7237(-2)	3.8339(-2)	3.4205(-2)	3.2224(-2)	3.1256(-2)	3.0778(-2)
2^{-6}	3.4781(-2)	2.4163(-2)	1.9509(-2)	1.7378(-2)	1.6363(-2)	1.5870(-2)
2^{-7}	3.0592(-2)	1.7728(-2)	1.2220(-2)	9.8407(-3)	8.7588(-3)	8.2459(-3)
2^{-8}	2.9947(-2)	1.5603(-2)	8.9492(-3)	6.1447(-3)	4.9420(-3)	4.3971(-3)
2^{-9}	2.9879(-2)	1.5298(-2)	7.8795(-3)	4.4961(-3)	3.0811(-3)	2.4765(-3)
2^{-10}	2.9850(-2)	1.5278(-2)	7.7316(-3)	3.9594(-3)	2.2534(-3)	1.5427(-3)
Results by Upwind method						
2^{-5}	2.4433(-1)	8.3405(-2)	4.1287(-2)	3.0357(-2)	3.0330(-2)	3.0317(-2)
2^{-6}	7.3163(-1)	2.3837(-1)	7.8895(-2)	3.6533(-2)	1.9683(-2)	1.5392(-2)
2^{-7}	3.9223(+14)	7.2461(-1)	2.3544(-1)	7.6687(-2)	3.4519(-2)	1.7332(-2)
2^{-8}	9.9962(+25)	7.2515(+29)	7.2113(-1)	2.3399(-1)	7.5596(-2)	3.3524(-2)
2^{-9}	1.8202(+36)	1.1027(+53)	2.4871(+60)	7.1940(-1)	2.3326(-1)	7.5054(-2)
2^{-10}	1.0796(+46)	7.8487(+73)	1.3443(+107)	2.9307(+121)	7.1853(-1)	2.3290(-1)

Table 2. Maximum absolute errors in the solution of Example 2

ε/h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
Present method						
2^{-5}	4.7237(-2)	3.8339(-2)	3.4205(-2)	3.2224(-2)	3.1256(-2)	3.0778(-2)
2^{-6}	3.4781(-2)	2.4163(-2)	1.9509(-2)	1.7378(-2)	1.6363(-2)	1.5870(-2)
2^{-7}	3.0592(-2)	1.7728(-2)	1.2220(-2)	9.8407(-3)	8.7588(-3)	8.2459(-3)
2^{-8}	2.9947(-2)	1.5603(-2)	8.9492(-3)	6.1447(-3)	4.9420(-3)	4.3971(-3)
2^{-9}	2.9879(-2)	1.5298(-2)	7.8795(-3)	4.4961(-3)	3.0811(-3)	2.4765(-3)
2^{-10}	2.9850(-2)	1.5278(-2)	7.7316(-3)	3.9594(-3)	2.2534(-3)	1.5427(-3)
Results by Upwind method						
2^{-5}	2.4433(-1)	8.3405(-2)	4.1287(-2)	3.0357(-2)	3.0330(-2)	3.0317(-2)
2^{-6}	7.3163(-1)	2.3837(-1)	7.8895(-2)	3.6533(-2)	1.9683(-2)	1.5392(-2)
2^{-7}	3.9223(+14)	7.2461(-1)	2.3544(-1)	7.6687(-2)	3.4519(-2)	1.7332(-2)
2^{-8}	9.9962(+25)	7.2515(+29)	7.2113(-1)	2.3399(-1)	7.5596(-2)	3.3524(-2)
2^{-9}	1.8202(+36)	1.1027(+53)	2.4871(+60)	7.1940(-1)	2.3326(-1)	7.5054(-2)
2^{-10}	1.0796(+46)	7.8487(+73)	1.3443(+107)	2.9307(+121)	7.1853(-1)	2.3290(-1)

Table 3. Maximum absolute errors in the solution of Example 3

ε/h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
Present method						
2^{-5}	3.4017(-2)	2.3185(-2)	1.8612(-2)	1.6555(-2)	1.5582(-2)	1.5109(-2)
2^{-6}	3.0772(-2)	1.7388(-2)	1.1909(-2)	9.5615(-3)	8.5030(-3)	8.0033(-3)
2^{-7}	3.0772(-2)	1.5551(-2)	8.8505(-3)	6.0478(-3)	4.8582(-3)	4.3206(-3)
2^{-8}	3.0772(-2)	1.5504(-2)	7.8646(-3)	4.4655(-3)	3.0527(-3)	2.4519(-3)
2^{-9}	3.0772(-2)	1.5504(-2)	7.7821(-3)	3.9549(-3)	2.2434(-3)	1.5346(-3)
2^{-10}	3.0772(-2)	1.5504(-2)	7.7821(-3)	3.8986(-3)	1.9832(-3)	1.1254(-3)
Results by Upwind method						
2^{-5}	8.1773(-1)	2.7050(-1)	9.3321(-2)	4.6263(-2)	2.7620(-2)	1.9552(-2)
2^{-6}	1.2988(+15)	8.0246(-1)	2.6278(-1)	8.7541(-2)	4.0995(-2)	2.2067(-2)
2^{-7}	7.6980(+26)	2.3934(+30)	7.9474(-1)	2.5889(-1)	8.4631(-2)	3.8343(-2)
2^{-8}	2.9988(+37)	8.4784(+53)	8.1953(+60)	7.9085(-1)	2.5695(-1)	8.3171(-2)
2^{-9}	3.6728(+47)	1.2920(+75)	1.0327(+108)	9.6489(+121)	7.8891(-1)	2.5597(-1)
2^{-10}	2.6304(+57)	1.9421(+95)	2.4033(+150)	1.5354(+216)	1.3403(+244)	7.8793(-1)

Table 4. Maximum absolute errors in the solution of Example 4

ε/h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
Present method						
2^{-5}	1.7505(-2)	9.2584(-3)	1.0326(-2)	1.0832(-2)	1.1078(-2)	1.1201(-2)
2^{-6}	1.9714(-2)	8.9213(-3)	4.6967(-3)	5.2189(-3)	5.4659(-3)	5.5859(-3)
2^{-7}	2.3024(-2)	1.0056(-2)	4.5035(-3)	2.3653(-3)	2.6235(-3)	2.7455(-3)
2^{-8}	2.6160(-2)	1.1759(-2)	5.0782(-3)	2.2625(-3)	1.1869(-3)	1.3152(-3)
2^{-9}	2.7985(-2)	1.3375(-2)	5.9419(-3)	2.5518(-3)	1.1340(-3)	5.9449(-4)
2^{-10}	2.8903(-2)	1.4317(-2)	6.7627(-3)	2.9867(-3)	1.2791(-3)	5.6766(-4)
Results by Upwind method						
2^{-5}	3.5674(-1)	1.0663(-1)	4.0186(-2)	1.2967(-2)	4.9421(-4)	5.4920(-3)
2^{-6}	1.1314(0)	3.6222(-1)	1.1219(-1)	4.5777(-2)	1.8571(-2)	6.1037(-3)
2^{-7}	6.1770(+14)	1.1333(0)	3.6503(-1)	1.1502(-1)	4.8612(-2)	2.1409(-2)
2^{-8}	1.5778(+26)	1.1446(+30)	1.1343(0)	3.6645(-1)	1.1645(-1)	5.0039(-2)
2^{-9}	2.8763(+36)	1.7425(+53)	3.9301(+60)	1.1348(0)	3.6716(-1)	1.1716(-1)
2^{-10}	1.7069(+46)	1.2409(+74)	2.1255(+107)	4.6336(+121)	1.1351(0)	3.6752(-1)

Table 5. Maximum absolute errors in the solution of Example 5

ε/h	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}	2^{-10}
Present method						
2^{-5}	7.3463(-2)	5.0839(-2)	4.0954(-2)	3.6410(-2)	3.4261(-2)	3.3230(-2)
2^{-6}	6.7110(-2)	3.8162(-2)	2.6095(-2)	2.0722(-2)	1.8219(-2)	1.7020(-2)
2^{-7}	7.0307(-2)	3.4406(-2)	1.9537(-2)	1.3262(-2)	1.0448(-2)	9.1275(-3)
2^{-8}	7.7196(-2)	3.5825(-2)	1.7483(-2)	9.9082(-3)	6.6966(-3)	5.2516(-3)
2^{-9}	8.0434(-2)	3.8850(-2)	1.8084(-2)	8.8236(-3)	4.9937(-3)	3.3671(-3)
2^{-10}	8.2035(-2)	4.0505(-2)	1.9488(-2)	9.0849(-3)	4.4330(-3)	2.5079(-3)
Results by Upwind method						
2^{-5}	6.5165(-1)	2.0101(-1)	8.1361(-2)	3.5948(-2)	3.2226(-2)	3.2242(-2)
2^{-6}	2.0007(0)	6.4194(-1)	2.0179(-1)	8.4893(-2)	3.7334(-2)	1.7275(-2)
2^{-7}	1.0744(+15)	1.9759(0)	6.3705(-1)	2.0218(-1)	8.6667(-2)	3.9350(-2)
2^{-8}	2.7276(+26)	1.9788(+30)	1.9634(0)	6.3459(-1)	2.0236(-1)	8.7556(-2)
2^{-9}	4.9573(+36)	3.0033(+53)	6.7738(+60)	1.9571	6.3335(-1)	2.0246(-1)
2^{-10}	2.9374(+46)	2.1355(+74)	3.6577(+107)	7.9741(+121)	1.9540(0)	6.3274(-1)

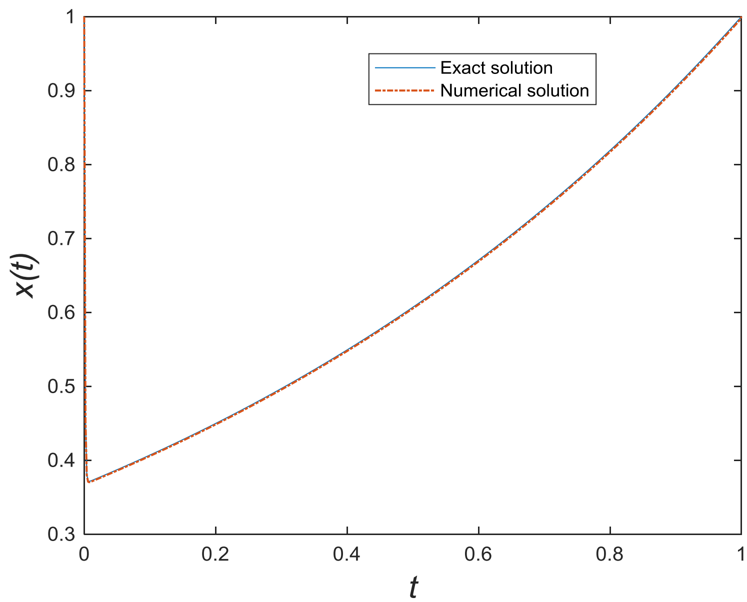

Figure 1. Solution profile in Example 1 for $\varepsilon = 2^{-10}, h = 2^{-7}$.

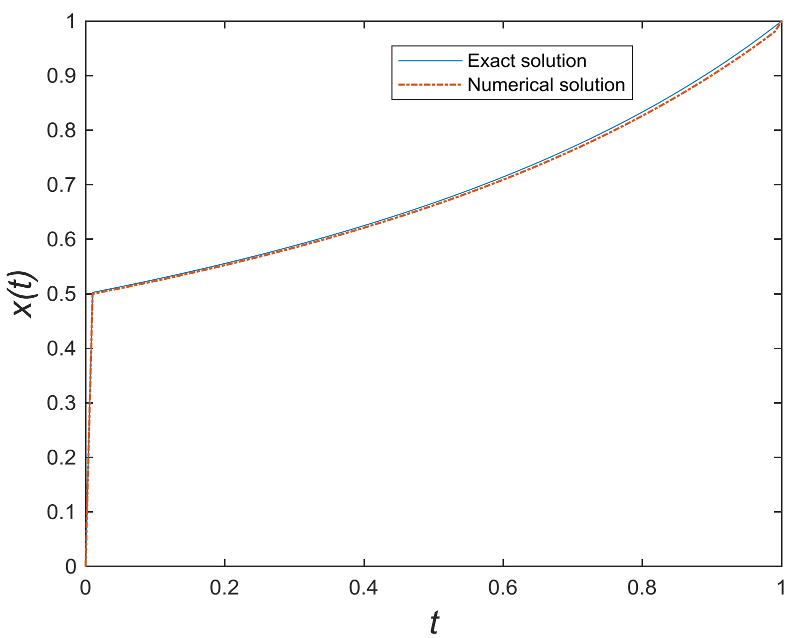

Figure 2. Solution profile in Example 2 for $\varepsilon = 2^{-10}, h = 2^{-7}$.

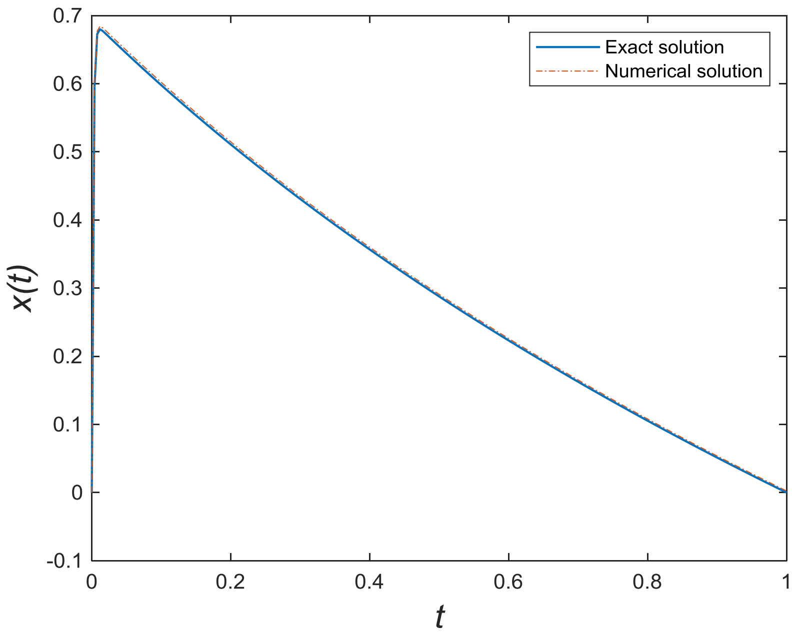

Figure 3. Solution profile in Example 3 for $\varepsilon = 2^{-10}, h = 2^{-7}$.

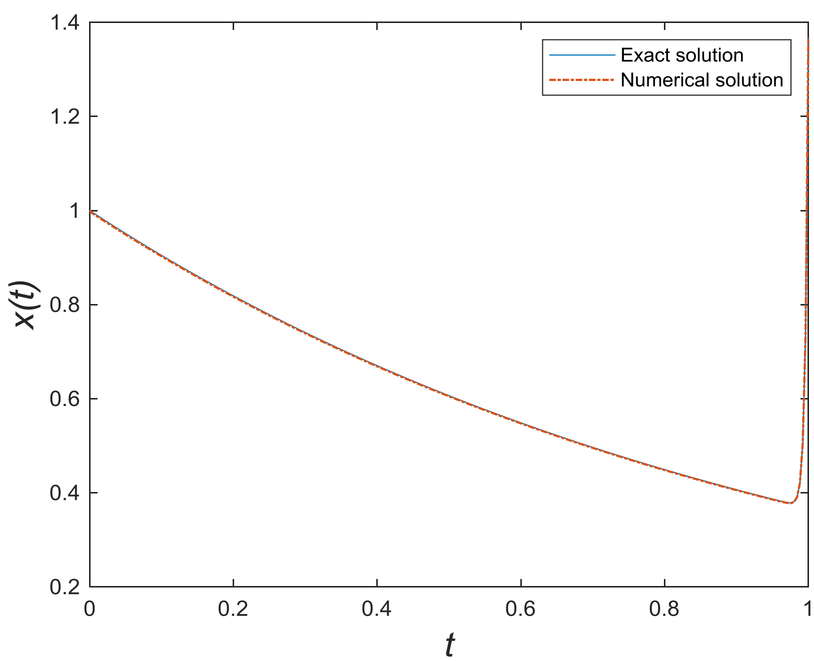

Figure 4. Solution profile in Example 4 for $\varepsilon = 2^{-10}, h = 2^{-7}$.

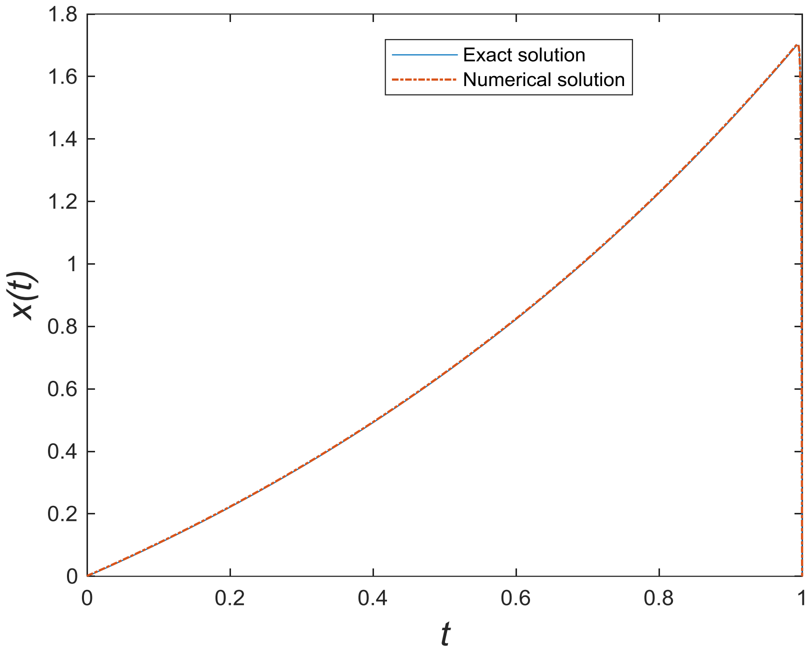

Figure 5. Solution profile in Example 5 for $\varepsilon = 2^{-10}, h = 2^{-7}$.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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