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UNIQUENESS AND VARIABLE SHARING OF q-SHIFT DIFFERENCE POLYNOMIALS OF ENTIRE FUNCTIONS

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Abstract. In this paper, we study weighted sharing and characteristics of q-shift difference polynomials of entire functions and obtain a uniqueness theorem concerning weighted z-sharing of certain q-shift difference-differential polynomials of entire functions.

Keywords: meromorphic function; difference polynomials; q-shift; Picard's exceptional values; uniqueness.

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1. Introduction

In this paper, every function like meromorphic function, entire function are defined on complex plane. We use the standard notations and definitions of the Nevanlinnas value distribution theory of meromorphic functions such as counting function $N(r,f)$, the characteristics function $T(r,f)$ etc, as explained in [7, 10, 22]. $S(r,f)$ is any quantity satisfying $S(r,f) = o(T(r,f))$ for all r outside a possible exceptional set of the finite logarithmic measure. The family of all meromorphic functions α which satisfy the relation $T(r,f) = S(r,f) = o(T(r,f))$, where $r \longrightarrow \infty$ outside of a possible exceptional set of the finite logarithmic measure is denoted by $\mathbb{S}(f)$. For our convenience we means that $\mathbb{S}(f)$ contains all constant functions and $\hat{\mathbb{S}} = \mathbb{S}(f) \cup \{\infty\}$.

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Let f and g be two meromorphic functions defined in the complex plane and a be a value in the extended complex plane. Now we say that f and g share that value a CM(counting multiplicities) if the zeros of $f-a$ and $g-a$ coincide in location and multiplicity and say that f and g share the value a IM if zeros of $f-a$ and $g-a$ coincide only in location but not in multiplicity. The counting function of zeros of $f-a$ where m-fold zero is counted m-times if $m \leq p$ and p times if $m > p$ is denoted by $N_p(r,a;f)$ where $p \in \mathbb{Z}^+$. We define difference operators for a moromorphic function by
$\Delta_c f(z) = f(z+c) - f(z), (c \neq 0)$ and
$\Delta_q f(z) = f(qz) - f(z), (q \neq 0)$.

Many mathematicians already worked out many research papers on entire, meromorphic functions, their differential polynomials and sharing(see [4, 12, 16, 17, 21]). Recently mathematicians are interested in studying q-shift difference polynomials, difference equations and their products in the complex plane . Already a numbers of papers have been published which have focused the uniqueness of difference analogue of Nevanlinna theory.(see [2, 3, 5, 6, 11, 13, 14, 17]).

2. Preliminaries

Definition 2.1. *Let $p(z) = \Sigma_{i=0}^{n} a_i z^i$ be a nonzero polynomial, where $a_i (i = 0,1,2,...,n)$ are complex constant and $a_n \neq 0$. Let m_1 is the numbers of single zeros of $p(z)$ and m_2 is the number of multiple zeros of $p(z)$ and Γ_1, Γ_2 defined by $\Gamma_1 = m_1 + m_2, \Gamma_2 = m_1 + 2m_2$ respectively. We denote $\gamma = gcd(\gamma_0, \gamma_1, ..., \gamma_n)$ where $\gamma_i = n+1$, if $a_i = 0$, $\gamma_i = i+1$, if $a_i \neq 0$.*

Definition 2.2. [8, 9] *Let p be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_p(a;f)$ the set of all a-point of f where an a-point of multiplicity m is counted m times if $m \leq p$ and $p+1$ times if $m > p$. If $E_p(a;f) = E_p(a;g)$ we say that f, g share the value a with weight p.*

In 2007, laine and Yang [11] studied zero distributions of difference polynomials of entire functions and obtained the following results.

Theorem A. [11] *Let f be a transcendental entire function of finite order and ζ be a nonzero complex constant. Then for $n \geq 2$, $f^n f(z+\zeta)$ assumes every nonzero value a in $\mathbb{C}$ infinitely often.*

The uniqueness result corresponding to Theorem A given by Qi, Yang and Liu [15].

Theorem B. [15] *Let f and g be two transcendental entire functions of finite order, and ζ be a nonzero complex constant, and let $n \geq 6$ be an integer. If $f^n(z)f(z+\zeta)$ and $g^n(z)g(z+\zeta)$ share 1 CM, then either $fg = t_1$ or $f = t_2 g$ for same constants t_1 and t_2 satisfying $t_1^{n+1} = t_2^{n+1} = 1$.*

Theorem C. [20] *Let f be a transcendental entire function of finite order and ζ be a fixed nonzero complex constant. Then for $n > \Gamma_1$, $P(f(z))f(z+\zeta) - \omega(z) = 0$ has infinitely many solutions, where $\omega(z) \in \mathbb{S}(f) \setminus \{0\}$.*

Theorem D. [20] *Let f and g be two transcendental entire functions of finite order, ζ be a nonzero complex constant and $n > 2\Gamma_2 + 1$ be an integer. If $p(f)f(z+\zeta)$ and $p(g)g(z+\zeta)$ share 1 CM, then one of the following results hold:*
i) $f = tg$, where $t^\gamma = 1$;
ii)f and g satisfy the algebraic equation $\Phi(f,g) = 0$, where $\Phi(\lambda_1, \lambda_2) = p(\lambda_1)\lambda_1(z+\zeta) - p(\lambda_2)\lambda_2(z+\zeta)$;
iii)$f = e^\xi$, $g = e^\psi$, where ξ and ψ are two polynomials and $\xi + \psi = d$, d is a complex constant satisfying $a_n^2 e^{(n+1)d} = 1$.

In 2010 Zhang and korhonen[23] obtain the following result on value distribution of q-shift difference polynomials of meromorphic functions.

Theorem E. [23] *Let f be a transcendental meromorphic(resp. entire) function of zero order and q be a nonzero complex constant. Then for $n \geq 6$(resp. $n \geq 2$) $f(z)^n f(qz)$ assume every nonzero value $c \in \mathbb{C}$ infinitely often.*

Theorem F. [23] *Let f and g be two transcendental meromorphic functions of zero order. Suppose that q is a nonzero complex constant and $n \geq 6$ is an integer. If $f(z)^n(f(z)-1)f(qz)$ and $g(z)^n(g(z)-1)g(qz)$ share 1 CM, then $f \equiv g$.*

In 2015 Xu, Liu and Cao [19] obtained the following result for a q-shift of a meromorphic function.

Theorem G. [19] *Let f be a zero order transcendental meromorphic(resp. entire) function, $q \in \mathbb{C} \setminus 0$, $\zeta \in \mathbb{C}$. Then for any positive integer $n > \Gamma_1 + 4$(resp.for entire $n > \Gamma_1$), $p(f(z))f(qz+\zeta) = \phi(z)$ has infinitely many solutions, where $\phi(z) \in \mathbb{S}(f) \setminus \{0\}$.*

Theorem H. [19] *Let f and g be two transcendental entire functions of zero order and let $q \in \mathbb{C} \setminus \{0\}$, $\zeta \in \mathbb{C}$. If $p(f(z))f(qz+\zeta)$ and $p(g(z))g(qz+\zeta)$ share 1 CM and $n > 2\Gamma_2 + 1$ be an integer, then one of the following results hold:*
i)$f = tg$ for a constant t such that $t^\gamma = 1$;
ii)f and g satisfy the algebraic equation $\Phi(f,g) = 0$, where $\Phi(\lambda_1,\lambda_2) = p(\lambda_1)\lambda_1(qz+\zeta) - p(\lambda_2)\lambda_2(qz+\zeta)$;
iii) $fg = d$, where d is a complex constant satisfying $a_n^2 d^{n+1} \equiv 1$.

Considering weighted sharing in 2001 Lahiri [8, 9] prove the following result,

Theorem I. [8, 9] *Let f and g be two transcendental entire functions of zero order and let $q \in \mathbb{C} \setminus \{0\}$, $\zeta \in \mathbb{C}$. If $E_l(1; p(f(z))f(qz+\zeta)) = E_l(1; p(g(z))g(qz+\zeta))$ and l,m,n are integers satisfy one of the following conditions:*
i)$l \geq 3$; $n > 2\Gamma_2 + 1$;
ii)$l = 2$; $n > \Gamma_1 + 2\Gamma_2 + 2 - \alpha$;
iii)$l = 1$; $n > 2\Gamma_1 + 2\Gamma_2 + 3 - 2\alpha$;
iv)$l = 0$; $n > 3\Gamma_1 + 2\Gamma_2 + 4 - 3\alpha$.
Then the conclusions of theorem H holds, where $\alpha = min\{\Theta(0,f), \Theta(0,g)\}$

Recently Sahoo and Biswas [18] prove the following theorem,

Theorem J. [18] *Let f and g be two transcendental entire functions of zero order and let $q \in \mathbb{C} \setminus \{0\}$, $\zeta \in \mathbb{C}$. If $E_l(1; (p(f(z))f(qz+\zeta))^{(k)}) = E_l(1; (p(g(z))g(qz+\zeta))^{(k)})$ and l,m,n are integers satisfy one of the following conditions:*
i)$l \geq 2$; $n > 2\Gamma_2 + 2km_2 + 1$;
ii)$l = 1$; $n > \frac{1}{2}(\Gamma_1 + 4\Gamma_2 + 5km_2 + 3)$;

iii) $l=0$; $n>3\Gamma_1+2\Gamma_2+5km_2+4$.

Then one of the following results holds:

i) $f=tg$ *for a constant* t *such that* $t^{\gamma}=1$;

ii) f *and* g *satisfy the algebraic equation* $\Phi(f,g)=0$, *where* $\Phi(\lambda_1,\lambda_2)=p(\lambda_1)\lambda_1(qz+\zeta)-p(\lambda_2)\lambda_2(qz+\zeta)$;

iii) $fg=d$, *where* d *is a complex constant satisfying* $a_n^2d^{n+1}\equiv 1$.

3. Main Results

Theorem 3.1. *Let* f *and* g *be two transcendental entire functions of zero order and let* $q\in\mathbb{C}\setminus 0$, $\zeta\in\mathbb{C}$. *If* $E_l(z;(p(f)f(qz+\zeta))^{(k)})=E_l(z;(p(g)g(qz+\zeta))^{(k)})$ *and* l,m,n *are integers satisfy one of the following conditions:*

i) $l\geq 2$; $n>2\Gamma_2+2km_2+1$;

ii) $l=1$; $n>\frac{1}{2}(\Gamma_1+4\Gamma_2+5km_2+3)$;

iii) $l=0$; $n>3\Gamma_1+2\Gamma_2+5km_2+4$.

Then one of the following results holds:

i) $f=tg$ *for a constant* t *such that* $t^{\gamma}=1$;

ii) f *and* g *satisfy the algebraic equation* $\Phi(f,g)=0$, *where* $\Phi(\lambda_1,\lambda_2)=p(\lambda_1)\lambda_1(qz+\zeta)-p(\lambda_2)\lambda_2(qz+\zeta)$;

iii) $f(z)=\mu_1e^{\frac{u}{2}z^2+vz}$ *and* $g(z)=\frac{\mu_1}{\mu}e^{-(\frac{u}{2}z^2+vz)}$ *where* μ_1,μ,u,v *are complex constant and not equal to zero. If* $A=(-1)a_n^2\mu^{n+1}$, *then* $u^2=\frac{1}{A(n+q^2)^2}$ *and* $v^2=\frac{\zeta^2q^2}{A(n+q)^2(n+q^2)^2}$.

4. Lemmas

In this section we present some necessary lemmas.

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows : $H=(\frac{F''}{F'}-\frac{2F'}{F-1})-(\frac{G''}{G'}-\frac{2G'}{G-1})$.

Lemma 4.1. [22] *Let* f *be a nonconstant meromorphic function, and* $P(f)=\sum_{i=0}^{n}a_if^i$, *where* $a_0,a_1,a_2,...,a_n$ *are complex constants and* $a_n\neq 0$. *Then* $T(r,p(f))=nT(r,f)+S(r,f)$.

Lemma 4.2. [24] *Let f be a nonconstant meromorphic function, and $p,k \in \mathbb{Z}^+$.Then*

$$N_p(r,\frac{1}{f^{(k)}}) \leq T(r,f^{(k)}) - T(r,f) + N_{p+k}(r,\frac{1}{f}) + S(r,f), \tag{4.1}$$

$$N_p(r,\frac{1}{f^{(k)}}) \leq k\overline{N}(r,f) + N_{p+k}(r,\frac{1}{f}) + S(r,f), \tag{4.2}$$

Lemma 4.3. [9] *Let f and g be two non-constant meromorphic functions. If $E_2(1;f) = E_2(1;g)$, then one of the following relation holds:*

i)$T(r) \leq N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{g}) + N_2(r,f) + N_2(r,g) + S(r)$;

ii)$f = g$;

iii)$fg = 1$;

where $T(r) = max\{T(r,f),T(r,g)\}$ and $S(r) = o\{T(r)\}$.

Lemma 4.4. [1] *Let F and G be two non-constant meromorphic functions such that $E_1(1;F) = E_1(1;G)$ and $H \not\equiv 0$, then*

$T(r,F) \leq N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G) + \frac{1}{2}\overline{N}(r,\frac{1}{F}) + \frac{1}{2}\overline{N}(r,F) + S(r,F) + S(r,G)$;

and we can deduce same result for $T(r,G)$.

Lemma 4.5. [1] *Let F and G be two non-constant meromorphic functions which are share 1 IM and $H \not\equiv 0$, then, $T(r,F) \leq N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G) + 2\overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) + 2\overline{N}(r,F) + \overline{N}(r,G) + S(r,F) + S(r,G)$.*

Lemma 4.6. [19] *Let f be a transcendental meromorphic function of order zero and q,ζ two nonzero complex constants. Then*

$T(r,f(qz+\zeta)) = T(r,f(z)) + S(r,f)$;

$N(r,f(qz+\zeta)) = N(r,f(z)) + S(r,f)$;

$N(r,\frac{1}{f(qz+\zeta)}) = N(r,\frac{1}{f(z)}) + S(r,f)$;

$\overline{N}(r,f(qz+\zeta)) = \overline{N}(r,f(z)) + S(r,f)$;

$\overline{N}(r,\frac{1}{f(qz+\zeta)}) = \overline{N}(r,\frac{1}{f(z)}) + S(r,f)$.

Lemma 4.7. [19] *Let f be a transcendental meromorphic function of order zero and $q(\neq 0),\zeta$ two nonzero complex constants. Then*

$(n-1)T(r,f) + S(r,f) \leq T(r,p(f)f(qz+\zeta)) \leq (n+1)T(r,f) + S(r,f)$.

In addition, if f is a transcendental entire function of zero order, then
$T(r,p(f)f(qz+\zeta)) = T(r,p(f)f(z)) + S(r,f) = (n+1)T(r,f) + S(r,f).$

Lemma 4.8. *Let f and g be two entire functions, q,ζ complex constants and $q \neq 0$; n,k are two positive integers and let $F = \frac{(p(f)f(qz+\zeta))^{(k)}}{z}$, $G = \frac{(p(g)g(qz+\zeta))^{(k)}}{z}$. If there exists two nonzero constants c_1 and c_2 such that*
$\overline{N}(r,c_1;F) = \overline{N}(r,\frac{1}{G})$ and $\overline{N}(r,c_2;G) = \overline{N}(r,\frac{1}{F})$, then $n \leq 2\Gamma_1 + km_2 + 1$.

Proof. Let $F_1 = P(f)f(qz+\zeta)$and $G_1 = P(g)g(qz+\zeta)$ and $F = \frac{F_1^{(k)}}{z}$ and $G = \frac{G_1^{(k)}}{z}$.
By the second main theorem of Nevanlinna, we have

$$T(r,F) \leq \overline{N}(r,\frac{1}{F}) + \overline{N}(r,c_1;F) + S(r,F) \leq \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{G}) + S(r,F), \tag{4.3}$$

using (4.1),(4.2),(4.3), lemma 4.1, lemma 4.6 and lemma 4.7, we get

$$\begin{aligned}
&(n+1)T(r,f)\\
&\quad\leq T(r,F) - \overline{N}(r,\frac{1}{F}) + N_{k+1}(r,\frac{1}{F_1}) + S(r,f)\\
&\quad\leq \overline{N}(r,\frac{1}{G}) + N_{k+1}(r,\frac{1}{F_1}) + S(r,f)\\
&\quad\leq N_{k+1}(r,\frac{1}{F_1}) + N_{k+1}(r,\frac{1}{G_1}) + S(r,g) + S(r,f)\\
&\quad\leq N_{k+1}(r,\frac{1}{p(f)}) + N_{k+1}(r,\frac{1}{p(g)}) + N_{k+1}(r,\frac{1}{f(qz+\zeta)})\\
&\quad+ N_{k+1}(r,\frac{1}{g(qz+\zeta)}) + S(r,g) + S(r,f)\\
&\quad\leq (m_1+m_2+km_2+1)(T(r,f)+T(r,g)) + S(r,f) + S(r,g), \qquad (4.4)
\end{aligned}$$

Similarly,

$$(n+1)T(r,g) \leq (m_1+m_2+km_2+1)(T(r,f)+T(r,g)) + S(r,f) + S(r,g), \tag{4.5}$$

In view of (4.4), (4.5), we have,
$(n-2m_1-2m_2-2km_2-1)(T(r,f)+T(r,g)) \leq S(r,f) + S(r,g).$
which gives $n \leq 2\Gamma_1 + 2km_2 + 1$. This proves the lemma. □

Lemma 4.9. *Let f and g be two transcendental entire functions of zero order and let $q \in \mathbb{C} \setminus 0$, $\zeta \in \mathbb{C}$. If $p(f)f(qz+\zeta) = p(g)g(qz+\zeta)$. Then one of the following results holds:*
i) $f = tg$ for a constant t such that $t^{\gamma} = 1$;
ii) f and g satisfy the algebraic equation $\Phi(f,g) = 0$, where $\Phi(\lambda_1,\lambda_2) = p(\lambda_1)\lambda_1(qz+\zeta) - p(\lambda_2)\lambda_2(qz+\zeta)$.

Proof. This lemma can be proved easily in the line of the proof of the Theorem 11 [19]. □

5. Proof of the Theorem 3.1

Proof. Let $F_1 = p(f)f(qz+\zeta)$, $G_1 = p(g)g(qz+\zeta)$ then, $E_l(z;F_1^{(k)}) = E_l(z;G_1^{(k)})$. Again let $F = \frac{F_1^{(k)}}{z}$ and $G = \frac{G_1^{(k)}}{z}$. Then F and G are transcendental meromorphic functions satisfy $E_l(1;F) = E_l(1;G)$. Now with help of lemma 4.7 and using (4.1) we have

$$\begin{aligned} N_2(r,\frac{1}{F}) &\leq N_2(r,\frac{1}{\frac{F_1^{(k)}}{z}}) + S(r,f) \\ &\leq T(r,F_1^{(k)}) - T(r,F_1) + N_{k+2}(r,\frac{1}{F_1}) + S(r,f) \\ &\leq T(r,F) - (n+1)T(r,f) + N_{k+2}(r,\frac{1}{F_1}) + S(r,f), \end{aligned}$$

Hence

$$(n+1)T(r,f) \leq T(r,F) - N_2(r,\frac{1}{F}) + N_{k+2}(r,\frac{1}{F_1}) + S(r,f), \tag{5.1}$$

we can show from (4.2)

$$\begin{aligned} N_2(r,\frac{1}{F}) &\leq N_2(r,\frac{1}{F^{(k)}}) + S(r,f) \\ &\leq N_{k+2}(r,\frac{1}{F_1}) + S(r,f), \end{aligned} \tag{5.2}$$

Now following three cases will be discuss separately.

Case I

Let $l \geq 2$. If possible we assume that (i) of lemma 4.3 holds. We can deduce from (5.1) with help of (5.2)

$$\begin{aligned} (n+1)T(r,f) &\leq N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G) + N_{k+2}(r,\frac{1}{F_1}) + S(r,f) + S(r,g) \\ &\leq N_{k+2}(r,\frac{1}{F_1}) + N_{k+2}(r,\frac{1}{G_1}) + S(r,f) + S(r,g) \\ &\leq (m_1 + 2m_2 + km_2 + 1)(T(r,f) + T(r,g)) + S(r,f) + S(r,g), \end{aligned} \tag{5.3}$$

Same we can show for $T(r,g)$ i.e

$$(n+1)T(r,g) \leq (m_1 + 2m_2 + km_2 + 1)(T(r,f) + T(r,g)) + S(r,f) + S(r,g), \tag{5.4}$$

We can obtain from (5.3) and (5.4)

$$(n - 2m_1 - 4m_2 - 2km_2 - 1)(T(r,f) + T(r,g)) \leq S(r,f) + S(r,g),$$

which contradict the fact $n > 2\Gamma_2 + 2km_2 + 1$. Then by lemma 4.3 we claim that either $FG = 1$ or $F = G$. Let $FG = 1$. Then,

$$(p(f)f(qz+\zeta))^{(k)}(p(g)g(qz+\zeta))^{(k)} = z^2, \tag{5.5}$$

If possible, let $p(z) = 0$ has m roots $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_m$ with multiplicity $n_1, n_2, n_3, ..., n_m$. Then we have $n_1 + n_2 + n_3 + ... + n_m = n$. Now

$$\begin{aligned} &[a_n(f-\alpha_1)^{n_1}(f-\alpha_2)^{n_2}...(f-\alpha_m)^{n_m} f(qz+\zeta)]^{(k)} \times \\ &[a_n(g-\alpha_1)^{n_1}(g-\alpha_2)^{n_2}...(g-\alpha_m)^{n_m} f(qz+\zeta)]^{(k)} = z^2, \end{aligned} \tag{5.6}$$

Since f and g are entire functions from (5.6), we see that $\alpha_1 = \alpha_2 = ... = \alpha_m = 0$. Also we can say that $\alpha_1, \alpha_2, ..., \alpha_m$ are picard's exceptional values. By picard's theorem of entire function, we have at least three picard's exceptional values of f and if $m \geq 2$ and $\alpha_i \neq 0 (i = 1, 2, ..., m)$, then we obtain a contradiction.Next we assume that $p(z) = 0$ has only one root. Then $p(f) = a_n(f-a)^n$ and $p(g) = a_n(g-a)^n$, where a is any complex constant. Now from (5.5) we can write

$$[a_n(f-a)^n f(qz+\zeta)]^{(k)}[a_n(g-a)^n g(qz+\zeta)]^{(k)} = z^2, \tag{5.7}$$

By picard's theorem and as f and g are transcendental entire functions, then we can say that $f-a=0$ and $g-a=0$ do not have zeros. Then, we obtain that $f(z)=e^{\alpha(z)}+a$ and $g(z)=e^{\beta}+a$, $\alpha(z),\beta(z)$ being nonconstant polynomials. From (5.7), we also see that $f(qz+\zeta)\neq 0$ and $g(qz+\zeta)\neq 0$ and therefore $a=0$. Thus $f(z)=e^{\alpha(z)}, g(z)=e^{\beta(z)}, p(z)=a_nz^n$ and

$$[a_ne^{n\alpha(z)+\alpha(qz+\zeta)}]^{(k)}[a_ne^{n\beta(z)+\beta(qz+\zeta)}]^{(k)}=z^2, \tag{5.8}$$

If $k=0$,then from (5.8) we have

$$a_n^2e^{n(\alpha(z)+\beta(z))+\alpha(qz+\zeta)+\beta(qz+\zeta)}=z^2.$$

which is a contradiction as for no value of $\alpha(z)$ and $\beta(z)$ we can compare both side.

If $k=1$, then from (5.8) we have

$$[a_ne^{n\alpha(z)+\alpha(qz+\zeta)}(n\alpha'(z)+q\alpha'(qz+\zeta))][a_ne^{n\beta(z)+\beta(qz+\zeta)}(n\beta'(z)+q\beta'(qz+\zeta))]=z^2, \tag{5.9}$$

i.e

$$a_n^2e^{n(\alpha+\beta)+\alpha(qz+\zeta)+\beta(qz+\zeta)}(n\alpha'(z)+q\alpha'(qz+\zeta))(n\beta'(z)+q\beta'(qz+\zeta))=z^2,$$

Now the relation can be hold if $\alpha+\beta=c$; c is complex constant. Then $\alpha'+\beta'=0$, i.e $\beta'=-\alpha'$. Then from (5.9) we have

$$(-1)a_n^2e^{(n+1)c}(n\alpha'(z)+q\alpha'(qz+\zeta))^2=z^2, \tag{5.10}$$

Now if $\alpha'(z)$ be one degree polynomials, i.e $\alpha'(z)=uz+v$, then $\alpha'(qz+\zeta)=uqz+u\zeta+v$. Let $A=(-1)a_n^2e^{(n+1)c}=(-1)a_n^2\mu^{n+1}$, where $\mu=e^c$. Then we can show from (5.10) that $u^2=\frac{1}{A(n+q^2)^2}$ and $v^2=\frac{\zeta^2q^2}{A(n+q)^2(n+q^2)^2}$. Now $\alpha'=uz+v$ i.e $\alpha=\frac{u}{2}z^2+vz+w$, then $f(z)=\mu_1e^{\frac{u}{2}z^2+vz}$and $g(z)=\frac{\mu}{\mu_1}e^{-(\frac{u}{2}z^2+vz)}$ where $\mu_1=e^w$.

If $k\geq 2$, then we get

$$[a_ne^{n\alpha(z)+\alpha(qz+\zeta)}]^{(k)}=z^2a_ne^{n\alpha(z)+\alpha(qz+\zeta)}p(\alpha'\alpha'_\zeta,...,\alpha^{(k)}\alpha_\zeta^{(k)}),$$

where $\alpha_\zeta = \alpha(qz+\zeta)$. Obviously, $p(\alpha'\alpha'_\zeta,...,\alpha^{(k)}\alpha_\zeta^{(k)})$ has infinitely many zeros, and which contradict with (5.8).

Now let $F = G$. Then
$\frac{(p(f)f(qz+\zeta))^{(k)}}{z} = \frac{(p(g)g(qz+\zeta))^{(k)}}{z}$ i.e $(p(f)f(qz+\zeta))^{(k)} = (p(g)g(qz+\zeta))^{(k)}$.
Integrating one time we have

$$(p(f)f(qz+\zeta))^{(k-1)} = (p(g)g(qz+\zeta))^{(k-1)} + c_{k-1}.$$

where c_{k-1} is a constant. If $c_{k-1} \neq 0$ using lemma 4.8 we say that $n \leq 2\Gamma_1 + 2km_2 + 1$, which contradict with the fact that $n > 2\Gamma_2 + 2km_2 + 1$ $(\Gamma_2 \geq \Gamma_1)$. Hence $c_{k-1} = 0$. Now repeating the process upto k-times, we can established $p(f)f(qz+\zeta) = p(g)g(qz+\zeta)$. Hence by lemma 4.9 we have either $f = tg$ for a constant t such that $t^\gamma = 1$, or f and g satisfy the algebraic equation $\Phi(f,g) = 0$ where,

$$\Phi(\lambda_1,\lambda_2) = p(\lambda_1)\lambda_1(qz+\zeta) - p(\lambda_2)\lambda_2(qz+\zeta).$$

Case II

Let $l = 1$ and $H \not\equiv 0$. Using lemma 4.4 and (5.2) we can established from (5.1)

$$\begin{aligned}
(n+1)T(r,f) &= N_2(r,\frac{1}{G}) + N_2(r,F) + N_2(r,G) + \frac{1}{2}\overline{N}(r,\frac{1}{F}) \\
&+ \frac{1}{2}\overline{N}(r,F) + N_{k+2}(r,\frac{1}{F_1}) + S(r,f) + S(r,g) \\
&\leq N_{k+2}(r,\frac{1}{F_1}) + N_{k+2}(r,\frac{1}{G_1}) + \frac{1}{2}N_{k+1}(r,\frac{1}{F_1}) \\
&+ S(r,f) + S(r,g) \\
&\leq \frac{1}{2}[3m_1 + (3k+5)m_2 + 3]T(r,f) + [m_1 + (k+2)m_2 + 1]T(r,g) \\
&+ S(r,f) + S(r,g) \\
&\leq \frac{1}{2}[5m_1 + (5k+9)m_2 + 5]T(r) + S(r).
\end{aligned}$$

where $T(r)$ and $S(r)$ two inequalities, defined in lemma 4.3. Similarly we can show that

$$(n+1)T(r,g) \leq \frac{1}{2}[5m_1 + (5k+9)m_2 + 5]T(r) + S(r),$$

we have from two inequalities,

$$(n-\frac{5m_1+(5k+9)m_2+3}{2})T(r)\leq S(r),$$

which contradict the fact $n>\frac{\Gamma_1+4\Gamma_2+5km_2+3}{2}$.

Now, let $H\equiv 0$, i.e $(\frac{F''}{F'}-\frac{2F'}{F-1})-(\frac{G''}{G'}-\frac{2G'}{G-1})=0$. After two times integration we have

$$\frac{1}{F-1}=\frac{A}{G-1}+B, \tag{5.11}$$

where A,B are constants and $A\neq 0$. From (5.9) it is clear that F,G share the value 1 CM and then they share (1,2) and hence $(p(f)f(qz+\zeta))^{(k)}$ and $(p(g)g(qz+\zeta))^{(k)}$ share (z,2). Hence we have $n>2\Gamma_2+2km_2+1$. Now we study the following cases.

Subcase I

Let $B\neq 0$ and $A=B$. Then from (5.9) we get

$$\frac{1}{F-1}=\frac{BG}{G-1}, \tag{5.12}$$

If $B=-1$, then from (5.12), $FG=1$ i.e $(p(f)f(qz+\zeta))^{(k)}(p(g)g(qz+\zeta))^{(k)}=z^2$ then we obtain the same result as in Case I.

Now if $B\neq -1$.Then from (5.12), we have,$\frac{1}{F}=\frac{BG}{(1+B)G-1}$ and then, $\overline{N}(r,\frac{1}{1+B};G)=\overline{N}(r,\frac{1}{F})$. Now from the second main theorem of Nevalinna, we get using (4.1) and (4.3) that

$$\begin{aligned} T(r,G) &= \overline{N}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{1+B};G)+\overline{N}(r,G)+S(r,G) \\ &\leq \overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})+\overline{N}(r,G)+S(r,G) \\ &\leq N_{k+1}(r,\frac{1}{F_1})+T(r,G)+N_{k+1}(r,\frac{1}{G_1})-(n+1)T(r,g)+S(r,g). \end{aligned}$$

This gives,

$$(n+1)T(r,g)\leq(m_1+(k+1)m_2+1)(T(r,f)+T(r,g))+S(r,g),$$

we can show same result for $T(r,f)$ i.e

$$(n+1)T(r,f)\leq(m_1+(k+1)m_2+1)(T(r,f)+T(r,g))+S(r,f),$$

Thus we obtain

$$(n-2m_1-2(k+1)m_2-1)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g),$$

a contradiction as $n > 2\Gamma_2+2km_2+1$.

Subcase II

Let $A \neq 0$ and $B = 0$. Now from (5.11) we have $F = \frac{G+A-1}{A}$ and $G = AF-(A-1)$.If $A \neq 1$, we have $\overline{N}(r, \frac{A-1}{A}; F) = \overline{N}(r, \frac{1}{G})$ and $\overline{N}(r, 1-A; G) = \overline{N}(r, \frac{1}{F})$.Then by lemma 4.8, we have $n \leq 2\Gamma_1+2km_2+1$, which is a contradiction. Thus $A = 1$ and $F = G$, then the result follows from the Case I.

Subcase III

Let $A \neq 0$ and $A \neq B$. Then from (5.11), we obtain $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and therefore $\overline{N}(r, \frac{B-A+1}{B+1}; G) = \overline{N}(r, \frac{1}{F})$. Proceeding similarly as in Subcase I , we can get a contradiction.

Case III

Let $l = 0$ and $H \not\equiv 0$, we can established from (5.1) after using lemma 4.5 and (5.2)

$$\begin{aligned}
(n+1)T(r,f) &= N_2(r,\frac{1}{G})+N_2(r,F)+N_2(r,G)+2\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})+2\overline{N}(r,F) \\
&+ \overline{N}(r,G)+N_{k+2}(r,\frac{1}{F_1})+S(r,f)+S(r,g) \\
&\leq N_{k+2}(r,\frac{1}{F_1})+N_{k+2}(r,\frac{1}{G_1})+2N_{k+1}(r,\frac{1}{F_1})+N_{k+1}(r,\frac{1}{G_1}) \\
&+ S(r,f)+S(r,g) \\
&\leq [3m_1+(3k+4)m_2+3]T(r,f)+[2m_1+(2k+3)m_2+2]T(r,g) \\
&+ S(r,f)+S(r,g) \\
&\leq [5m_1+(5k+7)m_2+5]T(r)+S(r),
\end{aligned}$$

Similarly it follows that $(n+1)T(r,g) \leq [5m_1+(5k+7)m_2+5]T(r)+S(r)$.

From the above two inequalities we have $(n-5m_1-(5k+7)m_2-4]T(r) \leq S(r)$, which contradict with our assumption that $n > 3\Gamma_1 + 2\Gamma_2 + 5km_2 + 4$. Therefore $H = 0$ and then proceeding in similar manner as Case II, we get the results. This complete the proof of the theorem. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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