Available online at http://scik.org
J. Math. Comput. Sci. 2 (2012), No. 5, 1464-1474
ISSN: 1927-5307

COINCIDENCE POINTS AND COMMON FIXED POINTS IN CONE BANACH SPACES

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Abstract: In this manuscript we obtain coincidence points and common fixed points in cone Banach spaces. Our result generalizes and extends the result of Thabet Ableljwal, Erdal Karapinar and KenanTas [3].

Keywords: Cone normed spaces, Coincidence points, Common fixed points.

2000 AMS Mathematics Subject Classification: 54H25, 47H10.

1. Introduction:

In 2007, Huang and Zhang [5] introduced the concept of cone metric space, replacing the set of real numbers by Banach space ordered by a cone and proved some fixed point theorems for function satisfying contractive conditions in these spaces. In this setting, Bogdan Rzepecki [11] generalized the fixed point theorems of Maia type [9] and Shy-Der Lin [8] considered some results of Khan and Imdad [7] Huang and Zhang [5] also discussed some properties of convergence of sequences and proved the fixed point theorems of contractive mapping for cone metric spaces: Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality

$$d(Tx, Ty) \leq k\, d(x, y)$$

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Received June14, 2012

for all $x,y \in X$, has a unique fixed point.

Recently, Thabet Abdeljawad et. al. [3] proved some fixed point theorems for self maps satisfying some contraction principles on a cone Banach space. More precisely they proved that for a closed and convex subset C of a cone Banach space with the norm $\| \|_p$, and letting $d: X \times X \to E$ with $d(x,y) = \| x-y \|_p$, if there exist a,b,c,s and $T: C \to C$ satisfies the conditions $0 \le \frac{s+a-2b-c}{2(a+b)} < 1$ and $a\, d(Tx, Ty) + b(d(x, Tx) + d(y, Ty)) + c\, d(y, Tx) \le s\, d(x,y)$ for all $x,y \in C$, then T has at least one fixed point.

Here we will give some generalization of this theorem

2. Preliminaries:

Let E be a real Banach space. A subset P of E is said to be a cone if and only if

i. P is closed, nonempty and $P \neq \{0\}$.

ii. $ax+by \in P$ for all $x,y \in P$ and non-negative real numbers a,b.

iii. $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, me can define a partial ordering $\le$ with respect to P by $x \le y$ if and only if $y-x \in P$. $x < y$ will stand for $x \le y$ and $x \neq y$, while $x \ll y$ will stand for $y-x \in \text{int } P$, where int P denotes the interior of P.

The cone P is called normal if there is a number $M > 0$ such that for all $x,y \in E$,

$0 \le x \le y$ implies $\|x\| \le M \|y\|$.

The least positive number satisfying the above is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is , if $\{x_n\}$, is a sequence such that $x_1 \le x_2 \le \ldots\ldots \le y$ for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent.

Lemma 2.1 [4, 10] (i) Every regular cone is normal.

(ii) For each $k > 1$, there is a normal cone with normal constant $K > k$

Definition 2.2 [5] Let X be a nonempty set. Then any map $d: X \times X \rightarrow E$ is said to be cone metric on X if for all $x,y,z \in X$, d satisfies.

i. $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$.

ii. $d(x,y) = d(y,x)$

iii. $d(x,y) \leq d(x,z) + d(z,y)$.

Pair (X,d) is called as cone metric space (CMS).

We denote set of all reals by R

Example 2.3 Let $E = R^2$, $P = \{(x,y) \in E : x,y \geq 0\}$ and $X = R$.

Define $d : X \times X \rightarrow E$ by $d(x,y) = (\alpha|x-y| , \beta|x-y|)$,

where α,β are positive constants. Then (X,d) is a CMS.

It is quite natural to consider cone normed spaces (CNS).

Defintion2.4 [1, 16] Let X be a linear space over R and $\|\,.\,\|_p: X \rightarrow E$ be a map which satisfies

i. $\|x\|_p > 0$ for all $x \in X$,

ii. $\|x\|_p = 0$ if and only if $x = 0$,

iii. $\|x + y\|_p \leq \|x\|_p + \|y\|_p$ for all $x,y \in X$,

iv. $\|kx\|_p = |k| \, \|x\|_p$ for all $k \in R$, Then $\|\cdot\|_p$ is called cone norm on X, and pair $(X, \|.\|_p)$ is called cone normed space (CNS).

Note that each CNS is CMS. Indeed, $d(x,y) = \|x-y\|_p$.

Definition 2.5 Let $\{x_n\}_{n\geq 1}$ be a sequence in CNS $(X, \|.\|_p)$. Then

i. It is said to be a convergent sequence if for every $c \in E$ with $c \geq 0$ there is a natural number N such that for all $n \geq N$, $\| x_n - x \|_p \leq c$ for some fixed $x \in X$.

ii. It is said to be a Cauchy sequence if for every $c \in E$ with $c \geq 0$ there is a natural number N such that for all $n,m \geq N$, $\|x_n - x_m\|_p \leq c$.

iii. CNS $(X, \|.\|_p)$ is said to be complete if every Cauchy sequence in X is convergent.

Lemma 2.6 [6] Let $(X, \|.\|_p)$ be a CNS and P be a normal cone with normal constant K. If $\{x_n\}$ is a sequence in X, then

i. $\{x_n\}$ converges to x if and only if $\|x_n - x\|_p \to 0$, as $n\to\infty$

ii. $\{x_n\}$ is a Cauchy sequence if and only if $\|x_n - x_m\|_p \to 0$ as $n , m\to\infty$.

iii. $\{x_n\}$ converges to x and sequence $\{y_n\}$ converges to y, then $\|x_n - y_n\|_p \to \|x - y\|_p$.

Lemma 2.7 [14, 15, 6] Let $(X, \|.\|_p)$ be a CNS over a cone P in E. Then

i. $Int(P) + Int(P) \subseteq Int(P)$ and $\lambda\, Int(P) \subseteq Int(P)$, $\lambda > 0$.

ii. If $c >> 0$ then there exists $\delta > 0$ such that $\|b\| < \delta$ implies $b << c$.

iii. For any given $c >> 0$ and $c_o >> 0$ there exists a natural number n_o such that $c_o/n_o << c$.

iv. If a_n ,b_n are sequences in E such that $a_n \to a$, $b_n \to b$ and $a_n \leq b_n$, for all n, then $a \leq b$.

Definition2.8 [4] Cone P is called minihedral cone if sup{x,y} exists for all $x,y \in E$ and strongly minihedral if every subset of E which is bounded from above has a supremum.

Lemma 2.9 [2] Every strongly minihedral normal cone is regular

For $T : X \to X$, the set of fixed points of T is denoted by $F(T) = \{z \in X : Tz = z\}$

Definition 2.10 [13] Let C be a closed and convex subset of a cone Banach space with the norm $\|x\|_p = d(x,0)$ and $T : C \to C$ a map. Then T is called non expansive if

$$\|Tx - Tz\|_p \le \|x - z\|_p \text{ for all } x,z \in C$$

and T is called quasi-nonexpansive if

$$\|Tx - z\|_p \le \|x - z\|_p \text{ for all } x \in C,\ z \in F(T)$$

3. Main Results :

Theorem 3.1

Let C be a closed convex subset of a cone Banach space X with norm $\|x\|_p$. Suppose $E = (E \|\ .\ \|)$ is a real Banach space and let $d : X \times X \to E$ be a mapping such that $d(x,y) = \|x - y\|_p$.

If there exist a,b,c,e and $T : C \to C$ satisfying the conditions

$$0 \le \frac{e+a-2b-c}{2a+2b+c} < 1,\ a+b+c \ne 0\ ,a+b+c > 0 \text{ and } e \ge 0 \tag{3.1}$$

$$a\, d(Tx , Ty) + b\{d(x,Tx) + d(y,Ty)\} + c\{d(y,Tx) + d(x,Ty)\} \le e\, d(x,y)$$
(3.2)

hold for all $x,y \in C$. Then T has at least one fixed point.

Proof :

Pick $x_o \in C$ and define a sequence $\{x_n\}$ in the following way :

$$x_{n+1} = \frac{x_n + Tx_n}{2}\ , n = 0, 1, 2,\ldots\ldots \tag{3.3}$$

Notice that

$$x_n - Tx_n = 2\left(x_n - \left(\frac{x_n + Tx_n}{2}\right)\right) = 2(x_n - x_{n+1}) \quad (3.4)$$

which yields that

$$d(x_n , Tx_n) = \| x_n - Tx_n \|_p = 2\|x_n - x_{n+1}\|_p = 2\, d(x_n , x_{n+1}) \quad (3.5)$$

for n = 0, 1, 2,…. Analogously, for n = 0,1,2,3…, one can get

$d(x_{n-1} , Tx_{n-1}) = 2d(x_{n-1} , x_n)$, and

$$d(x_n, Tx_{n-1}) = \frac{1}{2}\, d(x_{n-1}, Tx_{n-1}) = d(x_{n-1}, x_n), \quad (3.6)$$

and by the triangle inequality

$$d(x_n, Tx_n) - d(x_n, Tx_{n-1}) \le d(Tx_{n-1}, Tx_n). \quad (3.7)$$

We put $x = x_{n-1}$ and $y = x_n$ in inequality (3.2),

$$a\, d(Tx_{n-1}, Tx_n) + b[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] + c[d(x_n, Tx_{n-1}) + d(x_{n-1}, Tx_n)] \le e\, d(x_{n-1}, x_n). \quad (3.8)$$

for all a,b,c,e that satisfy (3.1). Taking into account (3.5) and (3.6) one can observe.

$$a\, d(Tx_{n-1}, Tx_n) + b[2d(x_{n-1}, x_n) + 2d(x_n, x_{n+1})] + c[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \le e\, d(x_{n-1}, x_n). \quad (3.9)$$

which is equivalent to

$$a\, d(Tx_{n-1}, Tx_n) \le e\, d(x_{n-1}, x_n) - 2b[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] - c[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]. \quad (3.10)$$

By using (3.7), the statement (3.10) turns into

$$a\,[d(x_n,Tx_n) - d(x_n, Tx_{n-1})] \le e\, d(x_{n-1}, x_n) - 2b[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] - c[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]. \quad (3.11)$$

Regarding (3.5) and (3.6), in (3.11),

$$2a\, d(x_n, x_{n+1}) - a\, d(x_{n-1}, x_n) \le e\, d(x_{n-1}, x_n) - 2b\, d(x_{n-1}, x_n) - 2b\, d(x_n, x_{n+1}) - c\, d(x_{n-1}, x_n) - c\, d(x_n, x_{n+1}).$$

$$\Rightarrow (2a+2b+c)\, d(x_n, x_{n+1}) \le (e+a-2b-c)\, d(x_{n-1}, x_n)$$

Since $a+b+c \neq 0$, we get $d(x_n, x_{n+1}) \leq \frac{e+a-2b-c}{2a+2b+c} d(x_{n-1}, x_n)$.

⇨ $d(x_n, x_{n+1}) \leq K\, d(x_{n-1}, x_n)$, where $K = \frac{e+a-2b-c}{2a+2b+c}$

Thus the sequence $\{x_n\}$ is a Cauchy sequence that converges to some element of C, say z. We claim that z is a fixed point of T. When we substitute $x = z$ and $y = x_n$ in (3.2).

$a\, d(Tz, Tx_n) + b\{d(z, Tz) + d(x_n, Tx_n)\} + c\{d(x_n, Tz) + d(z, Tx_n)\} \leq e\, d(z, x_n)$

Due to the equation (3.3) and $x_n \to z$, we have $Tx_n \to z$

⇨ $a\, d(Tz, z) + b\, d(z, Tz) + c\, d(z, Tz) \leq 0$ as $n \to \infty$

⇨ $(a+b+c)\; d(z, Tz) \leq 0$

⇨ $Tz = z$ as $a+b+c > 0$.

Definition 3.2 Let (X, d) be a complete metric space and S, T be self maps on X. A point $z \in X$ is said to be a coincidence point of S, T if $Sz = Tz$ and it is called common fixed point of S,T if $Sz = Tz = z$.

More over a pair (S,T) of self maps is called weakly compatible on X if they commute at their coincidence points i.e. $z \in X$, $Sz = Tz$ implies $STz = TSz$

Theorem 3.3 Let C be a closed convex subset of a cone Banach space X with norm $\| \|_p$ and let $d : X \times X \to E$ with $d(x,y) = \|x-y\|_p$. If T and S are self maps on C that satisfy the conditions.

(3.31) $T(C) \subseteq S(C)$

(3.32) S(C) is a complete subspace

(3.33) $a\, d(Tx, Ty) + b\{d(Sx, Tx) + d(Sy, Ty)\} + c\{ d(Sy, Tx) + d(Sx, Ty)\} \leq r\, d(Sx, Sy)$.

for $a+b+c \neq 0$, $0 \leq r < a+2b$, $r < b$, $a \neq r$.

hold for all $x,y \in C$, then S and T have a common coincidence point. Moreover if S and T are weakly compatible, then they have a unique common fixed point in C.

Proof : Pick $x_o \in C$. By (3.31) we can find a point in C, say x_1, such that $T(x_o) = Sx_1$. Since S, T are self maps, there exists $y_o \in C$ such that $y_o = Tx_o = Sx_1$.

Inductively we can define a sequence $\{y_n\}$ and sequence $\{x_n\}$ in C such that

(3.34) $y_n = Sx_{n+1} = Tx_n$, $n = 0,1,2,\ldots$.

We put $x= x_n$ and $y = x_{n+1}$ in inequality (3.33), it implies that

$a\, d(Tx_n, Tx_{n+1}) + b\{d(Sx_n, Tx_n) + d(Sx_{n+1}, Tx_{n+1})\} + c\{ d(Sx_{n+1}, Tx_n) + d(Sx_n, Tx_{n+1})\} \le r\, d(Sx_n, Sx_{n+1})$

⇨ $a\, d(y_n, y_{n+1}) + b\{ d(y_{n-1}, y_n) + d(y_n, y_{n+1})\} + c\{ d(y_n, y_n) + d(y_{n-1}, y_{n+1})\} \le r\, d(y_{n-1}, y_n)$

By using triangle inequality and suitable choices of a,b,c, it implies,

$(a+b)\, d(y_n, y_{n+1}) + b\, d(y_{n-1}, y_n) + c\, d(y_{n-1}, y_n) + c\, d(y_n, y_{n+1}) \le r\, d(y_{n-1}, y_n)$

⇨ $d(y_n, y_{n+1}) \le \frac{r-b-c}{a+b+c}\, d(y_{n-1}, y_n) = k\, d(y_{n-1}, y_n)$

where $k = \frac{r-b-c}{a+b+c}$. Similarly $d(y_{n-1}, y_n) \le k\, d(y_{n-2}, y_{n-1})$

Since $0 \le r < a+2b$, $r < b$, then $0 \le k < 1$.

By routine calculations,

(3.35) $d(y_n, y_{n+1}) \le k^n\, d(y_o, y_1)$.

We claim that $\{y_n\}$ is a Cauchy sequence. Let $n > m$,

Then by (3.35) and the triangle inequality.

$d(y_n, y_m) \le d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \ldots\ldots + d(y_{m+1}, y_m)$.

$\le k^{n-1} d(y_o, y_1) + k^{n-2}\, d(y_o, y_1) + \ldots\ldots + k^m\, d(y_o, y_1)$.

$\le \quad k^m \quad d(y_o, y_1)$

$\overline{\quad}$
(1-k)

Therefore $\{y_n\}$ is a Cauchy sequence. Since S(C) is complete, then $\{y_n = Sx_{n+1} = Tx_n\}$ converges to some point in S(C) , say z

Now by replacing x with p and y with x_{n+1} in (3.33), we get

$a\, d(Tp, Tx_{n+1}) + b\{Sp, Tp) + d(Sx_{n+1}, Tx_{n+1})\} + c\{d(Sx_{n+1}, Tp) + d(Sp, Tx_{n+1})\} \leq r\, d(Sp, Sx_{n+1})$.

⇨ $a\, d(Tp, y_{n+1}) + b\{d(z, Tp) + d(y_n, y_{n+1})\} + c\{ d(y_n, Tp) + d(z, y_{n+1})\} \leq r\, d(z, y_n)$

As $n\rightarrow\infty$, it becomes

$a\, d(Tp, z) + b\, d(z, Tp) + c\, d(z, Tp) \leq 0$.

Since $a+b+c \neq 0$, then Tp = z. Hence Tp = z = Sp.

i.e. p is a coincidence point of S and T.

If S and T are weakly compatible, then they commute at a coincidence point. Therefore ,Tp = z = Sp ==>STp = TSp for some $p \in C$, which implies Tz = Sz. We claim that z is a common fixed point of S and T.

Substitute x = p and y = Tp = z in (3.33), to give

$a\, d(Tp, TTp) + b\{d(Sp, Tp) + d(STp, TTp)\} + c\, \{d(STp, Tp) + d(Sp, TTp)\} \leq r\, d(Sp, STp)$.

which is equivalent to

$a\, d(z, Tz) + b\{ d(z, z) + d(Sz, Tz)\} + c\{ d(Sz, z) + d(z, Tz)\} \leq r\, d(z, Sz)$.

⇨ $(a + 2c - r)\, d(z, Tz) \leq 0$.

Since $a + 2c - r \neq 0$, then z = Tz = Sz.

To prove uniqueness, suppose the contrary, that w is another common fixed point of S and T. Put x by z and y by w in the inequality (3.33), one can get.

$a\, d(Tz, Tw) + b\{ d(Sz, Tz) + d(Sw, Tw)\} + c\{d(Sw, Tz) + d(Sz, Tw)\} \leq r\, d(Sz, Sw)$.

$\Rightarrow$ a d(z, w) + 2c d(z, w) $\leq$ r d(z, w)

$\Leftrightarrow$ (a + 2c – r) d(z, w) $\leq$ 0.

which is a contradiction since a + 2c – r $\neq$ 0. Hence the common fixed point of S and T is unique.

Acknowledgment :

The authors express their gratitude to the referees for constructive and useful remarks and suggestions.

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