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ON THE PATH COSPECTRAL GRAPHS AND PATH SIGNLESS LAPLACIAN MATRIX OF GRAPHS

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Abstract. In this paper, we explore path cospectral graphs and obtain some result concerning to these graphs. Also, we give nonisomorphic path cospectral graphs on $5 \leq n \leq 6$ vertices and $3 \leq m \leq 10$ edges. Further, we define path signless Laplacian matrix of a graph and investigate its properties.

Keywords: Real symmetric matrix; eigenvalues; cospectral graphs.

2010 AMS Subject Classification: 05C50.

1. Introduction

Let G be a graph with $V(G) = \{1,...,n\}$ and $E(G) = \{e_1,...,e_n\}$. The adjacency matrix of G, denoted by $A(G)$, is the $n \times n$ matrix defined as follows. The rows and the columns of $A(G)$ are indexed by $V(G)$. If $i \neq j$ then the (i, j)-entry of $A(G)$ is 0 for vertices i and j non-adjacent, and the (i, j)-entry is 1 for i and j adjacent. If G is simple, the (i,i)-entry of $A(G)$ is 0 for $i = 1,\ldots,n$. We often denoted $A(G)$ simply by A. The eigenvalues of a matrix A are called as the eigenvalues of the graph G. The spectrum of a finite graph G is its set of eigenvalues together with their multiplicities. Several properties of eigenvalues of graphs and their applications have

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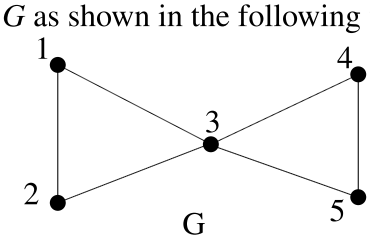
been explored in [3,4].

We define a new matrix, called the path matrix [1,2] of a graph, in the following way.

Definition 1.1. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. Define the matrix $P = (p_{ij})$ of size $n \times n$ such that p_{ij} is equal to the maximum number of vertex disjoint paths from v_i to v_j if $i \neq j$, and $p_{ij} = 0$ if $i = j$.

We call $P = P(G)$ as the *path matrix* of the graph G. By definition, P is a real and symmetric matrix. Therefore its eigenvalues are real. We call the eigenvalues of P the *path eigenvalues* of G, forming its path spectrum $Spec_P(G)$. For convenience, the eigenvalues of the adjacency matrix of G will be refereed to as the ordinary eigenvalues of G, forming its ordinary spectrum $\text{Spec}(G)$.

Example 1.2. Consider the graph G as shown in the following figure and its path matrix P.



$$\mathbf{P} = \begin{bmatrix} 0 & 2 & 2 & 1 & 1 \\ 2 & 0 & 2 & 1 & 1 \\ 2 & 2 & 0 & 2 & 2 \\ 1 & 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 2 & 0 \end{bmatrix}.$$

The characteristic polynomial of P is $= C_P(x) = |P - xI| = -x(x+2)^2(x^2 - 4x - 16)$. The path eigenvalues of G are $6.472,\ 0,\ -2,\ -2$ and -2.472. The ordinary eigenvalues of G are $2.562,\ 1,\ -1,\ -1$ and -1.562. For terminology in graph theory, we refer [3,4,6] and for matrix theory, we refer [5].

2. PATH COSPECTRAL GRAPHS

Proposition 2.1. All trees on n vertices are path cospectral.

Proof. Let T_1 and T_2 be two trees on n vertices. Then $P(T_1) = P(T_2)$ because the maximum number of vertex disjoint paths between any two vertices of T_i is 1, for $i = 1,2$. Thus, $Spec_P(T_1) = Spec_P(T_2)$. □

Proposition 2.2. All graphs on n vertices each of which has exactly one cycle of length k, where k is fixed and $3 \leq k \leq n$ are cospectral.

Proof. Let G_1 and G_2 be two graphs on n vertives each of which has exactly one cycle of length k, where k is fixed and $3 \leq k \leq n$. After a relabeling of vertices (rows and columns) of G_1 and G_2 if necessary, we arrive at a situation where $P(G_1) = P(G_2)$. Thus, $Spec_P(G_1) = Spec_P(G_2)$. □

Definition 2.3. A graph G of order n is called a bicyclic graph if G is connected and the number of edges of G is $n+1$. Any bicyclic graph on n vertices has minimum two cycles and maximum three cycles.

Let G be a bicyclic graph on n vertices without pendent vertices, then there are three types of such bicyclic graphs.

(1) G has two vertex disjoint cycles, joined by a path.

(2) G has two cycles with one vertex in common.

(3) G has two cycles with more than one vertex in common.

These three types of bicyclic graphs without pendent vertices are depicted in the following figure:

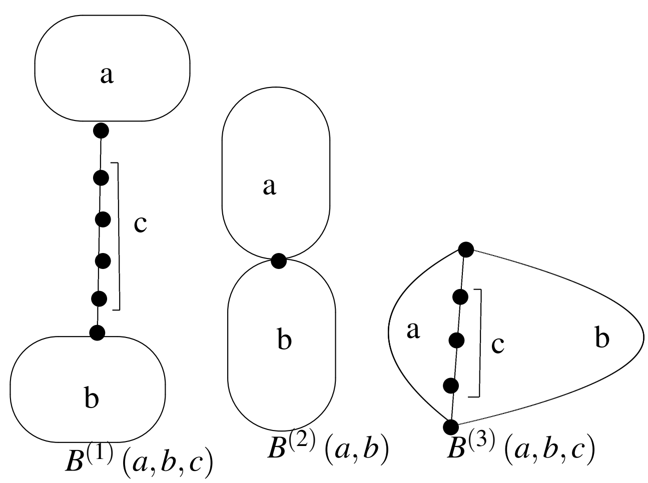


Fig. Types of bicyclic graphs

Note that $B^{(1)}(a,b,c)$ possesses $a+b+c$ vertices, $B^{(2)}(a,b)$ possesses $a+b-1$ vertices, and $B^{(3)}(a,b,c)$ possesses $a+b-c-2$ vertices, and that $a,b \geq 3, c \geq 0$.
For $i = 1,2,3$, denote by $\mathbf{B}^{(i)}$ the set of all connected bicyclic graphs without pendant vertices, of type $B^{(i)}$.
Denote by $\boldsymbol{B}_n$ the set of connected bicyclic graphs with n vertices.

Proposition 2.4. Let G_1 and G_2 be two bicyclic graphs on n vertices. Suppose that G_1 and G_2 have same cycle structure and the corresponding cycles have same length. Then G_1 and G_2 are path cospectral.

Proof. We make the following three cases to prove the proposition.
case 1. G_1 and G_2 have exactly two vertex disjoint cycles. i.e. $G_1, G_2 \in B^{(1)}$.
Let C_1 and C_2 be two cycles in G_i $(i = 1,2)$ of lengths n_1 and n_2, respectively and k be the number of vertices which are not on any of the cycles C_1 and C_2. Label the verices of C_1 as $1,2,...,n_1$, label the verices of C_2 as $n_1+1, n_1+2, ..., n_1+n_2$ and label the remaining vertices as $n_1+n_2+1, n_1+n_2+2, ..., n_1+n_2+k = n$. Then the path matrix $P(G_i)$ $(i = 1,2)$ can be writeen as

$$\mathbf{P(G_i)} = \begin{bmatrix}
0 & 2 & 2 & ... & 2 & 1 & 1 & ... & 1 & 1 & 1 & ... & 1 & 1 \\
2 & 0 & 2 & ... & 2 & 1 & 1 & ... & 1 & 1 & 1 & ... & 1 & 1 \\
2 & 2 & 0 & ... & 2 & 1 & 1 & ... & 1 & 1 & 1 & ... & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2 & 2 & 2 & ... & 0 & 1 & 1 & ... & 1 & 1 & 1 & ... & 1 & 1 \\
1 & 1 & 1 & ... & 1 & 0 & 2 & ... & 2 & 1 & 1 & ... & 1 & 1 \\
1 & 1 & 1 & ... & 1 & 2 & 0 & ... & 2 & 1 & 1 & ... & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & ... & 1 & 2 & 2 & ... & 0 & 1 & 1 & ... & 1 & 1 \\
1 & 1 & 1 & ... & 1 & 1 & 1 & ... & 1 & 0 & 1 & ... & 1 & 1 \\
1 & 1 & 1 & ... & 1 & 1 & 1 & ... & 1 & 1 & 0 & ... & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & ... & 1 & 1 & 1 & ... & 1 & 1 & 1 & ... & 0 & 1 \\
1 & 1 & 1 & ... & 1 & 1 & 1 & ... & 1 & 1 & 1 & ... & 1 & 0
\end{bmatrix}.$$

case 2. G_1 and G_2 have two cycles with one vertex in common. i.e. $G_1, G_2 \in B^{(2)}$.
Let C_1 and C_2 be two cycles in G_i $(i = 1,2)$ of lengths n_1 and n_2, respectively. Let $v \in V(C_1) \cap V(C_2)$. Label v as 1, label the remaining (other than v_1) verices of C_1 as $2,3,...,n_1$, label the remaing vertices of C_2 as $n_1+1, n_1+2, ..., n_1+n_2-1 = n$. Then the path matrix $P(G_i)$ $(i = 1,2)$ is given by

$$\mathbf{P(G_i)} = \begin{bmatrix} 0 & 2 & 2 & ... & 2 & 2 & 2 & ... & 2 & 2 \\ 2 & 0 & 2 & ... & 2 & 1 & 1 & ... & 1 & 1 \\ 2 & 2 & 0 & ... & 2 & 1 & 1 & ... & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & 2 & ... & 0 & 1 & 1 & ... & 1 & 1 \\ 2 & 1 & 1 & ... & 1 & 0 & 2 & ... & 2 & 2 \\ 2 & 1 & 1 & ... & 1 & 2 & 0 & ... & 2 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 1 & 1 & ... & 1 & 2 & 2 & ... & 0 & 2 \\ 2 & 1 & 1 & ... & 1 & 2 & 2 & ... & 2 & 0 \end{bmatrix}.$$

case 3. G_1 and G_2 have two cycles C_1 and C_2 of lengths n_1 and n_2, respectively having k vertices common. i.e. $G_1, G_2 \in B^{(3)}$.
Label the vertices of C_1 as $1,2,...,k,k+1,...,k+(n_1-k) = n_1$, label the vertices of C_2 as $1,2,...,k,k+(n_1-k+1),...,k+(n_2-k) = n_2$. Then the path matrix $P(G_i)$ $(i = 1,2)$ has the form

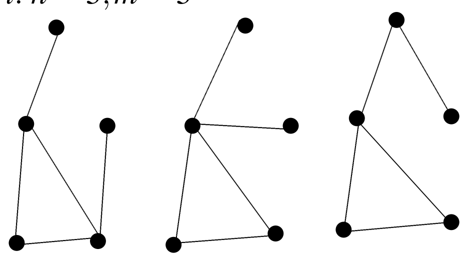
$$\mathbf{P(G_i)} = \begin{bmatrix} 0 & 2 & ... & 3 & 2 & ... & 2 & 2 & ... & 2 \\ 2 & 0 & ... & 2 & 2 & ... & 2 & 2 & ... & 2 \\ ... & ... & \ddots & ... & ... & ... & ... & ... & ... & ... \\ 3 & 2 & ... & 0 & 2 & ... & 2 & 2 & ... & 2 \\ 2 & 2 & ... & 2 & 0 & ... & 2 & 2 & ... & 2 \\ ... & ... & ... & ... & ... & \ddots & ... & ... & ... & ... \\ 2 & 2 & ... & 2 & 2 & ... & 0 & 2 & ... & 2 \\ 2 & 2 & ... & 2 & 2 & ... & 2 & 0 & ... & 2 \\ ... & ... & ... & ... & ... & ... & ... & ... & \ddots & ... \\ 2 & 2 & ... & 2 & 2 & ... & 2 & 2 & ... & 0 \end{bmatrix}.$$

In all the three cases, we can see that the path matrices of $P(G_1)$ and $P(G_2)$ are same. Hence G_1 and G_2 are path cospectral. □

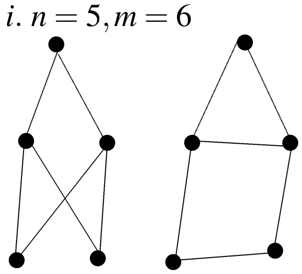
Now, we draw nonisomorphic path cospectral graphs on $5 \leq n \leq 6$ vertices and $3 \leq m \leq 10$ edges.

2.5. Pairs of Nonisomorphic Path Cospectral Graphs on $5 \leq n \leq 6$ Vertices and $5 \leq m \leq 10$ Edges

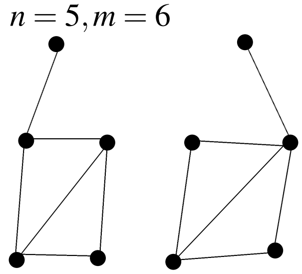
$i.\ n = 5, m = 5$



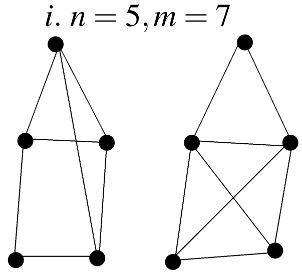
$i.\ n = 5, m = 6$



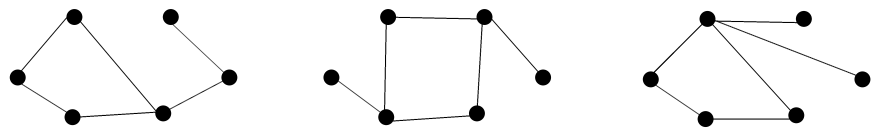
$ii.\ n = 5, m = 6$



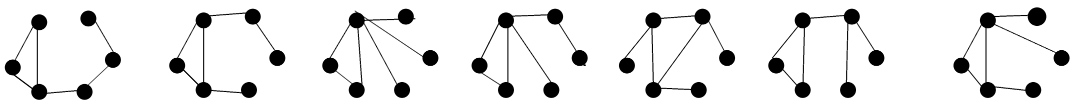
$i.\ n = 5, m = 7$



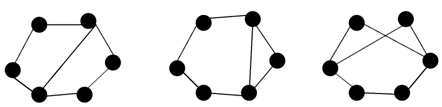
$i.\ n = 6, m = 6$



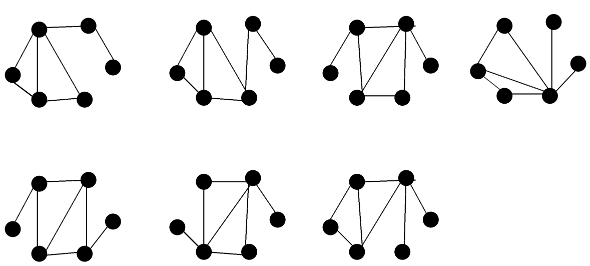
$ii.\ n = 6, m = 6$



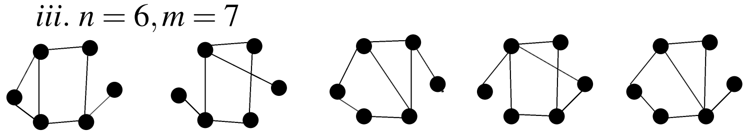
$i.\ n = 6, m = 7$



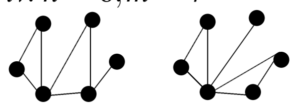
$ii.\ n = 6, m = 7$



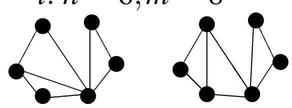
$iii.\ n = 6, m = 7$



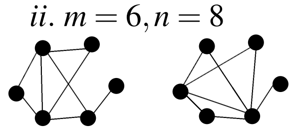
iv. $n = 6, m = 7$



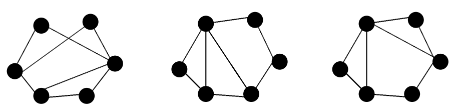
i. $n = 6, m = 8$



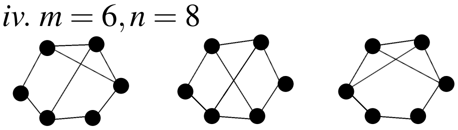
ii. $m = 6, n = 8$



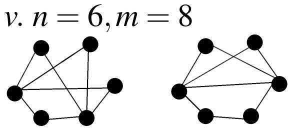
iii. $n = 6, m = 8$



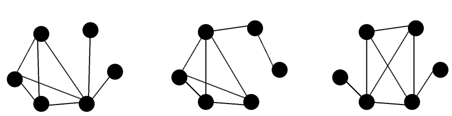
iv. $m = 6, n = 8$



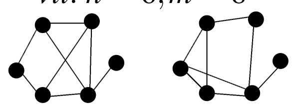
v. $n = 6, m = 8$



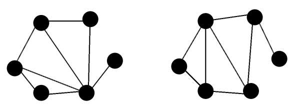
vi. $n = 6, m = 8$



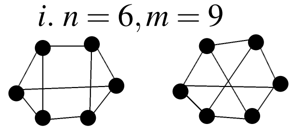
vii. $n = 6, m = 8$



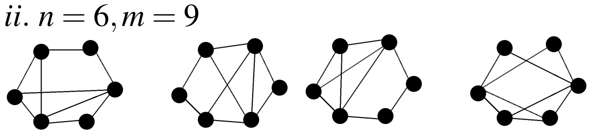
viii. $m = 6, n = 8$



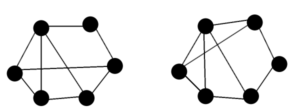
i. $n = 6, m = 9$



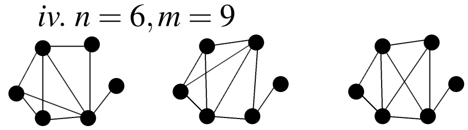
ii. $n = 6, m = 9$



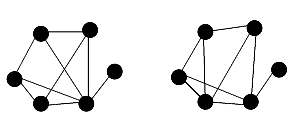
iii. $n = 6, m = 9$



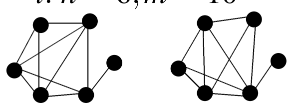
iv. $n = 6, m = 9$



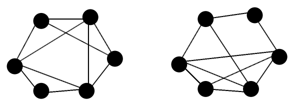
v. $n = 6, m = 9$



i. $n = 6, m = 10$



ii. $n = 6, m = 10$

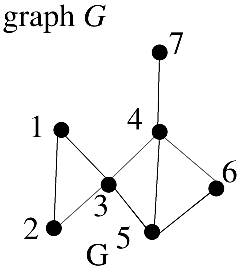


3. PATH SIGNLESS LAPLACIAN MATRIX

The ordinary signless Laplacian matrix [7, 8] of the graph G is defined by $SL(G) = D(G) + A(G)$, where $A(G)$ is the adjacency matrix of a graph G and $D(G)$ is the diagonal matrix of vertex degrees of the graph G.

Definition 3.1. The path signless Laplacian matrix of a graph G is defined as $D + P$, where D is the diagonal matrix of vertex degrees and P is the path matrix of G. We denote the path signless Laplacian matrix of G by $PSL(G)$.

Example 3.2. Consider the following graph G



Then the path signless laplacian matrix of G is

$$\mathbf{PSL(G)} = \begin{bmatrix} 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & 2 & 4 & 2 & 2 & 2 & 1 \\ 1 & 1 & 2 & 4 & 3 & 2 & 1 \\ 1 & 1 & 2 & 3 & 3 & 2 & 1 \\ 1 & 1 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The eigenvalues of $PSL(G)$ are $12.080, 3.437, 1.251, 0.735, 0.343, 0.153$ and 0. The signless Laplacian matrix of G, $SL(G)$ is given by

$$\mathbf{SL(G)} = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues of $SL(G)$ are $6.318, 4.269, 2.681, 2, 1.160, 1, 0.572$.

Proposition 3.3. Let G be a r-regular, r-connected graph with n vertices. Then the eigenvalues of path signless Laplacian matrix of a graph G are rn with multiplicity 1 and 0 with multiplicity $n-1$.

Proof. The path signless Laplacian matrix of a graph G is $D+P = rJ_n$. The eigenvalues of J_n are n with multiplicity 1 and 0 with multiplicity $n-1$. Hence the eigenvalues of path signless Laplacian matrix of a graph G are rn with multiplicity 1 and 0 with multiplicity $n-1$. □

Proposition 3.4. All eigenvalues of path signless Laplacian matrix of a connected graph G are non negative.

Proof. We prove that the path signless Laplacian matrix $D+P$ of G is a positive semidefinite matrix. Let $D+P = (a_{ij})$ and let $x \in \mathbb{R}^n$.
$x^T(D+P)x = \sum_{i=1}^n a_{ii}x_i^2 + 2[a_{12}x_1x_2 + a_{13}x_1x_3 + ... + a_{1n}x_1x_n + a_{23}x_2x_3 + ... + a_{n-1n}x_{n-1}x_n]$.
Let $k = \min\{a_{ij}\} \geq 0$. Thus $x^T(D+P)x \geq k[\sum_{i=1}^n x_i^2 + 2[x_1x_2 + x_1x_3 + ... + x_1x_n + x_2x_3 + ... + x_{n-1}x_n]] = k(x_1 + x_2 + ... + x_n)^2 \geq 0$. $D+P$ is positive semidefinite. Hence all eigenvalues of path signless Laplacian matrix are non negative. □

The following Theorem is known.

Theorem 3.5. Let $A, B \in M_n$ be symmetric where B is positive semidefinite. Let $\lambda_1(A) \geq \lambda_2(A) \geq ... \geq \lambda_n(A)$ and $\lambda_1(A+B) \geq \lambda_2(A+B) \geq ... \geq \lambda_n(A+B)$ be the eigenvalues of A and $A+B$, respectively. Then $\lambda_k(A) \leq \lambda_k(A+B)$, for $k = 1, 2, ..., n$.

We have the following result.

Proposition 3.6. Let G be a connected graph on n vertices with path matrix P and let D be the diagonal matrix of vertex degrees. Then $\lambda_k(P) \leq \lambda_k(D+P)$, for $k = 1,2,...,n$.

Proof. We show that D is positive semidefinite. Let $\mathbf{x} = (x_1,x_2,...,x_n) \in \mathbb{R}^n$ and $D = [d_1\ d_2\ ...\ d_n]$. Then $x^T Dx = d_1x_1^2 + d_2x_2^2 + ... + d_nx_n^2 \geq 0$ since $d_i > 0$, for $i = 1,2,...,n$. This implies that D is positive semidefinite. By Theorem 3, $\lambda_k(P) \leq \lambda_k(D+P)$, for $k = 1,2,...,n$. □

In the following Theorem, we obtain a lower bound for the largest eigenvalue of $PSL(G)$.

Theorem 3.7. Let G be a simple connected graph with n vertices and $PSL(G)$ be its path signless Laplacian matrix. Let $\Delta(G) = \max_i d_i$ and $\lambda_1(G)$ be the largest eigenvalue of $PSL(G)$. Then $\Delta(G) \leq \lambda_1(G)$. Equality holds for a star graph S_n.

Proof. Let $PSL = (a_{ij})$ be the path signless Laplacian matrix of G and $\mathbf{x} = (x_1,x_2,...,x_n)$ be an eigenvector of G corresponding to the eigenvalue $\lambda_1(G)$. Then $PSLx = \lambda_1 x$. This implies that $\lambda_1 x_i = a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n$. Now, all $a_{ij} \geq 0$ and since $\lambda_1 > 0$, by Perron-Frobenius theory, $\mathbf{x} > 0$. Therefore, $\lambda_1 x_i \geq x_1 + x_2 + ... + x_n$. We can write this as $\lambda_1 x_i \geq \sum_{j=1}^n x_j$, $i = 1,2,...,n$. $\lambda_1 x_1 + \lambda_1 x_2 + ... + \lambda_1 x_n \geq n\sum_{j=1}^n x_j$. Thus $\lambda_1 \sum_{i=1}^n x_i \geq n\sum_{j=1}^n x_j$ and $\lambda_1 \geq n \geq n-1$. In a simple graph G, $\Delta \leq n-1$. Hence $\lambda_1(G) \geq \Delta(G)$.
Now, if $G = S_n$, then $\lambda_1(S_n) = n-1 = \Delta(S_n)$. □

Theorem 3.8. Let A and B be two symmetric matrices of size n. Then for any $1 \leq k \leq n$,
$\sum_{i=1}^k \lambda_i(A+B) \leq \sum_{i=1}^k \lambda_i(A) + \sum_{i=1}^k \lambda_i(B)$
where, for a matrix M, $\lambda_i(M)$ denotes the largest i^{th} eigenvalue of M.

Corollary 3.9. Let G be a graph with n vertices, m edges and $PSL(G)$ be its path signless Laplacian matrix. Then $1 \leq k \leq n$,

$$\sum_{i=1}^{k} \lambda_i(PSL(G)) \leq 2m.$$

Proof. We know that, $\sum_{i=1}^n \lambda_i(P) = 0$, $\sum_{i=1}^n \lambda_i(D) = \sum_{i=1}^n d_i = 2m$. By Theorem 3, $\sum_{i=1}^k \lambda_i(PSL(G)) = \sum_{i=1}^k \lambda_i(P+D) \leq \sum_{i=1}^k \lambda_i(P) + \sum_{i=1}^k \lambda_i(D) \leq \sum_{i=1}^n \lambda_i(P) + \sum_{i=1}^n \lambda_i(D) = 0 + \sum_{i=1}^n \lambda_i(D) = 2m$. □

The proof of the following result for the path signless Laplacian matrix is along the lines of the proof of Perron-Frobenius theorem.

Theorem 3.10. Let G be a connected graph with $n \geq 2$ vertices, and let P be the corresponding path matrix and let D be a diagonal matrix of vertex degrees. Then the following statements hold:

(1) $D+P$ has an eigenvalue $\lambda > 0$ and an associated eigenvector $x > 0$. This eigenvalue will be referred to as the Perron eigenvalue of $D+P$.

(2) for any eigenvalue $\mu \neq \lambda$ of $D+P$, $-\lambda \leq \mu \leq \lambda$.

(3) if u is an eigenvector of $D+P$ for the eigenvalue λ, then $u = \alpha x$ for some α.

Theorem 3.11. Let G be a connected graph with n vertices and let P be the corresponding path matrix. Let D be a diagonal matrix of vertex degrees and let $\tau_1 \leq \tau_2 \leq ... \leq \tau_n$ be the eigenvalues of $D-P$. Then the algebraic multiplicity of τ_1 is 1 and there is a positive eigenvector of $D-P$ corresponding to τ_1.

Proof. Let $A = kI - (D-P)$, where $k > 0$ is sufficiently large so that $kI - D \geq 0$. The eigenvalues of A are $k-\tau_1 \geq k-\tau_2 \geq ... \geq k-\tau_n$. Since $A = (kI-D)+P$, by Theorem 3, $k-\tau_1$, which is the Perron eigenvalue of A, has algebraic multiplicity 1 and there is a positive eigenvector corresponding to this eigenvalue. It follows that τ_1, as an eigenvalue of $D-P$, has algebraic multiplicity 1 with an associated positive eigenvector. □

Conclusion. In the present paper, the concepts of path cospectral graphs and path signless laplacian matrix of graphs are given and studied.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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