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FIXED POINT THEOREMS OF WEAKLY COMPATIBLE MAPPINGS IN b_2-METRIC SPACE SATISFYING (ϕ,ψ) CONTRACTIVE CONDITIONS

THOKCHOM CHHATRAJIT SINGH[1,*], YUMNAM ROHEN SINGH[2], K. ANTHONY SINGH[3]

[1]Department of Mathematics, Manipur Technical University Takyelpat, Imphal-795004, India

[2]Department of Mathematics, National Institute of Technology Imphal, Manipur-795004, India

[3]Department of Mathematics, D.M. College of Science Imphal, Manipur-795001, India

Abstract. Generalising the concept of 2-metric space and b-metric space. Mustafa et. al. (Z. Mustafa. V, Paraneh, J. Razaei and Z. Kadulberg: b_2-metric spaces and some fixed point theorems, Fixed Point Theory and Applications 2014, 2014:144) introduced b_2-metric space. In this paper, we prove a common fixed point theorem for two pairs of weakly compatible mappings satisfying (ϕ,ψ) contractive condition in b_2-metric space. An example is also given to illustrate our result.

Keywords: 2-metric space; b-metric space; b_2-metric space; weakly compatible.

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1. INTRODUCTION

The study of metric fixed point have been an important research area for the last many years and many researchers had contributed a lot in this area. In order to strengthen this area various generalizations of metric space had been introduced. G*ä*hler [1] introduced a generalization of metric space. He called it 2-metric. But the claim of G*ä*hler that a 2-metric is generalization

*Corresponding author
E-mail address: chhatrajit@mtu.ac.in
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of usual metric was objected by many authors because there is no relation between these two functions.

Another generalization of metric space was introduced by Baktin [2] and extensively used by [3, 4]. For more results on generalization of metric space, one can see the research papers in [5–25] and references therein.

Generalizing the concept of both 2-metric and b-metric spaces. Mustafa et. al. [16] introduced the notion of b_2-metric space. They also noted that under certain condition b_2-metric space reduces to 2-metric space.

In this note, we prove a common fixed point theorem for two pairs of weakly compatible mappings satisfying (ϕ, ψ) contractive condition in b_2-metric space.
Following definitions was given by Gähler.

Definition 1.1. *[1] Let X be a nonempty set and let $d : X^3 \to R$ be a map satisfying the following conditions:*

(1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.
(2) If at least two of three points x,y,z are the same, then $d(x,y,z) = 0$.
(3) The symmetry: $d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x)$ for all $x,y,z \in X$
(4) The rectangle inequality: $d(x,y,z) = d(x,y,t) + d(y,z,t) + d(z,x,t)$ for all $x,y,z,t \in X$

Then d is called a 2-metric on X and (X,d) is called a 2-metric space.

2. PRELIMINARIES

Following definitions was given by Czerwik.

Definition 2.1. *[3] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a b-metric on X if for all $x,y,z \in X$, the following conditions hold:*

(1) $d(x,y) = 0$ if and only if $x = y$
(2) $d(x,y) = d(y,x)$
(3) $d(x,z) \leq s[d(x,y) + d(y,z)]$

In this case, the pair (X,d) is called a b-metric space.

Note that a b-metric is not always a continuous function of its variables (see,e.g., [4],Example 2), whereas an ordinary metric is.

Definition 2.2. *[1] Let (X,d) be a 2-metric space $a,b \in X$ and $r > 0$. The set $B(a,b,r) = \{x \in X : d(a,b,x) < r\}$ is called a 2-ball centered at a and b with radius r.*

The topology generated by the collection of all 2-balls as a sub-basis is called a 2-metric topology on X.

Remark 2.1. *[16]*

1. *(1) It is straightforward from Definition 1.2 that every 2-metric is non-negative and every b-metric space contains atleast three distinct points.*
2. *(2) A 2-metric d(x,y,z) is sequentially continuous in each argument. Moreover, if a 2-metric d(x, y,z) is sequentially continuous in two arguments, then it is sequentially continuous in all three arguments; see [6].*
3. *(3) A convergent sequence in a 2-metric space need not be a Cauchy sequence; see [6].*
4. *(4) In a 2-metric space (X,d), every convergent sequence is a Cauchy sequence if d is continuous; see [6].*
5. *(5) There exists a 2-metric space (X,d) such that every convergent sequence in it is a Cauchy sequence but d is not continuous; see [6]*

Following definitions was given by Mustafa et. al. [16]

Definition 2.3. *[16] Let X be a nonempty set, $s \geq 1$ be a real number and let $d : X^3 \to R$ be a map satisfying the following conditions:*

1. *(1) For every pair of distinct points $x,y \in X$, there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.*
2. *(2) If at least two of three points x, y, z are the same, then $d(x,y,z) = 0$*
3. *(3) The symmetry: $d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x)$ for all $x,y,z \in X$.*
4. *(4) The rectangle inequality: $d(x,y,z) \leq s[d(x,y,t) + d(y,z,t) + d(z,x,t)]$ for all $x,y,z,t \in X$.*

Then d is called a b_2-metric space with parameter s.

Obviously, for $s = 1, b_2$-metric reduces to $2-$metric space.

Definition 2.4. *[16] Let $\{x_n\}$ be sequence in a b_2-metric space (X,d).*

(1) $\{x_n\}$ is said to be b_2-convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$ if for all $a \in X$, $\lim_{n\to\infty} d(x_n,x,a) = 0$.

(2) $\{x_n\}$ is said to be b_2- Cauchy sequence in X if for all $a \in X$, $\lim_{n\to\infty} d(x_n,x_m,a) = 0$.

(3) (X,d) is said to b_2-complete if every b_2- Cauchy sequence is b_2-convergent sequence in X.

Example 2.1. *[16] Let $X = [0,\infty)$ and $d(x,y,z) = [xy+yz+zx]^p$ if $x \neq y \neq z \neq x$and otherwise $d(x,y,z) = 0$, where $p \geq 1$ is a real number. Evidently, from convexity of function $f(x) = x^p$ for $x \geq 0$, then by Jensen inequality, we have*

$$(a+b+c)^p \leq 3^{p-1}(a^p+b^p+c^p)$$

So, one can obtain the result that (X,d) is a b_2-metric space with $s \leq 3^{p-1}$.

Example 2.2. *[16] Let a mapping $d : R^3 \to [0,\infty)$ be defined by*

$$d(x,y,z) = \min\{|x-y|,|y-z|,|z-x|\}$$

Then d is a 2-metric on R, i.e., the following inequality holds:

$$d(x,y,z) = d(x,y,t)+d(y,z,t)+d(z,x,t)$$

for arbitrary real numbers x,y,z,t. Using convexity of the function $f(x) = x^p$ on $[0,\infty)$ for $p \geq 1$, we obtain that

$$d_p(x,y,z) = \big[\min\{|x-y|,|y-z|,|z-x|\}\big]^p$$

is a b_2-metric on R with $s < 3^{p-1}$.

Definition 2.5. *[16] Let (X,d) and (X',d') be two b_2-metric spaces and let $f : X \to X'$ be a mapping. Then f is said to be b_2-continuous at a point $z \in X$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $d(z,x,a) < \delta$ for all $a \in X$ imply that $d'(fz,fx,a) < \varepsilon$. The mapping f is b_2-continuous on X if it is b_2-continuous at all $z \in X$.*

Proposition 2.1. *[16] Let (X,d) and (X',d') be two b_2-metric spaces. Then a mapping $f : X \to X'$ is b_2-continuous at a point $x \in X$ if and only if it is b_2-sequentially continuous at x; that is, whenever $\{x_n\}$ is b_2-convergent to x, $\{fx_n\}$ is b_2-convergent to f(x).*

We will need the following simple lemma about the b_2-convergent sequences in the proof of our main results.

Lemma 2.1. *[16] Let (X,d) be a b_2-metric space and suppose that $\{x_n\}$ and $\{y_n\}$ are b_2-convergent to x and y, respectively. Then we have*

$$\frac{1}{s^2}d(x,y,a) \leq \lim_{n\to\infty}\inf d(x_n,y_n,a) = \lim_{n\to\infty}\sup d(x_n,y_n,a) = s^2 d(x,y,a)$$

for all $a \in X$. In particular, if $y_n = y$ is constant, then

$$\frac{1}{s}d(x,y,z) = \lim_{n\to\infty}\inf d(x_n,y,a) = \lim_{n\to\infty}\sup d(x_n,y,a) = sd(x,y,a)$$

for all $a \in X$.

Definition 2.6. *[16] A function $\phi : [0,+\infty) \to [0,+\infty)$ is called an altering distance function, if the following properties hold:*

(1) ϕ is continuous and nondecreasing

(2) $\phi(t) = 0$ if and only if t = 0

Let (X,d) be a b_2-metric space and let $f : X \to X$ be a mapping. For $x,y,a \in X$, set

$$M_a(x,y) = \max\left\{d(x,y,a), d(x,fx,a), d(y,fy,a), \frac{d(x,fy,a)+d(y,fx,a)}{2s}\right\}$$

and

$$N_a(x,y) = \max\left\{d(x,fx,a), d(x,fy,a), d(y,fy,a), d(y,fx,a), d(y,fy,a)\right\}$$

Definition 2.7. *[16] Let (X,d) be a b_2-metric space. We say that a mapping $f : X \to X$ is generalized $(\phi,\psi)_{s,a}$-contractive mapping if there exist two altering distance functions ψ and ϕ such that $\psi\big(sd(fx,fy,a)\big) \leq \psi\big(M_a(x,y)\big) - \phi\big(M_a(x,y)\big)$ for all $x,y,a \in X$.*

Definition 2.8. *[16] Let (X,d) be a b_2-metric space. Then the mappings $f,g : X \to X$ are weakly compatible if for every $x \in X$, $fgx = gfx$ holds whenever fx = gx.*

Definition 2.9. *Let (X,d) be a b_2-metric space.Two mappings f and g are said to be compatible if*

$$\lim_{n\to\infty} d(fgx_n, gfx_n, a) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$$

for some points $x \in X$.

3. MAIN RESULTS

Now we prove the following theorem.

Theorem 3.1. *Let $(X,\leq)$ be partially ordered set. Suppose that there exists a b_2-metric d on X such that (X,d) is a complete b_2-metric space. Also let self-mappings f, g, S,T on X satisfying the following conditions*

$$\psi\big(2s^4 d(fx,gy,a)\big) \leq \psi\big(M(x,y)\big) - \phi\big(M(x,y)\big) \tag{1}$$

for all comparable elements $x,y,z \in X$, where $\phi,\psi : [0,\infty) \to [0,\infty)$ are two mappings such that ψ is a continuous nondecreasing, ϕ is a lower semi-continuous function with $\psi(t) = \phi(t) = 0$ if and only if t = 0, and

$$M(x,y) = \max\Big\{d(Sx,Ty,a), d(fx,Sx,a), d(gy,Ty,a), d(fx,gy,a), \frac{d(fx,Ty,a)+d(gy,Sx,a)}{2s}\Big\}$$

If f, g are dominating S, T are dominating with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ and for a non-decreasing sequence $\{x_n\}$ with $y_n \leq x_n$ for all n and $y_n \to u$ implies that $u \leq x_n$ and

(1) one of $f(X)$ or $g(X)$ is closed subset of X,

(2) the pairs (f,S) and (g,T) are weakly compatible

then f, g, S and T have a common fixed point in X. Moreover, the set of common fixed points of f, g, S, T are well ordered if f, g, S, T have one and only one common fixed point.

Proof: Let x_0 be an arbitrary point in X. Since $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$, we can define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$\begin{aligned} y_{2n} &= fx_{2n} = Tx_{2n+1} \\ y_{2n+1} &= gx_{2n+1} = Sx_{2n+2} \end{aligned}$$

By the given assuption,

$$\begin{aligned} x_{2n+1} &\leq Tx_{2n+1} = fx_{2n} \leq x_{2n} \\ x_{2n+2} &\leq Tx_{2n+2} = fx_{2n+1} \leq x_{2n+1} \end{aligned}$$

Thus, for all $n \geq 1$, we have $y_{2n+1} \leq y_{2n}$. Let $y_{2n+1} \neq y_{2n}$ for every n. If not then $y_{2n} = y_{2n+1}$ for some n, then $d(y_{2n}, y_{2n+1}, a) = 0$ and from (1) we obtain

$$\begin{aligned} \psi(y_{2n}, y_{2n+1}, a) &= \psi\big(2s^4(y_{2n}, y_{2n+1}, a)\big) \\ &= \psi\big(2s^4(fx_{2n}, gx_{2n+1}, a)\big) \\ &\leq \psi\big(M(x_{2n}, x_{2n+1})\big) - \phi\big(M(x_{2n}, x_{2n+1})\big) \qquad (2) \end{aligned}$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\Big\{d(Sx_{2n}, Tx_{2n+1}, a), d(fx_{2n}, Sx_{2n}, a) \\ &\quad d(gx_{2n+1}, Tx_{2n+1}, a), d(fx_{2n}, gx_{2n+1}, a) \\ &\quad \frac{d(fx_{2n}, Tx_{2n+1}, a) + d(gx_{2n+1}, Sx_{2n}, a)}{2s}\Big\} \\ &= \max\Big\{d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n-1}, a) \\ &\quad d(y_{2n+1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a) \\ &\quad \frac{d(y_{2n}, y_{2n}, a) + d(y_{2n+1}, y_{2n-1}, a)}{2s}\Big\} \\ &= \max\Big\{d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a) \\ &\quad \frac{d(y_{2n-1}, y_{2n}, a) + d(y_{2n-1}, y_{2n}, a) + d(y_{2n-1}, y_{2n+1}, y_{2n})}{2}\Big\} \end{aligned}$$

If

$$\max\{d(y_{2n-1},y_{2n},a),d(y_{2n},y_{2n+1},a),\\ \frac{d(y_{2n-1},y_{2n},a)+d(y_{2n+1},y_{2n},a)+d(y_{2n-1},y_{2n+1},y_{2n})}{2}\}=d(y_{2n},y_{2n+1},a)$$

then by (2) we have

$$\psi\big(d(y_{2n},y_{2n+1},a)\big)\leq\psi\big(d(y_{2n},y_{2n+1},a)\big)-\phi\big(d(y_{2n},y_{2n+1},a)\big) \tag{3}$$

which gives a contradiction. If

$$d(y_{2n-1},y_{2n+1},a)=0$$

then

$$\max\{d(y_{2n-1},y_{2n},a),d(y_{2n},y_{2n+1},a),\\ \frac{d(y_{2n-1},y_{2n},a)+d(y_{2n+1},y_{2n},a)+d(y_{2n-1},y_{2n+1},y_{2n})}{2}\}=d(y_{2n},y_{2n+1},a)$$

therefore (1) becomes

$$\begin{aligned} d(y_{2n-1},y_{2n+1},a) &\leq \psi\big(d(y_{2n},y_{2n+1},a),\\ &\quad -\phi\max\{d(y_{2n-1},y_{2n},a),d(y_{2n},y_{2n+1},a),\\ &\quad \frac{d(y_{2n-1},y_{2n+1},a)}{2s}\}\big)\\ &\leq \psi d(y_{2n},y_{2n-1},a) \qquad (4)\end{aligned}$$

Thus $d(y_{2n},y_{2n+1},a);n\in\mathbf{N}\cup\{0\}$ is a non-increasing sequence of positive numbers. Hence, there exists $r\geq 0$ such that

$$\lim_{n\to\infty}d(y_{2n},y_{2n+1},a)=r.$$

Letting $n\to\infty$ in (3), we get

$$\begin{aligned}\psi(r) &= \psi(r)-\phi\big(\max\{r,r,\lim_{n\to\infty}\frac{d(y_{2n-1},y_{2n+1},a)}{2s}\}\big)\\ &= \psi(r)\end{aligned}$$

Therefore,

$$\phi\big(\max\{r,r,\lim_{n\to\infty}\frac{d(y_{2n-1},y_{2n+1},a)}{2s}\}\big)=0$$

and hence r = 0. Thus, we have

$$\lim_{n\to\infty} d(y_{2n}, y_{2n+1}, a) = 0 \tag{5}$$

for each $a \in X$. Note that if $d(y_{2n}, y_{2n+1}, a) \neq 0$ and

$$\begin{aligned} \max\{d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a), \quad & \frac{d(y_{2n-1}, y_{2n+1}, y_{2n}) + d(y_{2n+1}, y_{2n}, a) + d(y_{2n-1}, y_{2n}, a)}{2}\} \\ = \quad & \frac{d(y_{2n-1}, y_{2n+1}, y_{2n}) + d(y_{2n+1}, a, y_{2n}) + d(a, y_{2n-1}, y_{2n})}{2} \end{aligned}$$

Then by (1) and taking $a = y_{2n-1}$, we have

$$\begin{aligned} \psi d(y_{2n}, y_{2n+1}, y_{2n-1}) \quad &= \quad \psi(\frac{d(y_{2n-1}, y_{2n+1}, y_{2n}) + d(y_{2n+1}, y_{2n-1}, y_{2n}) + d(y_{2n-1}, y_{2n-1}, y_{2n})}{2}) \\ & \quad -\phi((\max\{d(y_{2n-1}, y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}, y_{2n-1}) \\ & \quad \frac{d(y_{2n-1}, y_{2n+1}, y_{2n+1})}{2s}\})) \\ \implies \psi(d(y_{2n}, y_{2n+1}, y_{2n-1})) \quad &= \quad \psi(d(y_{2n-1}, y_{2n+1}, y_{2n})) - \phi(d(y_{2n}, y_{2n+1}, y_{2n-1})) \end{aligned}$$

which gives $d(y_{2n}, y_{2n-1}, y_{2n+1}) = 0$, a contradiction. Next, we shall show that $\{y_n\}$ is a b_2-Cauchy sequence in X. For this it is sufficient to show that a subsequence $\{y_{2n}\}$ is Cauchy in X. For this purpose we use the following relation

$$d(y_i, y_j, d_k) = 0 \tag{6}$$

for all $i, j, k \in N$ (Note that this can be obtained as $\{d(y_{2n}, y_{2n+1}, a) : n \in N \cup \{0\}\}$ is a non-increasing sequence of positive numbers).

Suppose the contrary, that is, $\{x_n\}$ is not a b_2-Cauchy sequence. Then there exists $a \in X$ and $\varepsilon > 0$ for which we can find subsequences $\{y_{2m_i}\}$ and $\{y_{2n_i}\}$ of $\{y_{2n}\}$ such that n_i is the smallest index for which

$$2n_i > 2m_i > i, d(y_{2n}, y_{2m}, a) \geq \varepsilon \tag{7}$$

This means that

$$d(y_{2m_i}, y_{2n_i-1}, a) < \varepsilon \tag{8}$$

Using (8) and taking the upper limit as $i \to \infty$, we get

$$\lim_{i\to\infty} \sup d(y_{2m_i}, y_{2n_i-1}, a) \leq \varepsilon \tag{9}$$

On the other hand, we have

$$\begin{aligned} d(y_{2m_i}, y_{2n_i}, a) &\leq sd(y_{2m_i}, y_{2n_i}, y_{2n_i+1}) + sd(y_{2n}, a, y_{2m_i+1}) \\ &\quad + sd(a, y_{2m_i}, y_{2m_i+1}) \end{aligned}$$

as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq d(y_{2m_i+1}, y_{2n_i}, a) \tag{10}$$

Again, using the rectangular inequality, we have

$$\begin{aligned} d(y_{2m_i+1}, y_{2n_i-1}, a) &\leq sd(y_{2m_i+1}, y_{2n_i-1}, y_{2m_i}) + sd(y_{2n_i-1}, a, y_{2m_i}) \\ &\quad + sd(a, y_{2m_i+1}, y_{2m_i}) \end{aligned}$$

and

$$\begin{aligned} d(y_{2m}, y_{2n}, a) &\leq sd(y_{2m_i}, y_{2n_i}, y_{2n_i-1}) + sd(y_{2n_i}, a, y_{2n_i-1}) \\ &\quad + sd(a, y_{2m_i}, y_{2n_i-1}) \end{aligned}$$

Taking the upper limit as $i \to \infty$ in the first inequality above, we get

$$\lim_{i\to\infty} d(y_{2m_i+1}, y_{2n_i-1}, a) \leq \varepsilon s \tag{11}$$

Similarly, taking the upper limit as $i \to \infty$ in the inequality above, we get

$$\lim_{i\to\infty} \sup d(y_{2m_i}, y_{2n_i}, a) \leq \varepsilon s \tag{12}$$

we have

$$\begin{aligned} &\psi\big(2s^4 d(y_{2m_i+1}, y_{2n_i}, a)\big) \\ &= \psi\big(2s^4 d(fx_{2m_i+1}, gx_{2n_i}, a)\big) \\ &\leq \psi\big(M(x_{2m_i+1}, x_{2n_i})\big) - \phi\big(M(x_{2m_i+1}, x_{2n_i})\big) \qquad (13) \end{aligned}$$

where

$$\begin{aligned} M(x_{2m_i+1},x_{2n_i}) &= \max\big\{d(Sx_{2m_i+1},Tx_{2n_i},a),d(fx_{2m_i+1},Sx_{2n_i+1},a) \\ &\quad d(gx_{2n_i},Tx_{2n_i},a),d(fx_{2m_i+1},gx_{2n_i},a) \\ &\quad \frac{d(fx_{2m_i+1},Tx_{2n_i},a)+d(gx_{2m_i},Sx_{2m_i+1},a)}{2s}\big\} \\ &= \max\{d(x_{2m_i},x_{2n_i-1},a),d(x_{2m_i+1},x_{2m_i},a) \\ &\quad d(x_{2n_i},x_{2n_i-1},a),d(x_{2m_i+1},x_{2n_i},a) \\ &\quad \frac{d(x_{2m_i+1},x_{2n_i-1},a)+d(x_{2m_i},Sx_{2m_i},a)}{2s}\} \end{aligned}$$

Taking the upper limit as $i \to \infty$ in (13) and using (5),(9), (11) and (12), we get

$$\begin{aligned} \lim_{n\to\infty} M(x_{m_i},x_{n_i-1},a) &= \max\big\{\lim_{n\to\infty}\sup d(y_{2m_i},y_{2n_i-1},a),0,0,d(y_{2m_i+1},y_{2n_i},a), \\ &\quad \frac{1}{2s}\big[\lim_{n\to\infty}\sup d(y_{2m_i+1},y_{2n_i-1},a)+\lim_{n\to\infty}\sup d(y_{2n_i},y_{2m_i},a)\big]\big\} \\ &\le \max\big\{\varepsilon,0,0,\varepsilon,\frac{1}{2s}[\varepsilon s+\varepsilon s]\big\} \le \varepsilon \end{aligned} \tag{14}$$

So, we have

$$\lim_{n\to\infty}\sup M(x_{2m_i-1},x_{2n_i-1},a) \le \varepsilon \tag{15}$$

Now, taking the upper limit as $i \to \infty$ in (13) and using (10), (15) we have

$$\begin{aligned} \psi(s\frac{\varepsilon}{s}) &\le \psi\big(\lim_{n\to\infty}\sup d(y_{2m_i},y_{2n_i})\big) \\ &\le \psi\big(\lim_{n\to\infty}\sup M(x_{2m_i},x_{2m_i-1})\big)-\psi\big(\lim_{n\to\infty}\inf M(x_{2m_i},x_{2n_i-1})\big) \\ &= \psi(\varepsilon)-\psi\big(\lim_{n\to\infty}\inf M(x_{2m_i},x_{2n_i-1})\big) \end{aligned}$$

which further implies that

$$\phi\big(\lim_{n\to\infty}\inf M(x_{2m_i},x_{2n_i-1})\big)=0$$

so

$$\lim_{n\to\infty}\inf M(x_{2m_i},x_{2n_i-1})=0$$

a contradiction to (7).

Thus $\{y_{2n}\}$ is a b_2-Cauchy sequence in X. As X is a b_2-complete space,there exists $u \in X$ such that $y_{2n} \to u$ as $u \to \infty$, that is

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = u$$

Since X is complete, there exists $y \in X$ such that

$$\begin{aligned}\lim_{n\to\infty} y_{2n} &= \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} gx_{2n+1} \\ &= \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y\end{aligned}$$

Now, we show that y is a common fixed point of f, g, S and T.

Let $g(X)$ be a closed subset of X, since $g(X) \subset S(X)$, so there exists $u \in X$ such that $Su = y$. We prove that $fu = y$ since $gx_{2n+1} \leq x_{2n+1}$ and $gx_{2n+1} \to y$ as $n \to \infty, y \leq x_{2n+1}$ and $u \leq Su \leq y \leq x_{2n+1} \leq x_{2n}$, so from (1), we obtain

$$\begin{aligned}\psi\big(d(fu, gx_{2n+1}, a)\big) &\leq \psi\big(2s^4 d(fu, gx_{2n+1}, a)\big) \\ &\leq \psi\big(M(u, x_{2n+1})\big) - \phi\big(M(u, x_{2n+1})\big)\end{aligned} \tag{16}$$

where

$$\begin{aligned}\lim_{n\to\infty} M(u, x_{2n+1}) &= \max\Big\{d(Su, Tx_{2n+1}, a), d(fu, Su, a) \\ &\quad d(gx_{2n+1}, Tx_{2n+1}, a), d(fu, gx_{2n+1}, a) \\ &\quad \frac{d(fu, Tx_{2n+1}, a) + d(gx_{2m+1}, Su, a)}{2s}\Big\} \\ \implies M(u, x_{2n+1}) &= \max\Big\{d(Su, y, a), d(fu, Su, a), d(y, y, a), d(fu, y, a), \\ &\quad \frac{1}{2s}[d(fu, y, a) + d(y, Su, a)]\Big\} \\ &= \max\Big\{d(y, y, a), d(fu, y, a), 0, \\ &\quad d(y, y, a), \frac{1}{2s}[d(fu, y, a) + d(y, y, a)]\Big\} \\ &= d(fu, y, a)\end{aligned}$$

Therefore

$$\begin{aligned}\psi\big(d(fu,y,a)\big) &= \psi\big(d(fu,y,a)\big)-\phi\big(d(fu,y,a)\big)\\ &\le \psi\big(d(fu,y,a)\big)\end{aligned}$$

which is a contradiction.

Therefore $fu=y$.

Since the pairs (f,S) is weakly compatible we have, $fSu=Sfu$. Hence $fy=Sy$. We prove that fy = y, if $fy\neq y$, then from (1), we have

$$\begin{aligned}\psi\big(d(fy,gx_{2n+1},u)\big) &\le \psi\big(2s^4 d(fy,gx_{2n+1},u)\big)\\ &\le \psi\big(M(y,x_{2n+1})\big)-\phi\big(M(y,x_{2n+1})\big)\\ &= \psi d(fy,y,a)-\phi d(fy,y,a)\end{aligned}$$

a contradiction to $fy\neq y$. Therefore $fy=Sy=y$ and hence y is a common fixed point of f and S. Since $y=fy\in f(X)\subset T(X)$, hence there exists $v\in X$ such that $tv=y$. Now we have to show that gv = y. Since $v\le Tv=y\le x_{2n+1}$, hence from (1), we have

$$\begin{aligned}\psi\big(d(y,gv,a)\big) &= \psi\big(d(fy,gv,a)\big)\le\psi\big(2s^4 d(fy,gv,a)\big)\\ &\le \psi\big(M(y,v)\big)-\phi\big(M(y,v)\big)\end{aligned}$$

where

$$\begin{aligned}M(y,v) &= \max\Big\{d(Sy,Tv,a),d(fy,Sy,a)\\ &\quad d(gv,Tv,a),d(fy,gv,a),\\ &\quad \frac{d(fy,Tv,a)+d(gv,Sy,a)}{2s}\Big\}\\ &= \max\big\{d(y,y,a),d(y,y,a),d(gv,y,a),d(y,gv,a),\\ &\quad \frac{1}{2s}[d(y,y,a)+d(gv,y,a)]\big\}\\ &= \max\big\{0,0,d(gv,y,a),d(y,gv,a),\frac{1}{2s}d(gv,y,a)\big\}\\ &= d(gv,y,a)\end{aligned}$$

Therefore

$$\psi\big(d(y,gv,a)\big) \le \psi\big(d(gv,y,a)\big) - \phi\big(d(gv,y,a)\big)$$

which is a contradiction. Therefore

$$\begin{aligned} \big(d(y,gv,a)\big) &= 0 \\ \implies gv &= y \end{aligned}$$

By the weakly compatibility of the pairs (g,T). we have $Tgv = gTv$. Hence Ty = gy. We prove that gy = y, if $gy \neq y$, then from (1) we have

$$\begin{aligned} \psi\big(d(fy,gy,a)\big) &\le \psi\big(2s^4 d(fy,gy,a)\big) \\ &\le \psi\big(M(y,y)\big) - \phi\big(M(y,y)\big) \end{aligned}$$

where

$$\begin{aligned} M(y,y) &= \max\big\{d(Sy,Ty,a), d(fy,Sy,a), \\ &\quad d(gv,Ty,a), d(fy,gy,a), \\ &\quad \frac{d(fy,Ty,a)+d(gy,Sy,a)}{2s}\big\} \\ &= \max\big\{d(y,y,a), d(y,y,a), d(gy,y,a), d(y,gy,a), \\ &\quad \frac{1}{2s}[d(y,gy,a)+d(gy,y,a)]\big\} \\ &= \max\big\{d(y,gy,a), 0, 0, d(gy,y,a), \frac{1}{2s}d(y,gy,a)\big\} \\ &= d(y,gy,a) \end{aligned}$$

Therefore

$$\psi\big(d(y,gy,a)\big) \le \psi\big(d(y,gy,a)\big) - \phi\big(d(y,gy,a)\big)$$

which is a contradiction. Thus $gy = Ty = y$ and hence y is a common fixed point of g and T. Hence $fy = Sy = Ty = y$, thus y is a common fixed point of f,g,S and T. similarly if f(X) be a closed subset of X we can get the same result.

Here we give an example to illustrate Theorem 3.1.

Example 3.1. *Let* $X = [0,2]$ *be endowed with a* b_2-*metric* $d(x,y,z) = [xy+yz+zx]^2, x \neq y \neq z, d(x,y,z) = 0$ *otherwise. Also, let self-mappings* f,g,S,T *on* X *defined by* $f(x) = \begin{cases} 0, & \text{if } x=0; \\ \frac{x}{5}+1, & \text{otherwise.} \end{cases}$

$$g(x) = \begin{cases} 0, & \text{if } x=0; \\ \frac{2x}{5}+1, & \text{otherwise.} \end{cases}$$

$$S(x) = \begin{cases} 0, & \text{if } x=0; \\ \frac{3x}{5}+1, & \text{otherwise.} \end{cases}$$

$$T(x) = \begin{cases} 0, & \text{if } x=0; \\ \frac{4x}{5}+1, & \text{otherwise.} \end{cases}$$

where $x,y,z \in X$. $f(X) = [0,\frac{7}{5}], g(X) = [0,\frac{9}{5}], S(X) = [0,\frac{11}{5}], T(X) = [0,\frac{13}{5}]$. *Here* $f(X) \subseteq T(X), g(X) \subseteq S(X)$ *and* (f,S) *and* (g,T) *are weakly compatible at* $x = 0$. *Also,* $f(X)$ *or* $g(X)$ *is closed subset of* X.

Take $\psi(t) = t$ *and*

$$\phi(t) = \begin{cases} \frac{t}{100}, & \text{if } t \text{ greater than } 0; \\ 0, & \text{if } t=0. \end{cases}$$

Now,

$$\begin{aligned} \psi(2s^4 d(fx,gy,a)) &= \psi(2s^4 d((\frac{x}{5}+1),\frac{2y}{5}+1),0) \\ &= \psi(2s^4\{(\frac{x}{5}+1))\frac{2y}{5}+1)\}^2) \\ &= \frac{2s^4}{25}\{(x+5)(2y+5)\}^2 \end{aligned}$$

and

$$\begin{aligned} M(x,y) &= \max\{d(Sx,Ty,a), d(fx,Sx,a), d(gy,Ty,a), d(fx,gy,a), \frac{d(fx,Ty,a)+d(gy,Sx,a)}{2s}\} \\ &= \max\{((\frac{3x}{5}+1)(\frac{4y}{5}+1))^2, ((\frac{x}{5}+1)\frac{3x}{5}+1))^2, ((\frac{2y}{5}+1)(\frac{4y}{5}+1))^2, ((\frac{x}{5}+1)(\frac{2y}{5}+1))^2, \\ &\quad \frac{((\frac{x}{5}+1)(\frac{2y}{5}+1))^2 + ((\frac{2y}{5}+1)(\frac{3x}{5}+1))^2}{2s}\} \end{aligned}$$

Then

$$\psi\big(2s^4 d(fx,gy,a)\big) \leq \psi\big(M(x,y)\big) - \phi\big(M(x,y)\big)$$

Hence all the conditions of Theorem 2.1 *hold and* f, g, S, T *have the common fixed point at* $x = 0$ *in* X.

4. Conclusion

We prove a common fixed point theorem for two pairs of weakly compatible mappings satisfying (ϕ, ψ) contractive condition in b_2-metric space. Our results generalise the concept of 2-metric space and b-metric space.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] V.S. Gahler, 2-metrische Rume und ihre topologische struktur, Math. Nachr. 26 (1963), 115-118

[2] I.A. Bakhtin, The contraction mapping principle in quasimetric spaces, Funct. Anal. 30 (1989), 26-37.

[3] S. Czerwik, Contraction in b-metric spaces, Acta Math. Inform. Univ. Ostrav, 1(1993), 5-11.

[4] S. Czerwik, Nonlinear set-valued contraction mapping in b-metric spaces, Atti Semin. Mat. Fis. Univ. Modena, 46(1998), 263-276.

[5] N. Hussain, V. Parvaneh, Z. Kadelberg, Fixed Points of cyclic weakly (ψ, ϕ, L, A, B)-contractive mappings in ordered b-metric spaces with applications, Fixed Point Theory Appl. 2013 (2013), 256.

[6] V.R. Naidu, J.R. Prasad, Fixed point theorems in 2-metric spaces, Indian J. Pure Appl. Math. 17(1986), 974-993.

[7] N. V. Dung, V.T. Le Hang, Fixed point theorems for weak c-contractions in partially ordered 2-metric spaces, Fixed Point Theory Appl. 2013 (2013), Article ID 161.

[8] A. Aliouche, C. Simpson, Fixed points and lines in 2-metric spaces, Adv. Math. 229 (2012), 668-690.

[9] B. Deshpande, S. Chouhan, Common fixed point theorems for hybrid pairs of mappings with some weaker conditions in 2-metric spaces, Fasc. Math. 46 (2011), 37-55.

[10] N. Malhotra, B. Bansal, Some Common coupled fixed point theorems for generalised contraction in b-metric spaces, J. Nonlinear Sci. Appl. 8 (2015), 8-16

[11] K. Iseki, Fixed Point theorems in 2-metric spaces, Math. Semin. Notes, 3 (1975), 133-136.

[12] B.K. Lahiri, P. Das, L.K Dey, Cantors theorems in 2-metric spaces and its applications to fixed point problems, Taiwan. J. Math. 15 (2011), 337-352.

[13] S.N. Lal, A.K. Singh, An analouge of Banachs contraction principle in 2-metric spaces, Bull. Aust. Math. Soc. 18 (1978), 137-143

[14] V. Popa, M. Imdad, A. Javid, Using implicit relations to unified fixed point theorems in metric and 2-metric spaces, Bull. Malaysia Math. Soc. 33 (2010), 105-120.

[15] M.A. Ahmed, Common fixed point theorems for expansive mappings in 2-metric spaces and its application, Chaos Solutions Fractals, 42 (2009), 2914-2920.

[16] Z. Mustafa, V. Paraneh, J. Rezaei, and Z. Kadulberg: b_2-metric spaces and some fixed point theorems, Fixed Point Theory Appl, 2014 (2014), 144.

[17] Th. Chhatrajit, Y. Rohen, A comparative study of relationship among various types of spaces, Int. J. Appl. Math. 28 (2015), 29-36.

[18] N Priyobarta, Y. Rohen, R. Stojan, Fixed Point theorems on parametric A-metric space, Amer. J. Appl. Math. Stat. 6 (2018), 1-5.

[19] N Priyobarta, Y. Rohen, M. Nabil, Complex valued S_b metric space, J. Math. Anal. 8 (2017), 13-24.

[20] A. H. Ansari, D. Dhamodharan, Y. Rohen, R. Krishnakumar, C-class function on new contractive conditions of integral type on complete S-metric space, J. Glob. Res. Math. Arch. 5 (2018), 46-63.

[21] Y. Rohen, D. Tatjan, R. Stojan, A Note on the paper "A Fixed point Theorems in S_b Metric Space", Filomat, 31 (2017), 3335-3346.

[22] M. Nabil, Y. Rohen, Some coupled fixed point theorems in partially ordered A_b metric space, J. Nonlinear Sci. Appl. 10 (2017), 1731-1743.

[23] L. Ciric, M. O. Olatinwo, D. Gopal, Akinbo, Coupled fixed point theorems for mappings satisfying a contractive condition of rational type on a partially ordered metric space, Adv. Fixed Point Theory, 2 (2012), 1-8.

[24] F. Rouzkard, M. Imdad, D. Gopal, Some existence and uniqueness theorems on ordered metric spaces via generalized distances, Fixed Point Theory Appl. (2013) (2013), 45.

[25] D. K. Patel, P. Kumam, D. Gopal, Some discussion on the existence of common fixed points for a pair of maps, Fixed Point Theory Appl. 2013 (2013), 187.