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ARITHMETIC INTEGER ADDITIVE SET-VALUED GRAPHS: A CREATIVE REVIEW

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Abstract. For a non-empty ground set X, finite or infinite, the *set-valuation* or *set-labeling* of a given graph G is an injective function $f : V(G) \to \mathscr{P}(X)$, where $\mathscr{P}(X)$ is the power set of the set X. A set-indexer of a graph G is an injective set-valued function $f : V(G) \to \mathscr{P}(X)$ such that the function $f^* : E(G) \to \mathscr{P}(X) - \{\emptyset\}$ defined by $f^*(uv) = f(u) * f(v)$ for every $uv \in E(G)$ is also injective, where $*$ is a binary operation on sets. Let $\mathbb{N}_0$ be the set of all non-negative integers and $\mathscr{P}(\mathbb{N}_0)$ is its power set. An integer additive set-labeling (IASL) of a graph G is an injective function $f : V(G) \to \mathscr{P}(\mathbb{N}_0)$ such that the induced function $f^+ : E(G) \to \mathscr{P}(\mathbb{N}_0)$ is defined by $f^+(uv) = f(u) + f(v)$, where $f(u) + f(v)$ is the sumset of the sets $f(u)$ and $f(v)$. An IASL f of a graph G is said to be an integer additive set-indexer (IASI) of G if the induced function f^+ is also injective. In this paper, we critically and creatively review the concepts and properties of a particular type integer additive set-valuation, called arithmetic integer additive set-valuation of graphs.

Keywords: set-indexers; integer additive set-indexers; weak integer additive set-indexers; strong integer additive set-indexers.

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1. INTRODUCTION

For all terms and definitions, not defined specifically in this paper, we refer to [6, 10, 18, 36]. For more about graph classes, we further refer to [7] and [13]. Unless otherwise mentioned, all graphs considered here are simple, finite and have no isolated vertices.

1.1. Set-Valued Graphs.

The researches on graph labeling problems commenced with the introduction of the concept of number valuations of graphs in [25]. Since then, the studies on graph labeling have contributed significantly to the researches in graph theory and associated fields. Graph labeling problems have numerous theoretical and practical applications. Many types of graph labeling are surveyed and listed in [13].

The notion of set-valuation of graphs has been introduced in [1], motivated by social interactions and social networks problems. For a non-empty ground set X, finite or infinite, the *set-valuation* or *set-labeling* of a given graph G is an injective function $f : V(G) \to \mathscr{P}(X)$, where $\mathscr{P}(X)$ is the power set of the set X.

Also, a *set-indexer* of a graph G is defined, in [1], as an injective set-valued function $f : V(G) \to \mathscr{P}(X)$ such that the function $f^* : E(G) \to \mathscr{P}(X) - \{\emptyset\}$ defined by $f^*(uv) = f(u) * f(v)$ for every $uv \in E(G)$ is also injective, where $\mathscr{P}(X)$ is the set of all subsets of X and $*$ is a binary operation on sets.

Taking the symmetric difference of two sets as the operation between two set-labels of the vertices of G, the following theorem was proved in [1].

Theorem 1.1. [1] *Every graph has a set-indexer.*

1.2. Integer Additive Set-Valued Graphs.

If A and B are two non-empty sets, then their *sumset* is the set $A+B$ defined by $A+B = \{a+b : a \in A, b \in B\}$. Using the concepts of sumsets, the notion of integer additive set-labeling of a given graph is introduced as follows.

Definition 1.2. [26] Let $\mathbb{N}_0$ denote the set of all non-negative integers and $\mathscr{P}(\mathbb{N}_0)$ be its power set. An *integer additive set-labeling* (IASL, in short) of a graph G is defined as an injective function $f : V(G) \to \mathscr{P}(\mathbb{N}_0)$ which induces a function $f^+ : E(G) \to \mathscr{P}(\mathbb{N}_0)$ such that $f^+(uv) = f(u) + f(v)$, $uv \in E(G)$. A graph which admits an IASL is called an *integer additive set-labeled graph* (IASL-graph).

The notion of an integer additive set-indexers of graphs has been introduced in [14] as follows.

Definition 1.3. [14] An integer additive set-labeling $f : V(G) \to \mathscr{P}(\mathbb{N}_0)$ of a graph G is said to be an *integer additive set-indexer* (IASI) if the induced function $f^+ : E(G) \to \mathscr{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective. A graph which admits an IASI is called an *integer additive set-indexed graph* (IASI-graph).

Certain studies about integer additive set-indexed graphs have been done in [2, 15, 27] and [26].

Analogous to Theorem 1.1, the following theorem established the existence of integer additive set-labeling for graphs, in general.

Theorem 1.4. [26] *Every graph has an integer additive set-labeling (or an integer additive set-indexer).*

The properties of the set labels of the elements of integer additive set-labeled graphs have been studied in [26, 15, 27] and the important notions and facts in this area are as follows:

If either $f(u)$ or $f(v)$ is countably infinite, then $f(u) + f(v)$ is also countably infinite and hence the study about the cardinality of $f(u) + f(v)$ becomes trivial. Hence, we restrict our discussion to finite sets of non-negative integers. We denote the cardinality of a set A by $|A|$.

The cardinality of the set-label of an element (vertex or edge) of a graph G is called the *set-indexing number* of that element. If the set-labels of all vertices of G have the same cardinality, then the vertex set $V(G)$ is said to be *uniformly set-indexed*.

An IASL (or IASI) is said to be *k-uniform* if $|f^+(e)| = k$ for all $e \in E(G)$. That is, a connected graph G is said to have a k-uniform IASL (or IASI) if all of its edges have the same set-indexing number k. If G is a graph which admits a k-uniform IASI and $V(G)$ is l-uniformly set-indexed, then G is said to have a (k,l)*-completely uniform IASI* or simply a *completely uniform IASI*.

Let f be an IASI defined on a graph G and let u, v be any two adjacent vertices in G. Two ordered pairs (a,b) and (c,d) in $f(u) \times f(v)$ are said to be *compatible* if $a+b = c+d$. If (a,b) and (c,d) are compatible, then we write $(a,b) \sim (c,d)$. Clearly, $\sim$ is an equivalence relation.

A *compatibility class* of an ordered pair (a,b) in $f(u)\times f(v)$ with respect to the integer $k=a+b$ is the subset of $f(u)\times f(v)$ defined by $\{(c,d)\in f(u)\times f(v):(a,b)\sim(c,d)\}$ and is denoted by C_k. Since $f(u)$ and $f(v)$ are finite sets, then each compatibility class C_k in $f(u)\times f(v)$ contains finite number of elements.

It is to be noted that no compatibility class in $f(u)\times f(v)$ can be non-empty. If a compatibility class C_k contains only one element, then it is called a *trivial class*. A compatibility class C_k that contains maximum number of elements are called a *maximal compatibility class*.

Lemma 1.5. [26] *For a compatibility class C_k in $f(u)\times f(v)$, we have $1\le|\mathsf{C}_k|\le\min(|f(u)|,|f(v)|)$.*

A compatibility class which contain the largest possible number of elements is called a *saturated class*. Hence, the cardinality of a saturated class in $f(u)\times f(v)$ is $\min(|f(u)|,|f(v)|)$.

It is to be noted that all saturated classes in $f(u)\times f(v)$ are maximal compatible classes, but a maximal compatible class need not be a saturated class of $f(u)\times f(v)$. That is, the existence of a saturated class depends on the nature of elements in the set-labels $f(u)$ and $f(v)$.

Based on the relation between the set-indexing numbers of an edge and its end vertices in G, the following notion is introduced in [27].

A *weak IASI* is an IASI f such that $|f^+(uv)|=\max(|f(u)|,|f(v)|)$ for all $uv\in E(G)$. A graph which admits a weak IASI may be called a *weak IASI-graph* (see [15]).

Theorem 1.6. [15] *An IASI f of a graph G is a weak IASL (or weak IASI) if and only if the set-indexing number of at least one end vertex of every edge of G is* 1.

A *strong IASI* is an IASI f such that $|f^+(uv)|=|f(u)|\,|f(v)|$ for all $u,v\in V(G)$. A graph which admits a strong IASI may be called a *strong IASI-graph* (see [27]).

It is to be noted that if f is a weak IASI (or a strong IASI) of G then all compatible classes in $f(u)\times f(v)$ are trivial classes, where $uv\in E(G)$.

2. Arithmetic IASI of Graphs

2.1. Introduction. In this article, we review the studies made on the characteristics of certain graphs, the elements of whose set-labels are in arithmetic progressions. By the term, an *arithmetically progressive set*, (AP-set, in short), we mean a set whose elements are in arithmetic progression. We call the common difference of the set-label of an element of a given graph G, the *deterministic index* of that element. The *deterministic ratio* of an edge of G is the ratio $k \geq 1$, of the deterministic indices of its end vertices. Now, we have the following notions.

Definition 2.1. [28] Let $f : V(G) \to \mathscr{P}(\mathbb{N}_0)$ be an IASI on G. For any vertex v of G, if $f(v)$ is an AP-set, then f is called a *vertex-arithmetic IASI* of G.

Definition 2.2. [28] For an IASI f of G, if $f^+(e)$ is an AP-set, for all $e \in E(G)$, then f is called an *edge-arithmetic IASI* of G.

A graph that admits vertex-arithmetic IASI (respectively, an edge-arithmetic IASI) is called a *vertex-arithmetic IASI-graph* (respectively, an *edge-arithmetic IASI-graph*).

Definition 2.3. [28] An IASI is said to be an *arithmetic integer additive set-indexer* if it is both vertex-arithmetic and edge-arithmetic.

In other words, an arithmetic IASI is an IASI f, under which the set-labels of all elements (vertices and edges) of a given graph G are AP-sets. A graph that admits an arithmetic IASI is called an *arithmetic IASI-graph*.

Since the set-labels of elements of a graph G we consider, are in AP-sets, the set-labels must consists of at least three elements. Therefore, we note by Theorem 1.6, that an arithmetic IASI of a graph G can never be a weak IASI.

Also, note that the sumset $A+B$ of two sets A and B is an AP-set need not imply A and B are AP-sets. Moreover, the sumset of two AP-sets need not be an AP-set. In view of this fact, we introduced the following notion.

Definition 2.4. [28] An IASI of a graph G with respect to which the set-labels of all vertices of a graph G are AP-sets and the set-labels of edges are not AP-sets, is called a *semi-arithmetic IASI*.

Hence, we note that the existence of vertex arithmetic IASIs and edge arithmetic IASIs for any given graph G do not imply each other, in general. In this context, the conditions required for two set-labels (which are AP-sets) of the adjacent vertices of G to have an AP-set as their sumset arouse much interest. The following theorem has provided a necessary and sufficient condition for an IASI of G to be an arithmetic IASI of G.

Theorem 2.5. [28] *A graph G admits an arithmetic IASI f if and only if f is a vertex arithmetic IASI and the deterministic ratio of any edge of G is a positive integer k, which is less than or equal to the set-indexing number of its end vertex having smaller deterministic index.*

In other words, if v_i and v_j are two adjacent vertices of a given graph G with deterministic indices d_i and d_j respectively with respect to an IASI f of G, where $d_i \leq d_j$, then f is an arithmetic IASI if and only if $d_j = k\,d_i$, where k is a positive integer such that $1 \leq k \leq |f(v_i)|$.

It is to noted that the deterministic index of an edge e of an arithmetic IASI-graph G, is always equal to the deterministic index of its end vertex having smaller deterministic index. That is, if d_u and d_v are the deterministic indices of two end vertices u and v respectively, of an edge $e = uv$ in G such that $k = \frac{d_v}{d_u}$ is a positive integer such that $1 \leq k \leq |f(v)|$, then the deterministic index of the edge e is equal to the deterministic index of the vertex u itself.

Let G be an arithmetic IASI-graph. Then, the set-indexing number of an edge of G, in terms of the set-indexing numbers of its end vertices, is established in the following theorem.

Theorem 2.6. [28] *Let G be a graph which admits an arithmetic IASI, say f and let d_i and d_j be the deterministic indices of two adjacent vertices v_i and v_j in G. If $|f(v_i)| \geq |f(v_j)|$, then for some positive integer $1 \leq k \leq |f(v_i)|$, the edge v_iv_j has the set-indexing number $|f(v_i)| + k(|f(v_j)| - 1)$.*

2.2. **Different Types of Arithmetic IASIs.**

In view of Theorem 2.5, we can categorise an arithmetic IASI into different types, according to the nature of the deterministic ratio k of the edges of G. In this section, we discuss the about the different types of arithmetic IASIs.

The first type arithmetic IASI is the one, called an isoarithmetic IASI, defined as follows.

Definition 2.7. [29] Let f be an arithmetic IASI defined on a given graph G. If all the elements of G have the same deterministic index with respect to f, then f is said to be an *isoarithmetic IASI* of G. A graph which admits an isoarithmetic IASI is called an *isoarithmetic IASI-graph*.

That is, an isoarithmetic IASI of a graph G is an arithmetic IASI, with respect to which, all the elements of G have the same deterministic indices and hence the deterministic ratio of every edge of G is 1.

The next type arithmetic IASI is an IASI with respect to which the deterministic ratios of the edges of a given graph G are greater than 1.

Definition 2.8. [29] A *biarithmetic IASI* of a graph G is an arithmetic IASI f of$G$, with respect to which the deterministic ratio of each edge of G is a positive integer greater than 1 and less than or equal to the set-indexing number of the end vertex of e having smaller deterministic index.

In other words, a biarithmetic IASI of a graph G is an arithmetic IASI f of G, for which the deterministic indices of any two adjacent vertices v_i and v_j in G, denoted by d_i and d_j respectively such that $d_i < d_j$, holds the condition $d_j = kd_i$ where k is a positive integer such that $1 < k \leq |f(v_i)|$.

One can immediately notice that, all edges of G may not have the same deterministic ratio with respect to a given arithmetic IASI. Keeping this fact in mind, we have introduced the following notion of a particular type of biarithmetic IASI in [29] as follows.

Definition 2.9. [29] Let f be a biarithmetic IASI defined on a graph G. If the deterministic ratio of every edge of G is the same, say k, then f is called an *identical biarithmetic IASI* of G and G is called an *identical biarithmetic IASI-graph*.

Definition 2.10. [29] A *prime arithmetic integer additive set-indexer* of a graph G is an arithmetic IASI $f : V(G) \to \mathscr{P}(\mathbb{N}_0)$, with respect to which the deterministic ratio of every edge of G is a prime integer.

In other words, a prime arithmetic IASI of a graph G is an arithmetic IASI such that for any two adjacent vertices in G, the deterministic index of one vertex is a prime integer multiple of the deterministic index of the other, where this prime integer is less than or equal to the set-indexing number of the its end vertex having smaller deterministic index.

In view of Theorem 2.5, we understand that a vertex arithmetic IASI is not an arithmetic IASI because of two reasons. One of them is that the deterministic ratio of the edges of G are not positive integers and the other reason is that the deterministic ratio of an edge of G is greater than the set-indexing number of its end vertex having smaller deterministic index. Hence, we have two types of semi-arithmetic IASIs which are

Definition 2.11. [33] A vertex-arithmetic IASI f of a graph G, under which the differences d_i and d_j of the set-labels $f(v_i)$ and $f(v_j)$ respectively for two adjacent vertices v_i and v_j of G, holds the conditions $d_j = kd_i$ and k is a non-negative integer greater than $|f(v_i)|$ is called the *semi-arithmetic IASI of the first kind*.

Definition 2.12. [32] A vertex-arithmetic IASI f of a graph G, under which the differences d_i and d_j of the set-labels $f(v_i)$ and $f(v_j)$ respectively for two adjacent vertices v_i and v_j of G are not multiples of each other, is called the *semi-arithmetic IASI of the second kind*.

Now that we have come across the different types of arithmetic IASIs and semi-arithmetic IASIs, we can now proceed to check the characteristics of the graphs which admit these IASIs and conditions for their existence.

3. Characterisation of Arithmetic IASIs

The existence of an arithmetic IASI for a given graph G depends upon the relation between cardinality of set-labels of vertices of G and the adjacency between these vertices of G. The following result establishes the admissibility of different types of arithmetic IASI by a given graph.

Proposition 3.1. [28, 29] *Every graph G admits an arithmetic (or isoarithmetic or biarithmetic) IASI.*

The above theorem can be proved by choosing the set-labels of vertices with sufficiently large in such a way that the deterministic ratio of adjacent edges can be chosen properly. The following theorem provides the set-indexing number of an edge of an isoarithmetic IASI-graph, in terms of the set-indexing numbers of its end vertices.

Theorem 3.2. [29] *Let G be a graph with an arithmetic IASI f defined on it. Then, f is an isoarithmetic IASI on G if and only if the set-indexing number of every edge of G is one less than the sum of the set-indexing numbers of its end vertices.*

Proof of the above theorem is immediate from Theorem 2.6 with $k = 1$. The following theorem is an immediate consequence of Theorem 3.2.

Theorem 3.3. [29] *Let f be an arithmetic IASI defined on a given graph G such that $V(G)$ is l-uniformly set-indexed. Then, f is an isoarithmetic IASI of G if and only if G is a $(2l-1)$-uniform IASI-graph.*

The following theorem has established the necessary and sufficient condition for a complete graph to admit an arithmetic IASI.

Theorem 3.4. [28] *A complete graph admits an arithmetic IASI if and only if the deterministic indices of any vertex of K_n is either an integral multiple or divisor of the deterministic indices of all other vertex of K_n such that the deterministic ratio of every edge of K_n is less than or equal to the set-indexing number of its end vertex having smaller deterministic index.*

The following result addresses the question whether an isoarithmetic IASI could be a strong IASI.

Proposition 3.5. [29] *No isoarithmetic IASI defined on a given graph G can be a strong IASI of G.*

As the set-indexing number of any edge of an isoarithmetic IASI-graph G is $m+n-1$ and that of any edge of a strong IASI-graph G is mn, where m and n are the set-indexing numbers of its end vertices, the proof of the above theorem follows from the fact that for any two positive integers $m,n \geq 3$, $m+n-1 \neq mn$.

In view of Proposition 3.5, for any two adjacent vertices $v_i, v_j \in V(G)$, it can be seen that under an isoarithmetic IASI f on G, some compatibility classes in $f(v_i) \times f(v_j)$, contain more than one element. Then, the question about the number of elements in various compatibility classes arises much interest. The following theorem discusses the number of elements in the compatibility classes of $f(v_i) \times f(v_j)$ in G.

Theorem 3.6. [29] *Let G be a graph which admits an isoarithmetic IASI, say f. Then, the number of saturated classes in the Cartesian product of the set-labels of any two adjacent vertices in G is one greater than the difference between cardinality of the set-labels of these vertices. Moreover, exactly two compatibility classes, other than the saturated classes, have the same cardinality in the Cartesian product of the set-labels of these vertices.*

Therefore, we also have

Corollary 3.7. [29] *Let f be an isoarithmetic IASI defined on a graph G, under which $V(G)$ is l-uniformly set-indexed. Then, there is exactly one saturated class in the Cartesian product of the set-labels of any two adjacent vertices in G.*

Invoking the above facts, we have proposed a necessary and sufficient condition for an isoarithmetic IASI of a given graph G to be a uniform IASI of G.

Theorem 3.8. [29] *An isoarithmetic IASI of a G is a uniform IASI if and only if $V(G)$ is uniformly set-indexed or G is bipartite.*

We have already seen that no isoarithmetic IASI of a graph G can be a strong IASI of G, it is natural to ask whether a biarithmetic IASI of G can be a strong IASI of it. The following theorem provides a necessary and sufficient condition for a biarithmetic IASI of G to be a strong IASI.

Theorem 3.9. [29] *Let G be a graph which admits a biarithmetic IASI, say f. Then, f is a strong IASI of G if and only if the deterministic ratio of every edge of G is equal to the set-indexing number of its end vertex having smaller deterministic index.*

Invoking Theorem 3.9, the analogous condition for an identical biarithmetic IASI to be a strong IASI is established in the following theorem.

Theorem 3.10. [29] *An identical biarithmetic IASI of a graph G is a strong IASI of G if and only if one partition of $V(G)$ is k-uniformly set-indexed, where k is the deterministic ratio of the edges of G.*

The existence of saturated classes and the number of elements in each compatibility class in the cross product of the set-labels of any two adjacent vertices of a biarithmetic IASI-graph have been studied in [29] and has been established in [29] that there exists no saturated class in the cross product of the set-labels of any two adjacent vertices of a biarithmetic IASI-graph. The number of maximal compatibility classes and their cardinalities are provided in the following result.

Theorem 3.11. [29] *Let G be a graph that admits a biarithmetic IASI, say f. Let v_i and v_2 be two adjacent vertices of G, where v_i has the smaller deterministic index and $k \leq |f(v_i)|$, be the deterministic ratio of the edge v_iv_j. If $|f(v_i)| = pk+q$, where p,q are non-negative integers such that $p \leq (|f(v_j)|-1)$ and $q < k$, then*

(i) if $q = 0$, then $(|f(v_j)|-p+1)k$ compatibility classes are maximal compatibility classes and contain p elements.

(ii) if $q > 0$, then $(|f(v_j)|-p-1)k+q$ compatibility classes are compatibility classes and contain $(p+1)$ elements.

All given graphs may not have an identical biarithmetic IASI. Hence, the conditions required for a given graph to admit an identical biarithmetic IASI gain much importance. A necessary and sufficient condition for the existence of an identical biarithmetic IASI for a given graph.

Theorem 3.12. [29] *A graph G admits an identical biarithmetic IASI if and only if it is bipartite.*

The following theorem have established a necessary and sufficient condition for a given graph G to have a prime arithmetic IASI.

Theorem 3.13. [31] *A graph G admits a prime arithmetic IASI if and only if it is bipartite.*

In view of Theorem 3.13, the paths, trees, even cycles and all acyclic graphs admit prime arithmetic IASIs.

Invoking Theorem 3.13 and 3.12, we can easily establish the following theorem.

Theorem 3.14. *The existence of an identical biarithmetic IASI for a given graph G implies the existence of a prime arithmetic IASI for G and vice versa.*

Proof. If G admits a prime arithmetic IASI, then by Theorem 3.13, G is bipartite. Then by Theorem 3.12 G admits an identical arithmetic IASI. Conversely, if G admits an identical arithmetic IASI, then by Theorem 3.12, G is bipartite. Then, by Theorem 3.13, G admits a prime arithmetic IASI. □

4. Arithmetic IASIs of Graph Operations

In this section, we go through the results regarding the admissibility of induced arithmetic IASI by the union and join of two arithmetic IASI-graphs.

The union of two given graphs $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$, denoted by $G_1 \cup G_2$, is the graph whose vertex set is $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2$. The existence of an (induced) arithmetic IASI for the union of two arithmetic IASI-graphs has been established in following result.

Proposition 4.1. [32] *The union of two arithmetic IASI-graphs admits an induced arithmetic IASI-graph.*

If f_1 and f_2 are the arithmetic IASIs of the graphs G_1 and G_2 then an induced IASI of $G_1 \cup G_2$ is the function f such that $f = f_1$ for all vertices of G_1 and $f = f_2$ for all vertices in G_2. It is to be noted that if G_1 and G_2 have a common vertex, say v, then we have $f_1(v) = f_2(v)$.

The corresponding theorem on the union of two isoarithmetic IASI-graphs is the following.

Theorem 4.2. [32] *The union of two isoarithmetic IASI-graphs admits an induced isoarithmetic IASI-graph if and only if all the vertices in both G_1 and G_2 have the same deterministic index.*

The admissibility of biarithmetic IASI by the union of two biarithmetic IASI-graphs is a particular case of Proposition 4.1. We also observe that the admissibility of a prime arithmetic IASI of a given graph is also a special case of Proposition 4.1.

As all possible cases regarding the graph union of two arithmetic IASI-graphs have been covered, let us consider the join of two graphs as the next operation for our study. The *join* of two graphs $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$, denoted by $G_1 + G_2$, is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup E_{ij}$, where $E_{ij} = \{u_iv_j : u_i \in G_1, v_j \in G_2\}$.

Note that, in $G_1 + G_2$, every vertex of G_1 is adjacent to all vertices of G_2 (and vice versa). Hence, all arithmetic IASIs of G_1 and G_2 need not constitute arithmetic IASIs for their join $G_1 + G_2$. The existence of an (induced) arithmetic IASI for the join of two arithmetic IASI-graphs is established in the following theorem.

Theorem 4.3. [32] *The join of two arithmetic IASI-graphs admits an arithmetic IASI if and only if the deterministic index of every vertex of one graph is an integral multiple or divisor of the deterministic index of every vertex of the other graph, where this integer is less than or equal to the set-indexing number of the vertex having smaller deterministic index.*

The join of two arithmetic IASI-graphs is an arithmetic IASI if and only if the deterministic ratio of every edge in the induced sub graph graph $\langle E_{ij} \rangle$ of $G_1 + G_2$ is a positive integer which lies between 1 and the set-indexing number of its end vertex having minimum deterministic index. Hence, we have

Theorem 4.4. [32] *The join G of two arithmetic IASI-graphs G_1 and G_2 admits an arithmetic IASI if and only if the induced subgraph $\langle E_{ij} \rangle$ of G admits an (induced) arithmetic IASI, where $E_{ij} = \{u_i v_j : u_i \in G_1, v_j \in G_2\}$.*

The following result establishes the admissibility of arithmetic IASIs by the join of two isoarithmetic IASI-graphs.

Theorem 4.5. [32] *The join of two isoarithmetic IASI-graphs is an arithmetic IASI-graph if and only if the deterministic index of the elements of one graph is a positive integral multiple of the deterministic index of the elements of the other, where this integer is less than or equal to the minimum of the set-indexing numbers of the elements of the graph whose vertices have the smaller deterministic index.*

The following result provides the necessary and sufficient condition required for the join of two isoarithmetic IASI-graphs to have an induced isoarithmetic IASI.

Proposition 4.6. [32] *The join of two isoarithmetic IASI-graphs admits an isoarithmetic IASI if and only if all the vertices in both G_1 and G_2 have the same deterministic index.*

Invoking Theorem 4.5 and Proposition 4.6, we have

Theorem 4.7. [32] *The join G of two isoarithmetic IASI-graphs admits an arithmetic IASI, that is not an isoarithmetic IASI, if and only if the induced IASI of the induced subgraph $\langle E_{ij} \rangle$ of G is an identical biarithmetic IASI of $\langle E_{ij} \rangle$.*

The admissibility of an induced biarithmetic IASI by the join of two biarithmetic IASI-graphs is a particular case of Theorem 4.3. It can also be observed that the join of two isoarithmetic IASI-graphs will never haves an induced biarithmetic IASI. Let us now consider the existence of identical biarithmetic IASI for the join of two identical biarithmetic IASI-graphs. Hence, we have

Proposition 4.8. [32] *The join of two identical arithmetic IASI-graphs will never admit an identical arithmetic IASI.*

The proof of the above theorem is obvious from the fact that the join of two bipartite graph is not a bipartite graph.

Let us now discuss the admissibility of arithmetic IASI by certain graph products. The *Cartesian product* of two graphs $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$, denoted by $G_1 \Box G_2$, is the graph with vertex set $V_1 \times V_2$ defined as follows. Let $u = (u_1,u_2)$ and $v = (v_1,v_2)$ be two points in $V_1 \times V_2$. Then, u and v are adjacent in $G_1 \Box G_2$ whenever [$u_1 = v_1$ and u_2 is adjacent to v_2] or [$u_2 = v_2$ and u_1 is adjacent to v_1].

The following theorem establishes the admissibility of arithmetic IASI by the Cartesian product of two arithmetic IASI-graphs.

Theorem 4.9. [32] *The Cartesian product of two arithmetic IASI-graphs G_1 and G_2 admits an arithmetic IASI if and only if the deterministic ratio of every edge between corresponding vertices of any two adjacent copies of G_1 (or G_2) in $G_1 \Box G_2$, is a positive integer which lies between 1 and l, where l is the set-indexing number of its end vertex having smaller deterministic index.*

Invoking Proposition 4.9, we have established the following result.

Corollary 4.10. [32] *The Cartesian product of two isoarithmetic IASI-graphs admits an isoarithmetic IASI if and only if all vertices in both G_1 and G_2 have the same deterministic index.*

The admissibility of biarithmetic IASI by the Cartesian product of two biarithmetic IASI-graphs is a particular case of Theorem 4.9. The admissibility of an identical biarithmetic IASI by the Cartesian product of two identical biarithmetic IASI-graphs has been established in the result given below.

Proposition 4.11. [32] *The Cartesian product of two identical biarithmetic IASI-graphs admits an identical biarithmetic IASI.*

The proof of above result immediately follows from the fact that the Cartesian product of two bipartite graphs is also a bipartite graph.

Another significant product of two graphs G_1 and G_2 is their corona, denoted by $G_1 \circ G_2$, is the graph obtained taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2 and then joining the i-th point of G_1 to every point in the i-th copy of G_2. The admissibility of arithmetic IASI by the corona of two graphs follows from the following result.

Theorem 4.12. [32] *Let G_1 and G_2 are an arithmetic IASI-graphs. Then, the corona $G_1 \circ G_2$ admits an arithmetic IASI if and only if the deterministic index of every vertex of one graph is an integral multiple or a divisor of the deterministic index of every vertex of the other, where this integer is less than or equal to the set-indexing number of the vertex having smaller deterministic index.*

In view of Proposition 4.9, we now establish the following results.

Proposition 4.13. [32] *The corona of two isoarithmetic IASI-graphs admits an isoarithmetic IASI if and only if all vertices in both G_1 and G_2 have the same deterministic index.*

Proposition 4.14. [32] *The corona of two identical biarithmetic IASI-graphs does not admit an identical biarithmetic IASI.*

The fact that the corona of any two graphs will never be a bipartite graph. Hence the above theorem follows form the Theorem 3.12.

Let us review the results regarding the admissibility of arithmetic IASI by the complements of given arithmetic IASI-graphs. Note that the vertices of a graph G and its complements have the same set-labels and hence the same deterministic indices.

Theorem 4.15. [32] *The complement of an arithmetic IASI-graph G admits an arithmetic IASI f if and only if the deterministic index of any vertex of G is an integral multiple or divisor of the deterministic index of every other vertex of G, where this integer lies between* 1 *and* $\min|f(v_i)|$ *for all* $v_i \in V(G)$.

The following theorem is an immediate consequence of the fact that the set-labels of the vertices of an isoarithmetic IASI-graph are AP-sets with the same common difference.

Proposition 4.16. [32] *The complement of an isoarithmetic IASI-graph admits an (induced) isoarithmetic IASI.*

The remaining case in this context is the verification of the existence of an identical biarithmetic IASI by the complement of an identical biarithmetic IASI. Hence, we have

Proposition 4.17. [32] *The complement of an identical biarithmetic IASI-graph never admits an identical biarithmetic IASI.*

The proof of the above proposition is a immediate consequence of the fact that the complement of a bipartite graph is not a bipartite graph.

Analogous to the above theorem, we can verify whether the complement of a prime arithmetic IASI-graph admits a prime arithmetic IASI as follows.

Proposition 4.18. [32] *The complement of a prime arithmetic IASI-graph does not admits a prime arithmetic IASI.*

5. Dispensing Number of Certain Graph Classes

By Theorem 3.13, a non-bipartite graph does not admit a prime AIASI. That is, some edges of a non-bipartite graph have non-prime deterministic ratio. Then, we define the following notion.

Definition 5.1. [31] The minimum possible number of edges in a graph G that do not have a prime deterministic ratio is called the *dispensing number* of G and is denoted by $\vartheta(G)$.

In other words, the dispensing number of a graph G is the minimum number of edges to be removed from G so that it admits a prime arithmetic IASI. Hence, we have

Theorem 5.2. [31] *If $b(G)$ is the number of edges in a maximal bipartite subgraph of a graph G, then $\vartheta(G) = |E(G)| - b(G)$.*

Invoking Theorem 5.2, we investigate the sparing number about the dispensing number of certain graph classes.

Proposition 5.3. [31] *The dispensing number of an odd cycle is* 1.

Theorem 5.4. [31] *The dispensing number of a complete graph K_n is*

$$\vartheta(K_n) = \begin{cases} \frac{1}{4}n(n-2) & \text{if } n \text{ is even} \\ \frac{1}{4}(n-1)^2 & \text{if } n \text{ is odd.} \end{cases}$$

Invoking Theorem 5.4, we estimate the dispensing number of a split graph and complete split graph in the following theorem.

Theorem 5.5. [31] *Let G be a split graph with a block K_r and an independent set S, where $K_r \cup \langle S \rangle$. Then,*

$$\vartheta(G) = \begin{cases} \frac{1}{4}r(r-2) & \text{if } r \text{ is even} \\ \frac{1}{4}(r-1)^2 & \text{if } r \text{ is odd.} \end{cases}$$

Next, we proceed to determine the dispensing number of the union of two graphs.

Theorem 5.6. [31] *Let G_1 and G_2 be two given IASI-graphs. Then, $\vartheta(G_1 \cup G_2) = \vartheta(G_1) + \vartheta(G_2) - \vartheta(G_1 \cap G_2)$. In particular, if G_1 and G_2 are two edge-disjoint graphs, then $\vartheta(G_1 \cup G_2) = \vartheta(G_1) + \vartheta(G_2)$.*

The following theorem provided a closed formula for finding the dispensing number of the join of two arbitrary graphs.

Theorem 5.7. [31] *Let $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ be two non-empty graphs. The dispensing number of G_1+G_2 is $\vartheta(G_1+G_2)=\min\{|E_1|+\vartheta(G_2),|E_2|+\vartheta(G_1)\}$.*

The dispensing number of certain standard graph products have also been determined in [31]. They are as follows.

Theorem 5.8. [31] *Let $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ be two non-empty graphs. The dispensing number of their Cartesian product $G_1\square G_2$ is $\vartheta(G_1\square G_2)=|V_1|\vartheta(G_2)+|V_2|\vartheta(G_1)$.*

Theorem 5.9. [31] *Let $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ be two non-empty graphs. The dispensing number of their corona product $G_1\odot G_2$ is $\vartheta(G_1\odot G_2)=\vartheta(G_1)+|V_1||E_2|$.*

In view of the above theorems, dispensing number of some standard graph classes has also been determined in [31]. Invoking Theorem 5.6, we establish the following theorem on the dispensing number of Eulerian graphs.

Theorem 5.10. [31] *The dispensing number of an Eulerian graph G is the number of odd cycles in G.*

A *cactus G* is a connected graph in which any two simple cycles have at most one vertex in common. That is, every edge in a cactus belongs to at most one simple cycle. The cycles in a cactus are edge disjoint. Hence, we have

Theorem 5.11. [31] *The dispensing number of a cactus G is the number of odd cycles in G.*

Another graph class similar to cactus graphs is a *block graph* (or a *clique tree*) which is defined as an undirected graph in which every biconnected component (block) is a clique. The following theorem estimates the dispensing number of a block graph.

Theorem 5.12. [31] *The dispensing number of a block graph G is the sum of the dispensing numbers of the cliques in G.*

In view of Theorem 5.7, the dispensing number of wheel graphs, defined by $W_{n+1}=C_n+K_1$, has been determined [31] as follows.

Proposition 5.13. [31] *The dispensing number of a wheel graph W_{n+1} is given by $\lceil \frac{n}{2} \rceil$.*

A *split graph* is a graph whose vertex has two partitions of which one is an independent set, say S and the subgraph induced by the other is a block, say K_r. A split graph is said to be a *complete split graph* if every vertex of the independent set S is adjacent to all vertices of the block K_r. The dispensing number of a split graph has been determined as follows.

Theorem 5.14. *Let G be a split graph with a block K_r and an independent set S, where $K_r \cup \langle S \rangle$. Then,*

$$\vartheta(G) = \begin{cases} \frac{1}{4}r(r-2) & \text{if } r \text{ is even} \\ \frac{1}{4}(r-1)^2 & \text{if } r \text{ is odd.} \end{cases}$$

6. Arithmetic IASIs of Associated Graphs

If an IASI defined over a graph G can be extended or restricted by using certain rules to some graphs associated to the given graph, then these associated graphs are said to have induced IASIs (an IASI induced by the given IASI). The elements of an associated graph H of a given graph G which are common to the graph G preserve the set-labels, with respect to the induced IASI of H. If we replace an element of a given graph G by a vertex or an edge, which is not present in G, to form an associated graph H, then with respect to an induced IASI of H, it is customary to assign the same set-label of the removed element of G to the newly introduced element of H.

In the following discussions, we review certain results proposed in [33] regarding admissibility of induced arithmetic IASIs of different types by certain graphs associated to a given arithmetic IASI-graph. Let us first state the hereditary nature of the existence of arithmetic IASI.

Proposition 6.1. [33] *Let G be an arithmetic (or isoarithmetic or biarithmetic) IASI-graph. Then, any non-trivial subgraph of G is also an arithmetic (or isoarithmetic or biarithmetic) IASI Graph.*

Let f be an arithmetic IASI on G and let $H \subset G$. The proof follows from the fact that the restriction $f|_H$ of f to the subgraph H is an induced arithmetic IASI on H.

By *edge contraction operation* in G, we mean an edge, say e, is removed and its two incident vertices, u and v, are merged into a new vertex w, where the edges incident to w each correspond to an edge incident on either u or v.

We establish the following theorem for the graphs obtained by contracting the edges of a given graph G. The following theorem verifies the admissibility of arithmetic IASIs by the graphs obtained by contracting the edges of a given arithmetic IASI-graph G.

Theorem 6.2. [33] *Let G be an arithmetic (or isoarithmetic) IASI-graph and let e be an edge of G. Then, $G \circ e$ admits an arithmetic (or isoarithmetic) IASI.*

A relevant question in this context is whether the graph obtained by contracting the edges of a biarithmetic IASI-graph admits an induced biarithmetic IASI. The following theorem provides a solution to this question.

Theorem 6.3. *Let G be a biarithmetic IASI-graph and e be an edge of G with a deterministic index d_e. Then, the graph $G \circ e$ admits a biarithmetic IASI if and only if the deterministic index of every vertex v_i in G that is adjacent to the end vertices of e in G is an integral multiple (or a divisor) of the deterministic index of the end vertex of e, where this integer is less than or equal to the set-indexing number of the vertex among these having smaller deterministic index.*

Proof. Let G be a biarithmetic IASI-graph and let $e = uv$ be an edge of G. Note that the deterministic index d_e of the edge e is $\min(d_u, d_v)$, where d_u, d_v are the deterministic indices of the vertices u and v respectively. Without loss of generality, let the vertex u has the minimum deterministic index. Let $G' = G \circ e$ be the graph obtained by contracting the edge e and w be the new vertex in G' obtained by identifying the vertices u and v.

Assume, without loss of generality, that the deterministic index of a vertex v_i, which is adjacent either to u or to v, is an integral multiple of the deterministic index of u, where this integer is less than or equal to set-indexing number of the vertex, among u and v_i, having the smaller deterministic index. Then, the deterministic index of w in G' is the deterministic index of e (and hence that of u) in G and since every other edge of G' is also an edge in G, every edge in G' has a positive integral deterministic ratio, which is less than or equal to the set-indexing number of its end vertex having smaller deterministic index.

Assume, conversely, that $G' = G \circ e$ admits an induced biarithmetic IASI. Then, the deterministic index of w is the deterministic index of the edge e (and hence that of u) in G. Also, the vertex w is adjacent to all the vertices in G' corresponding to the vertices in G that are adjacent either to u or to v in G. If v_i is a vertex of G that is adjacent to either u or v in G, then v_i is also adjacent to w in G' and hence the deterministic index of v_i is n integral multiple (or a divisor) of the deterministic index of w, this integer being less than or equal to the set-indexing number of the vertex among v_i and w having smaller deterministic index. Therefore, the vertex v_i is an integral multiple (or divisor) of the deterministic index of the vertex u in G, where this integer is less than or equal to the set-indexing number of the vertex among u and v_i having smaller deterministic index. □

In view of the above theorem, we propose the following result.

Proposition 6.4. *A graph obtained by contracting an edge of an identical biarithmetic IASI-graph G does not admit an identical biarithmetic IASI.*

Proof. Let G be an identical biarithmetic IASI-graph. Then, every edge of G has a unique deterministic ratio, say k. Consider any edge $e = uv$ of G such that the deterministic indices of e, u, v are d_e, d_u, d_v respectively. Then, $d_e = \min(d_u, d_v)$. Without loss of generality, take $d_e = d_u$. Now, let $G' = G \circ e$ and w be the new vertex in G' obtained by identifying the vertices u and v of G. If v_i is a vertex in G that is adjacent to v, then $d_{v_i} = k^r\, d_u = k^r\, d_w$, where $r = \{0, 2\}$. Therefore, G' does not admit an induced identical biarithmetic IASI. □

Let G be a connected graph and let v be a vertex of G with $d(v) = 2$. Then, v is adjacent to two vertices u and w in G. If u and w are non-adjacent vertices in G, then delete v from G and add the edge uw to $G - \{v\}$. This operation is known as an *elementary topological reduction* on G. If H is a graph obtained by applying a finite number of elementary topological reductions on a given graph G, then H is said to be *homeomorphic* to G.

Theorem 6.5. *Let G be a graph which admits an isoarithmetic IASI. Then, any graph G', that is homeomorphic to G, also admits an isoarithmetic IASI.*

A graph obtained by finite number of topological reduction on a biarithmetic IASI-graph G need not be a biarithmetic IASI-graph. Then, it is interesting to enquire the conditions for a graph that is homeomorphic to a given biarithmetic IASI-graph to admit a biarithmetic IASI. We propose the required condition for the same as given below.

Proposition 6.6. *Let G be a biarithmetic IASI-graph and v be a vertex of G, not in any triangle of G, such that $d(v) = 2$. Then, the graph obtained by an elementary topological reduction on G with respect to the vertex v, admits a biarithmetic IASI if the deterministic index of one adjacent vertex of v in G is an integral multiple of the deterministic index of the other adjacent vertex of v where this integer is less than or equal to the set-indexing number of the vertex among these two having smaller deterministic index.*

Proof. Let the graph G admits a biarithmetic IASI f. Let v be a vertex of degree 2 in G having deterministic index d_v. Let u and w be the two non-adjacent vertices in G that are adjacent to v in G, having the deterministic indices d_u and d_w respectively. Also, let G' be the graph obtained by removing the vertex v from G and and joining the vertices u and w.

Without loss of generality, let $d_u = l\,d_w$, where $l \leq |f(w)|$ is a positive integer. Then, the deterministic ratio of the edge uw is l. Since uw is the only edge of G' that is not in the biarithmetic IASI-graph G, all edges in G' have positive integral deterministic ratios. That is, G' admits an induced biarithmetic IASI. □

Proposition 6.7. *A graph obtained from an identical biarithmetic IASI-graph by applying an elementary topological reduction on G, is not an identical biarithmetic IASI-graph.*

Proof. Let the graph G admits an identical biarithmetic IASI f. Let v be a vertex of degree 2 in G having deterministic index d_v. Let u and w be the two non-adjacent vertices in G that are adjacent to v in G, having the deterministic indices d_u and d_w respectively. Since G admits an identical biarithmetic IASI, the deterministic ratios of both edges uv and vw are k. Then, Therefore, we have $d_u = r\,d_v$, where $r = k^s : r \in \{-1, 1\}$. Similarly, $d_u = l\,d_v$, where $l = k^t : t \in \{-1, 1\}$. Let G' be the graph obtained by removing the vertex v from G and and joining the vertices u and w. Then, the deterministic ratio of the edge uv is either 1 or k^2. Therefore, G' does not admit an identical biarithmetic IASI. □

Another associated graph of a given graph G is its graph subdivision. A *subdivision* of a graph G is the graph obtained by adding vertices of degree two into some or all of its edges.

Theorem 6.8. [33] *The graph subdivision G^* of an arithmetic (or isoarithmetic) IASI-graph G also admits an induced arithmetic (or isoarithmetic) IASI.*

Let us now recall the following definition of line graph of a graph. For a given graph G, its line graph $L(G)$ is a graph in which each vertex of $L(G)$ corresponds to an edge of G (that is, there exists a one to one correspondence between $V(L(G))$ and $E(G)$) and two vertices of $L(G)$ are adjacent if and only if their corresponding edges in G are incident on a common vertex in G.

The following theorem addresses the question whether the line graph of an arithmetic (or an isoarithmetic) IASI-graph admits an arithmetic (or isoarithmetic) IASI.

Theorem 6.9. [33] *If G is an arithmetic (or isoarithmetic) IASI-graph, then its line graph $L(G)$ is also an arithmetic (or isoarithmetic) IASI-graph.*

The above theorem is an obvious consequence of the fact that the deterministic index of edges of an isoarithmetic IASI-graph are the same.

The following results provide the characteristics of the line graph of a biarithmetic IASI-graph.

Theorem 6.10. [33] *Let G be a biarithmetic IASI-graph. Then, its line graph $L(G)$ admits an isoarithmetic IASI if and only if G is bipartite.*

Proof. If G is bipartite with a bipartition (X,Y), then label the vertices in X by the AP-sets having common difference d and label the vertices in Y by distinct AP-sets having common difference kd, where d is a positive integer greater than 1 and k is a positive integer such that $1 < k \le \min|f(v_i)|;\ v_i \in X$. Then, all edges of $G$and hence all the vertices of $L(G)$ have the unique deterministic index d. Therefore, $L(G)$ admits an isoarithmetic IASI.

Conversely, if $L(G)$ admits an isoarithmetic IASI, then all vertices of $L(G)$ and all edges of G have the unique deterministic index, say d, where d is a positive integer greater than 1. Then, the deterministic index of one end vertex of every edge of G is d and that of the other end vertex is an integral multiple of d. Let X be the set of vertices of G having the deterministic index d and $Y = V(G) - X$. Then, (X,Y) will be a bipartition of G. $\square$

A necessary condition for the line graph of a biarithmetic IASI-graph G admits a biarithmetic IASI has been established in [33] as follows.

Theorem 6.11. [33] *If the line graph $L(G)$ of a biarithmetic IASI-graph G admits a biarithmetic IASI, then G is acyclic.*

The converse of the theorem need not be true. All acyclic graphs need not admit an induced biarithmetic IASI. For example, the graph $K_{1,n}$ admits a biarithmetic IASI and is acyclic with all its edges, but its line graph $L(K_{1,n}) = K_n$ does not admit an induced biarithmetic IASI. Then, finding a necessary and sufficient condition for the line graph of a biarithmetic IASI-graph to admit an induced biarithmetic IASI.The following theorem establishes the necessary and sufficient condition for a biarithmetic IASI-graph so that its line graph is a biarithmetic IASI-graph.

Theorem 6.12. [33] *The line graph of a biarithmetic IASI-graph admits a biarithmetic IASI if and only if G is a path.*

Another graph associated to a given graph G is its total graph which is defined as follows. The *total graph* of a graph G, denoted by $T(G)$, is the graph having the property that a one-to one correspondence can be defined between its points and the elements (vertices and edges) of G such that two points of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent (if both elements are edges or if both elements are vertices) or they are incident (if one element is an edge and the other is a vertex). Hence, it can be noted that both G and $L(G)$ are the subgraphs of $T(G)$.

The following result is about the admissibility of an induced arithmetic IASI by the total graph of a given arithmetic IASI-graph.

Theorem 6.13. [33] *If G is an arithmetic (or isoarithmetic) IASI-graph, then its total graph $T(G)$ is also an arithmetic (or isoarithmetic) IASI-graph.*

The non-existence of an induced biarithmetic IASI for the total graph of a given biarithmetic IASI-graph G has been established in [33] as

Theorem 6.14. [33] *The total graph of a biarithmetic IASI-graph is an arithmetic IASI-graph, but not a biarithmetic IASI-graph.*

Another popular graph associated with a given graph G is a *subdivision graph* which is defined as the graph obtained from G by introducing a new vertex to some or all of its edges. The existence of induced arithmetic IASIs for the subdivision graphs of given arithmetic IASI-graphs has been established in the following result.

Theorem 6.15. *A graph subdivision G^* of a given arithmetic (isoarithmetic) IASI-graph G also admits arithmetic (isoarithmetic) IASI.*

The following result established the non-existence of an induced biarithmetic IASI for a subdivision of a biarithmetic IASI.

Theorem 6.16. *The graph subdivision G^* of a given biarithmetic IASI-graph G does not admit a biarithmetic IASI.*

The above theorem is proved as a consequence of the fact explained below. Let $e = uv$ be an edge of a given biarithmetic IASI G. If we introduce a new vertex w to the edge e then, the edge e is deleted and two new edges uw and wv will be formed. If $d_u < d_v$, where d_u and d_v are the deterministic indices of u and v respectively, then $d_w = d_e = d_u$. Therefore, the deterministic ratio of the edge uw is 1. Hence the induced IASI is not a biarithmetic IASI.

Due to the above theorem, it is obvious that the subdivisions of an identical biarithmetic graph do not admit an induced identical biarithmetic IASIs.

In the following section, we review the results established on the semi-arithmetic IASIs in [32] and [33].

7. On Semi-Arithmetic IASI Graphs

We have already mentioned about the two types of semi-arithmetic IASIs. If the deterministic ratio of every edge of G is greater than the set-indexing number of its end vertex having smaller deterministic index, then we can see that in the sumset S of the set-labels of these vertices, first m elements have the deterministic d, but the difference between the m-th element and the

$(m+1)$-th element is not d. The next m elements, from $(m+1)$-th element to $2m+1)$-th element, have the deterministic index d and the difference between the $(2m+1)$-th element and the $(2m+2)$-th element is not d. This pattern can be seen till the final element of the sumset S. Hence, we have

Theorem 7.1. [33] *Every semi-arithmetic IASI of the first kind of a graph G is a strong IASI of G.*

As a result of the above theorem, we notice that no compatibility classes in the cross product of the set-labels of adjacent vertices, contain more than 1 element. Hence,

Corollary 7.2. [33] *Let f be a semi-arithmetic IASI of first kind of a graph G and let v_i and v_j be two adjacent vertices in G. Then, all the compatibility classes in $f(v_i) \times f(v_j)$ are trivial classes.*

An interesting question that arises in this context is about the existence of uniform semi-arithmetic IASIs. The following theorem establishes the necessary and sufficient condition for a semi-arithmetic IASI to be uniform.

Proposition 7.3. [33] *If f is a semi-arithmetic IASI of first kind of a given graph G, then no edge of G has a prime set-indexing number.*

The following theorem discusses the condition for a semi-arithmetic IASI of a graph G to be a uniform IASI.

Theorem 7.4. [33] *A semi-arithmetic IASI of first kind of a graph G is a uniform IASI if and only if either G is bipartite or $V(G)$ is uniformly set-indexed.*

What are the conditions required for a semi-arithmetic IASI of the second kind to be a strong IASI? The following theorem has provided an answer to this question.

Theorem 7.5. [33] *Let f be an semi-arithmetic IASI defined on G. Also, let $|f(v_j)| = q.|f(v_i)| + r, 0 < r < |f(v_i)|$. Then, f is a strong IASI if and only if $q > |f(v_i)|$ or the differencesd_i and d_j of two set labels $f(v_i)$ and $f(v_j)$ respectively, are relatively prime.*

We note that an arithmetic IASI with arbitrary differences need not have saturated classes. In the following discussion, we find the number of maximal compatible classes for a semi-arithmetic IASIs of second kind in the following theorem.

Theorem 7.6. [33] *Let f be an arithmetic IASI with arbitrary common differences on a graph G. Let $|f(v_j)| = q.|f(v_i)|+r$. Also, let q_1 and q_2 be the positive integers such that $q_1.|f(v_j) = q_2.r$. Then, the number of elements in a maximal compatible class of $f(v_i)\times f(v_j)$ is $\lfloor \frac{|f(v_j)|}{q_1} \rfloor$.*

The following is an analogous result of Proposition 6.1 to semi-arithmetic IASI-graph.

Proposition 7.7. [33] *Any subgraph of a semi-arithmetic IASI-graph is also a semi-arithmetic IASI-graph.*

That is, the existence of semi-arithmetic IASIs is also a hereditary property.

The following theorem discusses the admissibility of semi-arithmetic IASI by certain graphs that are homeomorphic to a given semi arithmetic IASI-graph G.

Proposition 7.8. [33] *Let G be a semi-arithmetic IASI-graph and let v be an arbitrary vertex of G with $d(v) = 2$ not contained in any triangle of G. Let $G' = (G-v)\cup\{uw\}$, where u and w are adjacent vertices of v in G. Then, G' admits a semi-arithmetic IASI if and only if the deterministic indices of one of u or w is a positive integer multiple of the deterministic index of the other, where this integer is greater than the cardinality of the latter.*

The following result established the non-existence of an induced arithmetic IASI for the graph obtained from a given semi-arithmetic IASI-graph by contracting one of its edges.

Proposition 7.9. [33] *A graph obtained by contracting an edge of a semi-arithmetic IASI-graph G does not admit a semi-arithmetic IASI.*

The result follows from the fact that the newly introduced vertex, say w, in the graph obtained by contacting an edge of a semi-arithmetic IASI-graph G assumes the set-label of the deleted edge e, which is not an AP-set. The following result is also a consequence of this fact.

Proposition 7.10. [33] *A subdivision of a semi-arithmetic graph G is not an semi-arithmetic IASI-graph.*

The non-existence of an induced semi arithmetic IASI by the line graphs of given semi-arithmetic IASI-graphs has been proposed in the following proposition.

Proposition 7.11. [33] *The line graph $L(G)$ of a semi-arithmetic graph never admits a semi-arithmetic IASI.*

The above result follows from the fact that the edges of a semi-arithmetic IASI-graphs does not have AP-sets as their set-labels. The following result is also an immediate consequence of this fact.

Proposition 7.12. [33] *The total graph $T(G)$ of a semi-arithmetic graph never admits a semi-arithmetic IASI.*

8. CONCLUSION

So far, we have reviewed the studies about different types of arithmetic integer additive set-indexers of certain graphs and their characteristics. We have also proposed some new results in this article. Certain problems regarding the admissibility of different types of arithmetic IASIs by various other graph classes, graph operations and graph products and finding their corresponding dispensing numbers are still open. Characterisation of edge arithmetic IASI-graphs are also open.

More properties and characteristics of different types of IASIs, both uniform and non-uniform, are yet to be investigated. There are some more open problems regarding the necessary and sufficient conditions for various graphs and graph classes to admit certain IASIs.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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