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EXISTENCE AND UNIQUENESS RESULTS ON MIXED TYPE SUMMATION-DIFFERENCE EQUATIONS IN CONE METRIC SPACE

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Abstract. In this paper we investigate the existence and uniqueness results for Summation-Difference type equations in cone metric spaces. The results are obtained by using some extensions of Banach's contraction principle in complete cone metric space.

Keywords: difference equation; Summation equation; existence of solution; cone metric space; contraction mapping; ordered Banach space.

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1. Introduction

Existence and uniqueness of solutions of the differential equations, integral equations and Integro-differential equations have been studied by many authors using different techniques. Some fixed point theorems in cone metric spaces have been studied in [1,7, 8, 9, 10, 11]. K.L. Bondar etal[3,4, 5, 6], studied existence and uniqueness of some difference equations and summation equations.

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The aim of this paper is to study the existence and uniqueness of solutions for the summation and Summation-Difference type equations of the form:

$$x(t) = f(t) + \sum_{s=0}^{t-1} k(t,s,x(s)) + \sum_{s=0}^{b-1} h(t,s,x(s)), \quad t \in J = [0,b] \tag{1.1}$$

and

$$\Delta x(t) = f(t) + \sum_{s=0}^{t-1} k(t,s,x(s)) + \sum_{s=0}^{b-1} h(t,s,x(s)), \; t \in J = [0,b] \tag{1.2}$$

$$x(0) = x_0. \tag{1.3}$$

Where $f : J \to Z, k, h : J \times J \times Z \to Z$ are function and the given x_0 is element of Z, Z is a Banach space with $\|.\|$

In section 2, we present the preliminaries and the statement of our results.Section 3 deals with main results. Finally in Section 4, we give example to illustrate the application of our results.

2. PRELIMINARIES

Let us recall the concepts of the cone metric space and we refer the reader to [1, 8, 9, 11] for the more details.

Definition 2.1. Let E be a real Banach space and P is a subset of E. Then P is called a cone if and only if,

1.P is closed, nonempty and $P \neq 0$.

2.$a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$.

3.$x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \in E$, we define a partial ordering relation $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$.We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$. Where int P denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $\leq x \leq y$ implies $\|x\| \leq k\|y\|$, for every $x, y \in E$. The least positive number satisfying above is called the normal constant of P.

In the following way, we always suppose E is a real Banach space, P is cone in E with int $P \neq \phi$, and $\leq$ is partial ordering with respect to P.

Definition 2.2. Let X a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:

(d_1) $0 \leq d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$

(d_2) $d(x,y) = d(y,x)$, for all $x,y \in X$;

(d_3) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x,y \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space. The concept of cone metric space is more general than that of metric space. The following example is a cone metric space, see [11].

Example 2.1. Let $E = \mathbb{R}^2, p = \{(x,y) \in E : x,y \geq 0\}, x = \mathbb{R}$, and $d : X \times X \to E$ such that $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constat and then (X,d) is cone metric space.

Definition 2.3. Let X be an ordered space. A function $\Phi : X \to X$ is said to a comparison function if every $x,y \in X, x \leq y$, implies that $\Phi(x) \leq \Phi(y), \Phi(x) \leq x$ and $\lim_{n\to\infty} \|\Phi^n(x)\| = 0$, for every $x \in X$.

Example 2.2. Let $E = \mathbb{R}^2, p = \{(x,y) \in E : x,y \geq 0\}$, it is easy to check that $\Phi : E \to E$ with $\Phi(x,y) = (ax,ay)$, for some $a \in (0,1)$ is a comparison function. also if Φ_1, Φ_2 are two comparison function over $\mathbb{R}$. then

$\Phi(x,y) = (\Phi_1(x), \Phi_2(y))$ is also a comparison function over E.

Let $B = c([0,b],Z)$ be the Banach space of all continuous function from $[0,b]$ into Z endowed with supremum norm

$$\|x\|_\infty = sup\{\|x(t)\| : t \in [0,b]\}$$

Let $P = (x,y) : x,y \geq 0 \subset E = \mathbb{R}^2$, and define

$$d(f,g) = (\|f-g\|_\infty, \alpha\|f-g\|_\infty)$$

for every $f,g \in B$, then it is easily seen that (B,d) is a cone metric space.

Definition 2.4. The $x \in B$ given by

$$x(t) = x_0 + \sum_{s=0}^{t-1} f(s) + \sum_{s=0}^{t-1} [\sum_{\tau=0}^{s-1} k(s,\tau,x(\tau)) + \sum_{\tau=0}^{b-1} h(s,\tau,x(\tau))]$$

is called the solution of the initial value problem $(1.2)-(1.3)$

We need the following theorem for further discussion:

Lemma 2.1. Let (X,d) be a complete cone metric space,where P is a normal cone with normal constant K. Let $f : X \to X$ be a function such that there exists a comparison function $\Phi : P \to P$

such that

$$d(f(x), f(y)) \leq \Phi(d(x,y))$$

for very $x, y \in X$.Then f has unique fixed point.

We list the following hypothesis for our convenience:

(H_1) There exist continuous function $p_1 \cdot p_2 : J \times J \to \mathbb{R}^+$ and a comparison function $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$(\|k(t,s,u) - k(t,s,v)\|, \alpha\|k(t,s,u) - k(t,s,v)\|) \leq p_1(t,s)\Phi(d(u,v)),$$

and

$$(\|h(t,s,u) - h(t,s,v)\|, \alpha\|h(t,s,u) - h(t,s,v)\|) \leq p_2(t,s)\Phi(d(u,v)),$$

for every $t, s \in J$ and $u, v \in Z$

(H_2)
$$\sup_{t \in J} \sum_{s=0}^{b-1} [p_1(t,s) + p_2(t,s)] = 1$$

(H_3)
$$\sum_{t=0}^{b-1} \sum_{s=0}^{b-1} [p_1(t,s) + p_2(t,s)] \leq 1$$

3. MAIN RESULTS

Following are the main results in this work:

Theorem 3.1 Assume that hypotheses $(H_1) - (H_2)$ hold.Then the Summation equation (1.1) has a unique solution x on J

Proof: The operartor $F : B \to B$ is defined by

$$Fx(t) = f(t) + \sum_{s=0}^{t-1} k(t,s,x(s)) + \sum_{s=0}^{b-1} h(t,s,x(s)), \quad t \in J \tag{3.1}$$

By using the hypothesis $(H_1) - (H_2)$, We have

$(\|Fx(t) - Fy(t)\|, \alpha\|Fx(t) - Fy(t)\|)$

$$\leq \Big(\|\sum_{s=0}^{t-1} k(t,s,x(s)) + \sum_{s=0}^{b-1} h(t,s,x(s)) - \sum_{s=0}^{t-1} k(t,s,y(s)) - \sum_{s=0}^{b-1} h(t,s,y(s)\|,$$

$$\alpha\|\sum_{s=0}^{t-1} k(t,s,x(s)) + \sum_{s=0}^{b-1} h(t,s,x(s)) - \sum_{s=0}^{t-1} k(t,s,y(s)) - \sum_{s=0}^{b-1} h(t,s,y(s)\|\Big)$$

$$\leq \Big(\sum_{s=0}^{t-1} \|k(t,s,x(s)) - k(t,s,y(s))\| + \sum_{s=0}^{b-1} \|h(t,s,x(s)) - h(t,s,y(s))\|,$$

$$\alpha \sum_{s=0}^{t-1} \|k(t,s,x(s)) - k(t,s,y(s))\| + \alpha \sum_{s=0}^{b-1} \|h(t,s,x(s)) - h(t,s,y(s))\|\Big)$$

$$\leq \Big(\sum_{s=0}^{t-1} \|k(t,s,x(s)) - k(t,s,y(s))\|, \alpha \sum_{s=0}^{t-1} \|k(t,s,x(s)) - k(t,s,y(s))\|\Big)$$

$$+ \Big(\sum_{s=0}^{b-1} \|h(t,s,x(s)) - h(t,s,y(s))\|, \alpha \sum_{s=0}^{b-1} \|h(t,s,x(s)) - h(t,s,y(s))\|\Big)$$

$$\leq \sum_{s=0}^{t-1} P_1(t,s)\Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty) + \sum_{s=0}^{b-1} P_2(t,s)\Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty)$$

$$\leq \sum_{s=0}^{b-1} P_1(t,s)\Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty) + \sum_{s=0}^{b-1} P_2(t,s)\Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty)$$

$$\leq \sum_{s=0}^{b-1} [P_1(t,s) + P_2(t,s)]\Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty)$$

$$\leq \Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty) \sum_{s=0}^{b-1} [P_1(t,s) + P_2(t,s)]$$

$$\leq \Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty) \tag{3.2}$$

for every $x, y \in B$. This implies that $d(Fx, Fy) \leq \Phi(d(x,y))$, for every $x, y \in B$. Now an application of Lemma 2.1, the operator has a unique point in B. Thus equation (1.1) has unique solution.

Theorem 3.2 Assume that hypotheses $(H_1) - (H_3)$ hold. Then the initial value problem $(1.2) - (1.3)$ has a unique solution x on J

Proof: The operartor $F : B \to B$ is defined by

$$Gx(t) = x_0 + \sum_{s=0}^{t-1} f(s) + \sum_{s=0}^{t-1}[\sum_{\tau=0}^{s-1} k(s,\tau,x(\tau)) + \sum_{\tau=0}^{b-1} h(s,\tau,x(\tau))], \quad t \in J \tag{3.3}$$

By using the hypothesis $(H_1) - (H_3)$, We have

$(\|Gx(t) - Gy(t)\|, \alpha\|Gx(t) - Gy(t)\|)$

$$\leq \Big(\| \sum_{s=0}^{t-1}[\sum_{\tau=0}^{s-1} k(s,\tau,x(\tau)) + \sum_{\tau=0}^{b-1} h(s,\tau,x(\tau))] - \sum_{s=0}^{t-1}[\sum_{\tau=0}^{s-1} k(s,\tau,y(\tau)) + \sum_{\tau=0}^{b-1} h(s,\tau,y(\tau))]\|,$$

$$\alpha\| \sum_{s=0}^{t-1}[\sum_{\tau=0}^{s-1} k(s\tau,x(\tau)) + \sum_{\tau=0}^{b-1} h(s,\tau,x(\tau))] - \sum_{s=0}^{t-1}[\sum_{\tau=0}^{s-1} k(s,\tau,y(\tau)) + \sum_{\tau=0}^{b-1} h(s,\tau,y(\tau))]\|\Big)$$

$$\leq \Big(\sum_{s=0}^{t-1}\sum_{\tau=0}^{s-1}\|k(s,\tau,x(\tau))-k(s,\tau,y(\tau))\|+\sum_{s=0}^{t-1}\sum_{\tau=0}^{b-1}\|h(s,\tau,x(\tau))-h(s,\tau,y(\tau))\|,$$

$$\alpha\sum_{s=0}^{t-1}\sum_{\tau=0}^{s-1}\|k(s,\tau,x(\tau))-k(s,\tau,y(\tau))\|+\alpha\sum_{s=0}^{t-1}\sum_{\tau=0}^{b-1}\|h(s,\tau,x(\tau))-h(s,\tau,y(\tau))\|\Big)$$

$$\leq \Big(\sum_{s=0}^{t-1}\sum_{\tau=0}^{s-1}\|k(s,\tau,x(\tau))-k(s,\tau,y(\tau))\|,\alpha\sum_{s=0}^{t-1}\sum_{\tau=0}^{s-1}\|k(s,\tau,x(\tau))-k(s,\tau,y(\tau))\|\Big)$$

$$+\Big(\sum_{s=0}^{t-1}\sum_{\tau=0}^{b-1}\|h(s,\tau,x(\tau))-h(s,\tau,y(\tau))\|,\alpha\sum_{s=0}^{t-1}\sum_{\tau=0}^{b-1}\|h(s,\tau,x(\tau))-h(s,\tau,y(\tau))\|\Big)$$

$$\leq \sum_{s=0}^{t-1}\sum_{\tau=0}^{s-1}P_1(t,s)\Phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)+\sum_{s=0}^{t-1}\sum_{\tau=0}^{b-1}P_2(t,s)\Phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)$$

$$\leq \sum_{s=0}^{b-1}\sum_{\tau=0}^{b-1}P_1(t,s)\Phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)+\sum_{s=0}^{b-1}\sum_{\tau=0}^{b-1}P_2(t,s)\Phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)$$

$$\leq \Phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty)\cdot\sum_{s=0}^{b-1}\sum_{\tau=0}^{b-1}[P_1(t,s)+P_1(t,s)]$$

$$\leq \Phi(\|x-y\|_\infty,\alpha\|x-y\|_\infty) \tag{3.4}$$

for every $x,y \in B$. This implies that $d(Fx,Fy) \leq \Phi(d(x,y))$, for every $x,y \in B$. Now an application of Lemma 1, the operator has a unique point in B. Thus equation $(1.2)-(1.3)$ has a unique solution x on J.

4. Application

In this section we give an exampleas an application of main results

Example 4.1: In equations (1.1) and (1.2)-(1.3), we define

$$k(t,s,x)=ts+\frac{xs}{b},\quad h(t,s,x)=(ts)^2+\frac{tsx^2}{6},\quad s,t\in[0,2],\quad x\in C([0,2],\mathbb{R})$$

and consider metric $d(x,y)=(\|x-y\|_\infty,\alpha\|x-y\|_\infty)$ on $C([0,2],\mathbb{R})$ and $\alpha\geq 0$.

Then clearly $C([0,2],\mathbb{R})$ is a complete cone metric space.

Now we have

$$(|k(t,s,x(s))-k(t,s,y(s))|,\alpha|k(t,s,x(s))-k(t,s,y(s))|)$$

$$=(|ts+\tfrac{xs}{6}-(ts+\tfrac{ys}{6})|,\alpha|ts+\tfrac{xs}{6}-(ts+\tfrac{ys}{6})|)$$

$$=(|ts+\tfrac{xs}{6}-ts-\tfrac{ys}{6}|,\alpha|ts+\tfrac{xs}{6}-ts-\tfrac{ys}{6}|)$$

$$\leq (\tfrac{s}{6}|x-y|, \alpha\tfrac{s}{6}|x-y|)$$
$$= \tfrac{s}{6}(\|x-y\|_\infty, \alpha\|x-y\|_\infty)$$
$$= p_1^*\Phi_1^*(\|x-y\|_\infty, \alpha\|x-y\|_\infty),$$

where $p_1^*(t,s) = \frac{s}{3}$,which is function of $[0,2]\times[0,2]$ into $\mathbb{R}^+$ and a comparison function $\Phi_1^* : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi_1^*(x,y) = \frac{1}{2}(x,y)$.Also we have,

$$(|h(t,s,x(s)) - h(t,s,y(s))|, \alpha|h(t,s,x(s)) - h(t,s,y(s))|)$$
$$= (|(ts)^2 + \tfrac{tsx^2}{6} - ((ts)^2 + \tfrac{tsy^2}{6})|, \alpha|(ts)^2 + \tfrac{tsx^2}{6} - ((ts)^2 + \tfrac{tsy^2}{6})|)$$
$$= (|(ts)^2 + \tfrac{tsx^2}{6} - (ts)^2 - \tfrac{tsy^2}{6}|, \alpha|(ts)^2 + \tfrac{tsx^2}{6} - (ts)^2 - \tfrac{tsy^2}{6}|)$$
$$= (|\tfrac{tsx^2}{6} - \tfrac{tsy^2}{6}|, \alpha|\tfrac{tsx^2}{6} - \tfrac{tsy^2}{6}|)$$
$$= (\tfrac{ts}{6}|x^2-y^2|, \alpha\tfrac{ts}{6}|x^2-y^2|)$$
$$\leq \tfrac{ts}{6}(\|x^2-y^2\|_\infty, \alpha\|x^2-y^2\|_\infty)$$
$$\leq \tfrac{ts}{6}(\|x-y\|_\infty, \alpha\|x-y\|_\infty)$$
$$= p_2^*\Phi_1^*(\|x-y\|_\infty, \alpha\|x-y\|_\infty),$$

where $p_2^*(t,s) = \frac{ts}{3}$, which is function of $[0,2]\times[0,2]$ into $\mathbb{R}^+$.

Moreover

$$\sum_{s=0}^{1}[p_1^*(t,s) + p_2^*(t,s)] = \sum_{s=0}^{1}[\frac{s}{3} + \frac{ts}{3}] = \frac{1}{3}(1+t)$$

$$\sup_{t\in[0,2]} \frac{1}{2}(1+t) = 1$$

Also.

$$\sum_{t=0}^{1}\sum_{s=0}^{1}[p_1^*(t,s) + p_2^*(t,s)] = \sum_{t=0}^{1}\sum_{s=0}^{1}[\frac{s}{3} + \frac{ts}{3}] = \sum_{t=0}^{1}[\frac{1}{3}(1+t)] \leq 1$$

Thus with these choices of functions, all requirements of Theorem 3.1 and Theorem 3.2 are satisfied hence the existence and uniqueness are verified

5. Conclusion

In this paper, the existence and uniqueness of solutions for Summation-Difference type equations in cone metric spaces have been studied. Moreover an application is given.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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