KARHUNEN-LOÉVE EXPANSION OF BROWNIAN MOTION FOR APPROXIMATE SOLUTIONS OF LINEAR STOCHASTIC DIFFERENTIAL MODELS USING PICARD ITERATION

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Abstract: This work presents an application of the Picard Iterative Method (PIM) to a class of Stochastic Differential Equations where the randomness in the equation is considered in terms of the Karhunen-Loéve Expansion finite series. Two applicable numerical examples are considered to illustrate the convergence of the approximate solutions to the exact solutions and also to check the efficiency of the method. The results obtained show clearly that accuracy will be more visible by increasing the number of terms in the iteration. Thus, it is recommended for nonlinear financial models of different classes of Stochastic Differential Equations.

Keywords: linear Stratonovich SDEs; Karhunen-Loéve expansion; approximate solution; PIM; Brownian motion.

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1. INTRODUCTION

Stochastic Differential Equations (SDEs) are used to model random phenomena. Recently, SDEs have become a typical model for quantifying financial products like options, stocks, interest rates,

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and some other underlying assets [1]. Merton's work brought phenomena advancement in the application of SDEs to financial problems [2]. Several other areas, such as population dynamic, option pricing, and so on have been discussed in [3]. The methods for solving different classes of stochastic differential equations are presented in [4, 5], but the explicit solutions are limited for such equations [6]. Picard Iteration Method (PIM) is one of the effective numerical methods that can be used for solving differential equations. It transforms a differential equation into an integral equation. This method can be used to get the approximate analytical solutions of linear and nonlinear class of differential equations. Youssef, in [7] used Picard's Iterative Method with Gauss-Seidel Method to solve some IVP. Xiaohui *et al*., in [8] applied PIM to singular fractional differential equation. In [9], Edeki *et al*., used PIM and Differential Transforms Method to obtain the solutions of linear and nonlinear differential equations. Rach, in his research work, used Adomian Decomposition Method (ADM) and PIM [10]. ADM and PIM were compared by Bellomo and Sarafyan [11], Antonis *et al.,* used Picard's for the approximation of stochastic differential equations by applying it to Libor models [12]. Little research has been done on the application of PIM to stochastic differential models. Therefore, in this paper, our concern is to modify this numerical method by considering the Stratonovich linear stochastic differential models where the random path of the SDEs are being taken care of by the Karhunen-Loeve expansion of Brownian Motion.

The rest of the work is organized as follows. In section 2, linear SDEs are discussed, the method of solution is discussed in section 3, two numerical examples are given in section 4, and lastly, a concluding remark is presented in section 5.

2. LINEAR STOCHASTIC DIFFERENTIAL MODEL

Considering the Stratonovich SDE of the form:

$$\begin{cases} dX = f\left(X,t\right)dt + \sum_{j=1}^{d} \sigma\left(X,t\right) \circ dw^{j}\left(t\right) \\ X\left(0\right) = X_{0}, \end{cases} \tag{2.1}$$

where, f and σ are the drift and volatility coefficients respectively, $X(t) = X \in \mathbb{R}^n$ represents the stochastic process and $W(t)$ is the standard Brownian motion (Wiener process), also the dimension of the Brownian Motion is denoted with d. We remark as follows that $W = \{W(t), 0 \le t \le T\}$ is a 1-dimensional Brownian motion [13].

Basically, the SDE in (2.1) can be written in Itô form as:

$$\begin{cases} dX = f(t,X)dt + \sum_{j=1}^{d} \sigma(t,X)dW^j(t) \\ X(0) = X_0. \end{cases} \qquad (2.2)$$

In integral form, the SDE in (2.1) is written as:

$$X(t) = X_0 + \int_0^t f(s, X(s))ds + \int_0^t \sigma(s, Y(s))dW(s). \qquad (2.3)$$

The first integral and second integral in (2.3) are known as the Riemann-Stiltjes and stochastic integral, respectively.

3. METHODS OF SOLUTION

This section presents the concepts of the Picard Iterative Method (PIM) and Karhunen-Loeve Expansion (K-LE).

3.1 Picard Iterative Method (PIM)

Let us consider the differential equation of this form:

$$\begin{cases} x' = f(t,x), \\ x(0) = x_0. \end{cases} \qquad (3.1)$$

First order differential equations fall under this type of equation, and PIM is one of the suitable methods for handling this type of differential equation. Now, by integrating both sides of (3.1), we get:

$$\int_0^t x'(s)ds = \int_0^t f(s, x(s))ds, \qquad (3.2)$$

therefore, following the basic concept of calculus (3.2) becomes:

$$x(t) - x(0) = \int_0^t f\big(s, x(s)\big)ds,$$

$$\Rightarrow x(t) = x(0) + \int_0^t f\big(s, x(s)\big)ds. \tag{3.3}$$

Since $x(t)$ is appearing on both sides of equation (3.3) for arbitrary t, we therefore adopt this iterative process by choosing an initial condition:

$$x(0) = x_0, \ \ for \ n \geq 1, \ n \in \mathbb{Z}^+:$$

$$\therefore \ x_{n+1} = x_0 + \int_0^t f\big(s, x_n(s)\big)ds. \tag{3.4}$$

In what follows, the approximation of (3.1) is:

$$x(t) = \lim x_{n+1}(t), \ \ as \ n \to \infty.$$

The existence and uniqueness of this method have been studied by [9, 14].

3.2 Karhunen-Loeve Expansion (K-LE) finite series of Brownian Motion

The K-L expansion is an extension of the Fourier Transform (FT). Its analysis is from deterministic functions to probabilistic form. It is also referred to as a bi-orthogonal stochastic process expansion [15]. Stefano [16], in his work, used the K-L expansion to characterize the Brownian Motion (Wiener process). In this expansion [16], $\{X(t, w), \ t \in T\}$ represents a stochastic process in terms of sequence of identically and independent sample variables $\{z_i, i \in N\}$.

The Wiener process, $W(t)$ has a trajectory belonging to $L^2([0, T])$ for almost all $W's$.

Therefore, The K-L expansion for Brownian Motion can be represented as:

$$\begin{cases} W(t) = W(t, \omega) = \sum_{i=0}^{\infty} z_j(w)\Psi_j(t), \ 0 \leq t \leq T, \\ \Psi_j(t) = \dfrac{2\sqrt{2T}}{(2j+1)} \sin\left(\dfrac{(2j+1)\pi t}{2T}\right), \end{cases} \tag{3.5}$$

where $\Psi_j(t)$ form a basis of orthogonal function [6, 16].

So, simplifying (3.5) with $T = 1$ gives:

$$W(t) = \sqrt{2} \sum_{j=0}^{\infty} \left[\frac{\sin\left[(j+0.5)\pi t \right]}{(j+0.5)\pi} \right] z_j \quad . \tag{3.6}$$

Thus,

$$dW(t) = \sqrt{2} \sum_{j=0}^{\infty} \left(\cos\left[(j+0.5)\pi t \right] \right) z_j \quad . \tag{3.7}$$

Replacing the random path of SDE (2.1) with (3.7), we obtain:

$$\begin{cases} dX(t) = f(X,t)dt + \sigma(X,t)d\left(\sum_{j=0}^{5} z_j \Psi_j(t) \right), \\ X(0) = X_0. \end{cases} \tag{3.8}$$

For the purpose of this research work, z_j is generated using a mathematical computer software.

4. NUMERICAL EXAMPLES

Example 4a: Consider the Linear Stratonovich Stochastic Differential Equation [12]:

$$\begin{cases} dX(t) = \beta X(t) \circ dW(t), \\ X(0) = 1. \end{cases} \tag{4.1}$$

The exact solution of (4.1) is

$$X_{exact} = X(0) \exp(\beta W(t)), \ t \in [0, T].$$

Next, we re-express (4.1) in an integral form of (3.4):

$$X_{n+1} = X_0 + \int_0^t f(s, X_n(s)) dt \tag{4.2}$$

$$X_0 = 1$$

$$X_1 = 1 + \int_0^t \left(\beta X_0 dW_s \right) ds,$$

$$= 1 + \int_0^t dW_s ds, \beta = 1,$$

$$= 1 + \int_0^t \sqrt{2} \sum_{j=0}^{\infty} \left(\cos\left[(j+0.5)\pi t \right] \right) z_j ds.$$

$$X_2 = 1 + \int_0^t \left((X_1) \, dW_s \right) ds$$

$$= 1 + \beta \int_0^t (X_1) \, dW_s \, ds,$$

$$= 1 + \int_0^t (X_1) \sqrt{2} \sum_{j=0}^{\infty} \left(\cos\left[(j+0.5)\pi t \right] \right) z_j \, ds$$

$$X_3 = 1 + \int_0^t \left((X_2) \, dW_s \right) ds$$

$$= 1 + \beta \int_0^t (X_2) \, dW_s \, ds,$$

$$= 1 + \int_0^t (X_2) \sqrt{2} \sum_{j=0}^{\infty} \left(\cos\left[(j+0.5)\pi t \right] \right) z_j \, ds$$

$$X_4 = 1 + \int_0^t \left((X_3) \, dW_s \right) ds$$

$$= 1 + \beta \int_0^t (X_3) \, dW_s \, ds,$$

$$= 1 + \int_0^t (X_3) \sqrt{2} \sum_{j=0}^{\infty} \left(\cos\left[(j+0.5)\pi t \right] \right) z_j \, ds$$

$$X_5 = 1 + \int_0^t \left((X_4) \, dW_s \right) ds$$

$$= 1 + \beta \int_0^t (X_4) \, dW_s \, ds,$$

$$= 1 + \int_0^t (X_4) \sqrt{2} \sum_{j=0}^{\infty} \left(\cos\left[(j+0.5)\pi t \right] \right) z_j \, ds$$

$$\vdots$$

$$X_{n+1} = X(0) + \int_0^t \left((X_n) \, dW_s \right) ds.$$

Example 4b: Consider the Linear Stochastic Differential Equation [18]:

$$\begin{cases} dX(t) = \dfrac{1}{2} X(t) dt + X(t) \circ dW(t), \\ X(0) = 1. \end{cases} \tag{4.3}$$

The Stratonovich exact solution of (4.3) is given as

$$X_{exact} = \exp\left(\frac{t}{2} + B(t) \right). \tag{4.4}$$

Next, we re-express (4.3) in an integral form as in (3.4):

$$X_{n+1} = X_0 + \int_0^t f\left(s, X_n\left(s\right)\right)dt. \tag{4.5}$$

Thus, the following are obtained:

$$X_0 = 1,$$

$$X_1 = 1 + \frac{1}{2}\int_0^t X_0 + \int_0^t \left(X_0 dW_s\right)ds,$$

$$= 1 + \frac{1}{2}\int_0^t X_0 + \int_0^t \left(X_0\right)\left(\sqrt{2}\sum_{j=0}^{\infty}\left(\cos\left[\left(j+0.5\right)\pi t\right]\right)z_j\right)ds.$$

$$X_2 = 1 + \frac{1}{2}\int_0^t X_1 + \int_0^t \left(X_1 dW_s\right)ds,$$

$$= 1 + \frac{1}{2}\int_0^t X_1 + \int_0^t \left(X_1\right)\left(\sqrt{2}\sum_{j=0}^{\infty}\left(\cos\left[\left(j+0.5\right)\pi t\right]\right)z_j\right)ds.$$

$$X_3 = 1 + \frac{1}{2}\int_0^t X_2 + \int_0^t \left(X_2 dW_s\right)ds,$$

$$= 1 + \frac{1}{2}\int_0^t X_2 + \int_0^t \left(X_2\right)\left(\sqrt{2}\sum_{j=0}^{\infty}\left(\cos\left[\left(j+0.5\right)\pi t\right]\right)z_j\right)ds.$$

$$X_4 = 1 + \frac{1}{2}\int_0^t X_3 + \int_0^t \left(X_3 dW_s\right)ds,$$

$$= 1 + \frac{1}{2}\int_0^t X_3 + \int_0^t \left(X_3\right)\left(\sqrt{2}\sum_{j=0}^{\infty}\left(\cos\left[\left(j+0.5\right)\pi t\right]\right)z_j\right)ds.$$

$$X_5 = 1 + \frac{1}{2}\int_0^t X_4 + \int_0^t \left(X_4 dW_s\right)ds,$$

$$= 1 + \frac{1}{2}\int_0^t X_4 + \int_0^t \left(X_4\right)\left(\sqrt{2}\sum_{j=0}^{\infty}\left(\cos\left[\left(j+0.5\right)\pi t\right]\right)z_j\right)ds.$$

$$X_6 = 1 + \frac{1}{2}\int_0^t X_5 + \int_0^t \left(X_5 dW_s\right)ds,$$

$$= 1 + \frac{1}{2}\int_0^t X_5 + \int_0^t \left(X_5\right)\left(\sqrt{2}\sum_{j=0}^{\infty}\left(\cos\left[\left(j+0.5\right)\pi t\right]\right)z_j\right)ds,$$

$$\vdots$$

$$X_{n+1} = X\left(0\right) + \int_0^t \left(\left(X_n\right)dW_s\right)ds.$$

The numerical results and error analysis are given in Tables 1-2. Also, a graph is plotted for each

example, respectively. Other solution methods for linear and or nonlinear differential models (ordinary or partial), including SDEs are referred [19-31].

Table 1: Error analysis of example 4a

t	X_{exact}	$X_5(t)$	$\left|X_{exact} - X_5\right|$
0.0	1.0000	1.0000	0.0000
0.1	1.5981	1.5981	0.0000
0.2	2.2045	2.2041	0.0004
0.3	2.2390	2.2385	0.0005
0.4	2.1791	2.1787	0.0004
0.5	3.0838	3.0804	0.0034
0.6	4.9279	4.8986	0.0293
0.7	5.1487	5.1139	0.0348
0.8	4.1756	4.1608	0.0148
0.9	4.2626	4.2464	0.0162
1.0	4.7210	4.6963	0.0247
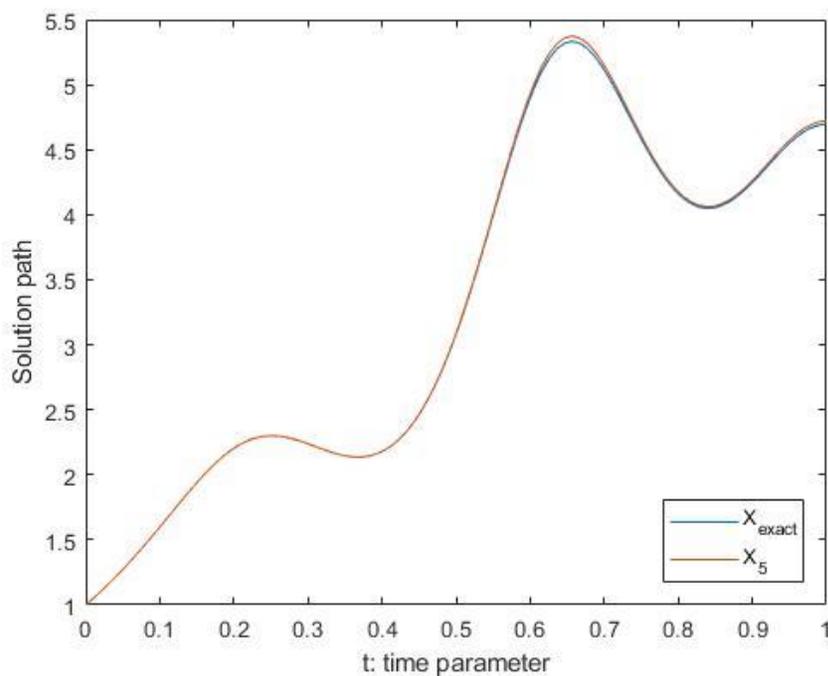
Fig. 1: Showing the graph of example 4a

Table 2: Error analysis of example 4b

t	X_{exact}	$X_6(t)$	$\left\|X_{exact} - X_6\right\|$
0.0	1.0000	1.0000	0.0000
0.1	1.6801	1.6801	0.0000
0.2	2.4363	2.4362	0.0001
0.3	2.6013	2.6011	0.0002
0.4	2.6615	2.6613	0.0002
0.5	3.9596	3.9574	0.0022
0.6	6.6520	6.6294	0.0226
0.7	7.3064	7.2742	0.0322
0.8	6.2292	6.2118	0.0174
0.9	6.6850	6.6620	0.0230
1.0	7.7837	7.7433	0.0404
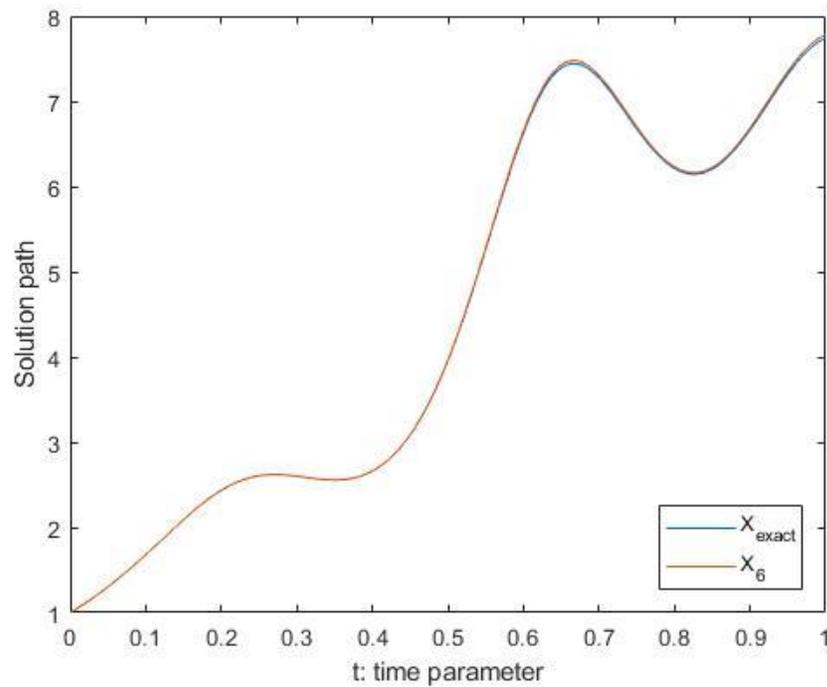
Fig. 2: Showing the graph of example 4b

4. CONCLUDING REMARKS

In this research paper, we have considered a procedure for obtaining an approximate analytical solution of a class of SDEs. The approach is by Picard Iterative Method (PIM), which transforms the differential equation into an equivalent integral equation, provided that the Lipchitz condition is satisfied and also, by converting the entire random sample into piecewise-continuous polynomials. In comparing the results obtained, it showed that more accuracy would be attained with an increasing number of terms. So based on these, further investigation can be carried out on the higher order of different classes of SDEs.

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CONFLICT OF INTERESTS

The authors declare no conflict of interest.

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