Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 4, 964-979
https://doi.org/10.28919/jmcs/4545
ISSN: 1927-5307

INTEGRATING FACTORS FOR NON-EXACT CONFORMABLE DIFFERENTIAL EQUATION

ASMA ALHABEES*, IMAN ALDARAWI

Department of Mathematics, The University of Jordan, Amman, Jordan

Abstract: Exact fractional differential equations are considered in this paper. The efforts were aimed to find, discuss and prove different cases of integrating factors that reduce a non-exact conformable fractional differential equation to exact conformable fractional differential one. Examples were explained to clarify some cases of integrating factors. What is worth saying is existence of some expressions in the differential equation may change the path of the solution, so the integrating factor can be found in different way. This was supported and illustrated by some examples in the paper.

Keywords: conformable derivative; exact conformable fractional differential equation; integrating factor.

2010 AMS Subject Classification: 26A33.

1. INTRODUCTION

Fractional calculus has taken a central place in many areas such as science, economic and engineering due to its potential in describing and modeling many phenomena in real world systems. [16, 10, 15].

---

*Corresponding author
E-mail address: a.habees@ju.edu.jo
Received February 26, 2020

There are many definitions of fractional derivative [14]. But the most used of these definitions are Riemann-Liouville and Caputo derivative. They were defined as follows:

(i) Riemann - Liouville Definition. For $\alpha \in$ [n-1, n), the α derivative of f is:

$$D_a^\alpha f(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

(ii) Caputo Definition. For $\alpha \in$ [n-1, n), the α derivative of f is:

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx$$

Properties for these definitions can be found in [12, 13].

Recently, authors in [9] presented a new definition for Fractional derivatives called conformable fractional derivative. Since then researchers pay attention to this new definition and use it to solve many equations. For example: the solutions of time and space fractional heat differential equations by conformable fractional derivative were found [6], as well as exact solutions to some conformable time fractional equations in Benjamin-Bona-Mohany family [11]. The solution of space-time fractional Fornberg–Whitham equation in series form was established [7]. Abel's formula and wronskian for conformable fractional differential equations were proposed [1]. Conformable fractional heat differential equation was solved [2]. Exact solutions of conformable fractional Harry Dym equation were found [4]. New Technique called conformable fractional reduced differential transform method (CFRDTM) and some of its Applications were given [3]. By this new defined conformable derivative, Total fractional differentials with applications to exact fractional differential equations was proposed [5]. In this paper the work has done in the sense of conformable derivative.

The paper is organized as follows: a brief review of some basic definitions and properties that used is in section 2; in section 3 we proposed main results about integrating factors with conformable sense; in section 4 examples are discussed; closing remarks and some examples are in section 5.

2. PRELIMINARIES

In this section basic properties of the conformable fractional derivative and $\alpha -$ Exact equation will be summarized.

Definition 2.1 [9]: Given a function $f:[0,\infty) \to \mathbb{R}$, and for all $t > 0, \alpha \in (0,1)$, then the conformable fractional derivative of order α is defined as:

$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}$,

T_α is called the conformable fractional derivative of f of order α .

Let $f^\alpha(t)$ stands for $T_\alpha(f)(t) = \frac{d^\alpha f}{dt^\alpha}$.

If f is α-differentiable in some $(0,b), b > 0$, and $\lim_{t \to 0^+} f^\alpha(t)$ exists, then by definition:

$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t)$.

Theorem 2.1 [9]: Let $\alpha \in (0,1)$ and f, g be α-differentiable at a point $t > 0$. Then

1. $T_\alpha(af + bg) = a\,T_\alpha(f) + b\,T_\alpha(g),\ for\ all\ a, b \in \mathbb{R}$.
2. $T_\alpha(t^p) = pt^{p-\alpha}\ for\ all\ p \in \mathbb{R}$.
3. $T_\alpha(\lambda) = 0\ for\ all\ constants\ functions\ f(t) = \lambda$.
4. $T_\alpha(fg) = f\,T_\alpha(g) + g\,T_\alpha(f)$.
5. $T_\alpha\left(\frac{f}{g}\right) = \frac{g\,T_\alpha(f) - f\,T_\alpha(g)}{g^2}$.
6. If, in addition, f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$.

Theorem 2.2 [11]: let f be differentiable and α-differentiable function in the conformable sense. Suppose that g is also differentiable and defined in the range of f. Then

$$T_\alpha(fog)\,(t) = t^{1-\alpha}\,g'(t)f'\big(g(t)\big) = g^{(\alpha)}(t)f'\big(g(t)\big).$$

More properties, definitions and theorems as Roll's Theorem and Mean Value Theorem for conformable fractional derivative can be found [9].

Definition 2.2 [5]: Let $0 < \alpha < 1. Whenever\ g$ is α-differentiable, let $d^\alpha g = g^{(\alpha)}(t)dt$. We call $d^\alpha g$ the fractional differential of g. If g is differentiable on $(0,\infty)$, then $d^\alpha g = t^{1-\alpha}g'(t)dt$.

And the fractional integral $I_\alpha^a(g)(t) = \int_a^t \frac{g(x)}{x^{1-\alpha}}dx$.

Definition 2.3 [5]: Let f be a function of two variables. We say f is α-differentiable at (x, y), if the α- increment of:

$$\Delta^\alpha f = f(x + x^{1-\alpha}\Delta x, y + y^{1-\alpha}\Delta y) - f(x, y)$$
$$= \frac{\partial^\alpha f}{\partial x^\alpha}(x, y)\Delta x + \frac{\partial^\alpha f}{\partial y^\alpha}(x, y)\Delta y + \in_1 \Delta x + \in_2 \Delta y$$

with $(\in_1, \in_2) \to (0,0)$ if $(\Delta x, \Delta y) \to (0,0)$.

The limit of $\Delta^\alpha f$ as $(\Delta x, \Delta y) \to (0,0)$, will be called the α-differentiable of f, and will be denoted by $d^\alpha f$.

For instance, $d^{\frac{1}{2}}(xy) = y\sqrt{x}\, dx + x\sqrt{y}\, dy$. Also one can notice that $d^\alpha f = 0$ if and only if $f(x, y)$ is constant.

Definition 2.4 [8] Let $f(x, y) = c$ be an equation that represents some curve in the xy −plane with $x > 0$. The Equation

$$\frac{y^\alpha - {y_0}^\alpha}{x^\alpha - {x_0}^\alpha} = \frac{{y_0}^{\alpha-1}}{{x_0}^{\alpha-1}}\, y^{(\alpha)}(x_0)$$

represents a curve passing through the point (x_0, y_0). Such a curve will be called fractional cord of the curve $f(x, y) = c$ at the point(x_0, y_0).

Remark 2.1[8]. If $\alpha = 1$, then the fractional cord equation is $\frac{y-y_0}{x-x_0} = y'(x_0)$, which is exactly the tangent line to the curve at (x_0, y_0).

Thus fractional cords represent deviation curves from the tangent line, in the sense

$\lim\limits_{\alpha\to 1} \frac{y^\alpha - {y_0}^\alpha}{x^\alpha - {x_0}^\alpha} = \lim\limits_{\alpha\to 1} \frac{{y_0}^{\alpha-1}}{{x_0}^{\alpha-1}}\, y^{(\alpha)}(x_0)$, which means $\frac{y-y_0}{x-x_0} = y'(x_0)$.

What is more interesting is, geometrical meaning of the conformable fractional derivative.

Theorem 2.3[8]. The conformable fractional derivative $y^{(\alpha)}(x_0)$ of the function $y(x)$ in the equation $f(x, y) = c$, is the slope of the tangent line to the fractional cords associated with the curve $f(x, y) = c$ at (x_0, y_0).

Definition 2.5 [5]: A first order differential equation of the form $Mdx + Ndy = 0$ is called α −Exact if there exists a function ϕ such that

$$\frac{\partial^\alpha \phi}{\partial x^\alpha} = M, and \frac{\partial^\alpha \phi}{\partial y^\alpha} = N \ .$$

Consequently,

$$d^\alpha \phi = Mdx + Ndy = 0.$$

From the properties of the conformable fractional derivative, we get ϕ is a constant function.

Proposition 2.1 [5]: Let all the first partial derivatives of M and N exist and continuous. Then $Mdx + Ndy = 0$ is α-exact if and only if

$$\frac{\partial^\alpha N}{\partial x^\alpha} = \frac{\partial^\alpha M}{\partial y^\alpha}.$$

3. MAIN RESULTS

Consider the equation:

$$M\,(x,y)dx \ + \ N(x,y)dy \ = \ 0 \qquad (*)$$

As in the case of classical differential equations, the previous equation may not be an α −exact equation. Now the question is: Can one reduce the non α −exact fractional differential equation to an α -exact equation? The answer is yes using what is called *integrating factors*.In this paper we will consider some cases for the integrating factors for equation (*).

Theorem 3.1: If $M = y^\alpha f(xy)^\alpha \ and \ N = x^\alpha g(xy)^\alpha$, for some functions f and g, where $f \neq g$, then the integrating factor is:

$$\mu = \frac{1}{(xy)^\alpha (f-g)}$$

Proof: Let $v = (xy)^\alpha$

$$\frac{\partial^\alpha}{\partial y^\alpha}(\mu M) = \frac{1}{x^\alpha}\frac{\partial^\alpha}{\partial y^\alpha}\left(\frac{f(v)}{f(v)-g(v)}\right) = \frac{1}{x^\alpha}\left[\frac{d}{dv}\left(\frac{f(v)}{f(v)-g(v)}\right)\right]\left[\frac{\partial v}{\partial y}\right] y^{1-\alpha} = \ \alpha(\frac{fg' - gf'}{(f-g)^2})$$

$$\frac{\partial^\alpha}{\partial x^\alpha}(\mu N) = \frac{1}{y^\alpha}\frac{\partial^\alpha}{\partial y^\alpha}\left(\frac{g(v)}{f(v)-g(v)}\right) = \frac{1}{y^\alpha}\left[\frac{d}{dv}\left(\frac{g(v)}{f(v)-g(v)}\right)\right]\left[\frac{\partial v}{\partial x}\right] x^{1-\alpha} = \ \alpha(\frac{fg' - gf'}{(f-g)^2})$$

Now

$$\frac{\partial^\alpha}{\partial y^\alpha}(\mu M) - \frac{\partial^\alpha}{\partial x^\alpha}(\mu N) = \ \alpha\left(\frac{fg' - gf'}{(f-g)^2}\right) - \ \alpha(\frac{fg' - gf'}{(f-g)^2}) = 0$$

Hence the equation $\mu M dx + \mu N dy = 0$ is $\alpha -$exact and $\mu = \frac{1}{(xy)^{\alpha}(f-g)}$ is an integrating factor.

Theorem 3.2: If $M = f(y/x)^{\alpha}\ and\ N = g(y/x)^{\alpha}$, for some functions f and g, then the integrating factor is:

$$\mu = \frac{1}{x^{\alpha}M + y^{\alpha}N}$$

Proof: Let $v = (y/x)^{\alpha}$

$$\frac{\partial^{\alpha}}{\partial y^{\alpha}}(\mu M) = \mu M_y^{\alpha} + M\mu_y^{\alpha}$$

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}(\mu N) = \mu N_x^{\alpha} + N\mu_x^{\alpha}$$

Now

$$\frac{\partial^{\alpha}}{\partial y^{\alpha}}(\mu M) - \frac{\partial^{\alpha}}{\partial x^{\alpha}}(\mu N) = \mu\left(M_y^{\alpha} - N_x^{\alpha}\right) - N\mu_x^{\alpha} + M\mu_y^{\alpha}$$

$$= \mu\left(M_y^{\alpha} - N_x^{\alpha}\right) + N\left(\frac{\alpha M + x^{\alpha}M_x^{\alpha} + y^{\alpha}N_x^{\alpha}}{(x^{\alpha}M + y^{\alpha}N)^2}\right) - M\left(\frac{\alpha N + x^{\alpha}M_y^{\alpha} + y^{\alpha}N_y^{\alpha}}{(x^{\alpha}M + y^{\alpha}N)^2}\right)$$

$$= \frac{\frac{\alpha f'}{x^{\alpha}} + \frac{\alpha y^{\alpha}g'}{x^{2\alpha}}}{x^{\alpha}f + y^{\alpha}g} - \alpha\left(\frac{\frac{y^{\alpha}gf'}{x^{\alpha}} + \frac{y^{2\alpha}gg'}{x^{2\alpha}} + ff' + \frac{y^{\alpha}fg'}{x^{\alpha}}}{(x^{\alpha}f + y^{\alpha}g)^2}\right)$$

$$= \frac{\alpha}{x^{2\alpha}}\left(\frac{x^{\alpha}f' + y^{\alpha}g'}{x^{\alpha}f + y^{\alpha}g}\right) - \alpha\left(\frac{f'\left(f + \frac{y^{\alpha}}{x^{\alpha}}g\right) + \frac{y^{\alpha}g'}{x^{\alpha}}(f + \frac{y^{\alpha}}{x^{\alpha}}g)}{(x^{\alpha}f + y^{\alpha}g)^2}\right)$$

$$= \frac{\alpha}{x^{2\alpha}}\left(\frac{x^{\alpha}f' + y^{\alpha}g'}{x^{\alpha}f + y^{\alpha}g}\right) - \alpha\left(\frac{\frac{f'}{x^{\alpha}}(x^{\alpha}\boldsymbol{f} + y^{\alpha}g) + \frac{y^{\alpha}g'}{x^{2\alpha}}(x^{\alpha}\boldsymbol{f} + y^{\alpha}g)}{(x^{\alpha}f + y^{\alpha}g)^2}\right)$$

$$= \frac{\alpha}{x^{2\alpha}}\left(\frac{x^{\alpha}f' + y^{\alpha}g'}{x^{\alpha}f + y^{\alpha}g}\right) - \alpha\left(\frac{\frac{f'}{x^{\alpha}} + \frac{y^{\alpha}g'}{x^{2\alpha}}}{x^{\alpha}f + y^{\alpha}g}\right)$$

$$= \frac{\alpha}{x^{2\alpha}}\left(\frac{x^{\alpha}f' + y^{\alpha}g'}{x^{\alpha}f + y^{\alpha}g}\right) - \frac{\alpha}{x^{2\alpha}}\left(\frac{x^{\alpha}f' + y^{\alpha}g'}{x^{\alpha}f + y^{\alpha}g}\right) = 0$$

Hence the equation $\mu M dx + \mu N dy = 0$ is an $\alpha -$exact and $\mu = \frac{1}{x^{\alpha}M + y^{\alpha}N}$ is an integrating factor.

Theorem 3.3 : Assume $\omega = x^{\alpha} - y^{\alpha}$ and $\frac{M_y^{\alpha} - N_x^{\alpha}}{\alpha(N+M)} = \varphi(x^{\alpha} - y^{\alpha}) = \varphi(\omega)$.

Then the integrating factor is:

$$\mu = e^{I_{\alpha}^{\omega}(\varphi)} = e^{\int(\varphi(\omega)/\omega^{1-\alpha})d\omega}$$

And

$$\mu M dx + \ \mu N dy = 0$$

Is an α −exact equation.

Proof: $\frac{\partial^{\alpha}}{\partial y^{\alpha}}(\mu M) = \frac{\partial^{\alpha}}{\partial y^{\alpha}}\left(e^{I_{\alpha}^{\omega}(\varphi)}M\right) = e^{I_{\alpha}^{\omega}(\varphi)}M_y^{\alpha} - \ \alpha M\varphi(\omega)\, e^{I_{\alpha}^{\omega}(\varphi)}$

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}(\mu N) = \frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(e^{I_{\alpha}^{\omega}(\varphi)}N\right) = e^{I_{\alpha}^{\omega}(\varphi)}N_x^{\alpha} + \alpha N\, \varphi(\omega)e^{I_{\alpha}^{\omega}(\varphi)}$$

Now

$$\frac{\partial^{\alpha}}{\partial y^{\alpha}}(\mu M) - \frac{\partial^{\alpha}}{\partial x^{\alpha}}(\mu N) = e^{I_{\alpha}^{\omega}(\varphi)}\left[M_y^{\alpha} - N_x^{\alpha}\right] - \alpha\, \varphi(\omega)e^{I_{\alpha}^{\omega}(\varphi)}[M+N]$$

$$= e^{I_{\alpha}^{\omega}(\varphi)}\alpha\, \varphi(\omega)[M+N] - \ \alpha\, \varphi(\omega)e^{I_{\alpha}^{\omega}(\varphi)}[M+N] = 0$$

Hence the equation $\mu M dx + \ \mu N dy = 0$ is α −exact and $\mu = e^{I_{\alpha}^{\omega}(\varphi)}$ is an integrating factor.

Theorem 3.4: Let $\omega = x^{\alpha} + y^{\alpha}$ and $\frac{M_y^{\alpha} - N_x^{\alpha}}{\alpha(N-M)} = \varphi(x^{\alpha} + y^{\alpha}) = \varphi(\omega)$.

Then the integrating factor is:

$$\mu = e^{I_{\alpha}^{\omega}(\varphi)} = e^{\int(\varphi(\omega)/\omega^{1-\alpha})d\omega}$$

Proof: Similar to the proof in Theorem 3.3.

Theorem 3.5: Assume $\omega = x^{2\alpha} + y^{2\alpha}$ and $\frac{M_y^{\alpha} - N_x^{\alpha}}{2\alpha(x^{\alpha}N - y^{\alpha}M)} = \varphi(x^{2\alpha} + y^{2\alpha}) = \varphi(\omega)$

Then the integrating factor is

$$\mu = e^{I_{\alpha}^{\omega}(\varphi)} = e^{\int(\varphi(\omega)/\omega^{1-\alpha})d\omega}$$

and

$$\mu M dx + \ \mu N dy = 0$$

is an α −exact equation.

Proof:

$$\frac{\partial^{\alpha}}{\partial y^{\alpha}}(\mu M) = \frac{\partial^{\alpha}}{\partial y^{\alpha}}\left(e^{I_{\alpha}^{\omega}(\varphi)}M\right) = e^{I_{\alpha}^{\omega}(\varphi)}M_y^{\alpha} + 2\alpha y^{\alpha}M\, \varphi(\omega)\, e^{I_{\alpha}^{\omega}(\varphi)}$$

$$\frac{\partial^\alpha}{\partial x^\alpha}(\mu N) = \frac{\partial^\alpha}{\partial x^\alpha}\left(e^{I_\alpha^\omega(\varphi)}N\right) = e^{I_\alpha^\omega(\varphi)}N_x^\alpha + 2\alpha x^\alpha N\,\varphi(\omega)e^{I_\alpha^\omega(\varphi)}$$

Now

$$\frac{\partial^\alpha}{\partial y^\alpha}(\mu M) - \frac{\partial^\alpha}{\partial x^\alpha}(\mu N) = e^{I_\alpha^\omega(\varphi)}\left[M_y^\alpha - N_x^\alpha\right] - 2\alpha\,\varphi(\omega)e^{I_\alpha^\omega(\varphi)}[x^\alpha N - y^\alpha M]$$

$$= 2\alpha\,\varphi(\omega)[x^\alpha N - y^\alpha M]e^{I_\alpha^\omega(\varphi)} -$$

$$2\alpha\,\varphi(\omega)e^{I_\alpha^\omega(\varphi)}[x^\alpha N - y^\alpha M] = 0$$

Hence the equation $\mu M dx + \mu N dy = 0$ is $\alpha -$exact and $\mu = e^{I_\alpha^\omega(\varphi)}$ is an integrating factor.

Theorem 3.6: Assume $\omega = (xy)^\alpha$ and $\frac{M_y^\alpha - N_x^\alpha}{\alpha(y^\alpha N - x^\alpha M)} = \varphi(xy)^\alpha = \varphi(\omega)$

Then the integrating factor is :

$$\mu = e^{I_\alpha^\omega(\varphi)} = e^{\int(\varphi(\omega)/\omega^{1-\alpha})d\omega}$$

and

$$\mu M dx + \mu N dy = 0$$

is an $\alpha -$exact equation.

Proof:

$$\frac{\partial^\alpha}{\partial y^\alpha}(\mu M) = \frac{\partial^\alpha}{\partial y^\alpha}\left(e^{I_\alpha^\omega(\varphi)}M\right) = e^{I_\alpha^\omega(\varphi)}M_y^\alpha + \alpha x^\alpha M\,\varphi(\omega)\,e^{I_\alpha^\omega(\varphi)}$$

$$\frac{\partial^\alpha}{\partial x^\alpha}(\mu N) = \frac{\partial^\alpha}{\partial x^\alpha}\left(e^{I_\alpha^\omega(\varphi)}N\right) = e^{I_\alpha^\omega(\varphi)}N_x^\alpha + \alpha y^\alpha N\,\varphi(\omega)e^{I_\alpha^\omega(\varphi)}$$

Now

$$\frac{\partial^\alpha}{\partial y^\alpha}(\mu M) - \frac{\partial^\alpha}{\partial x^\alpha}(\mu N) = e^{I_\alpha^\omega(\varphi)}\left[M_y^\alpha - N_x^\alpha\right] - 2\alpha\,\varphi(\omega)e^{I_\alpha^\omega(\varphi)}[x^\alpha N - y^\alpha M]$$

$$= \alpha\,\varphi(\omega)[y^\alpha N - x^\alpha M]e^{I_\alpha^\omega(\varphi)} - \alpha\,\varphi(\omega)e^{I_\alpha^\omega(\varphi)}[y^\alpha N - x^\alpha M]$$

$$= 0$$

Hence the equation $\mu M dx + \mu N dy = 0$ is $\alpha -$exact and $\mu = e^{I_\alpha^\omega(\varphi)}$ is an integrating factor.

Theorem 3.7: Assume $\omega = (x/y)^\alpha$ and $\frac{(M_y^\alpha - N_x^\alpha)y^{2\alpha}}{\alpha(y^\alpha N + x^\alpha M)} = \varphi(x/y)^\alpha = \varphi(\omega)$.

Then the integrating factor is:

$$\mu = e^{I_\alpha^\omega(\varphi)} = e^{\int(\varphi(\omega)/\omega^{1-\alpha})d\omega}$$

and

$$\mu M dx + \mu N dy = 0$$

is an $\alpha -$exact equation.

Proof:

$$\frac{\partial^\alpha}{\partial y^\alpha}(\mu M) = \frac{\partial^\alpha}{\partial y^\alpha}\left(e^{I_\alpha^\omega(\varphi)}M\right) = e^{I_\alpha^\omega(\varphi)}M_y^\alpha - \alpha x^\alpha y^{-2\alpha}M\,\varphi(\omega)\,e^{I_\alpha^\omega(\varphi)}$$

$$\frac{\partial^\alpha}{\partial x^\alpha}(\mu N) = \frac{\partial^\alpha}{\partial x^\alpha}\left(e^{I_\alpha^\omega(\varphi)}N\right) = e^{I_\alpha^\omega(\varphi)}N_x^\alpha + \alpha y^{-\alpha}N\,\varphi(\omega)e^{I_\alpha^\omega(\varphi)}$$

Now

$$\frac{\partial^\alpha}{\partial y^\alpha}(\mu M) - \frac{\partial^\alpha}{\partial x^\alpha}(\mu N) = e^{I_\alpha^\omega(\varphi)}\left[M_y^\alpha - N_x^\alpha\right] - \alpha\,\varphi(\omega)e^{I_\alpha^\omega(\varphi)}[x^\alpha y^{-2\alpha}M + y^{-\alpha}N]$$

$$= \alpha\,\varphi(\omega)\left[\frac{x^\alpha M + y^\alpha N}{y^{2\alpha}}\right]e^{I_\alpha^\omega(\varphi)} - \alpha\,\varphi(\omega)e^{I_\alpha^\omega(\varphi)}\left[\frac{x^\alpha M + y^\alpha N}{y^{2\alpha}}\right] = 0$$

Hence the equation $\mu M dx + \mu N dy = 0$ is an $\alpha -$exact and $\mu = e^{I_\alpha^\omega(\varphi)}$ is an integrating factor.

Theorem 3.8: Let $\omega = (y/x)^\alpha$ and $\frac{(N_x^\alpha - M_y^\alpha)x^{2\alpha}}{\alpha(y^\alpha N + x^\alpha M)} = \varphi(y/x)^\alpha = \varphi(\omega)$.

Then the integrating factor is :

$$\mu = e^{I_\alpha^\omega(\varphi)} = e^{\int(\varphi(\omega)/\omega^{1-\alpha})d\omega}$$

and

$$\mu M dx + \mu N dy = 0$$

is an $\alpha -$exact equation.

Proof: Similar to the proof in Theorem 3.7.

In [5] Authors proved the existence of two cases of integrating factors:

(i) If $\frac{M_y^\alpha - N_x^\alpha}{N} = \phi(x)$ then $\mu = e^{I_\alpha^x(\phi)} = e^{\int(\phi(x)/x^{1-\alpha})dx}$ is an integrating factor for (*).

That is

$$\mu M dx + \mu N dy = 0$$

is an α −exact equation.

(ii) If $\frac{M_y^\alpha - N_x^\alpha}{M} = \psi(y)$ then $\mu = e^{-I_\alpha^y(\psi)} = e^{-\int(\psi(y)/y^{1-\alpha})dy}$ is an integrating factor for (*).

That is

$$\mu M dx + \mu N dy = 0$$

is an α −exact equation.

4. Examples

In this section we present some examples to illustrate how to find some integrating factors.

Example 4.1: Consider

$$\left(x^{\frac{1}{2}} y^{\frac{3}{2}} + 2xy - y\right) dy + \left(xy + 2x^{\frac{3}{2}} y^{\frac{1}{2}} - 2x\right) dx = 0.$$

Let $\alpha = \frac{1}{2}$. This is equation not α- exact nor separable. But

$$M_y^{\frac{1}{2}} = \frac{3}{2} y\sqrt{x} + 2x\sqrt{y} - \sqrt{y} \quad , \qquad N_x^{\frac{1}{2}} = y\sqrt{x} + 3x\sqrt{y} - 2\sqrt{x},$$

So $M_y^{\frac{1}{2}} - N_x^{\frac{1}{2}} = \frac{1}{2} y\sqrt{x} - x\sqrt{y} - \sqrt{y} + 2\sqrt{x}$

and $y^{\frac{1}{2}} N - x^{\frac{1}{2}} M = \sqrt{x}\sqrt{y}\ (-2\sqrt{x} + \sqrt{y})$

Now $\frac{M_y^{\frac{1}{2}} - N_x^{\frac{1}{2}}}{\frac{1}{2}\left(y^{\frac{1}{2}} N - x^{\frac{1}{2}} M\right)} = 1 - \frac{2}{\sqrt{xy}} = \varphi(xy)^{\frac{1}{2}} = \varphi(\omega) = 1 - \frac{2}{\sqrt{\omega}}$

and $\mu = e^{I_{1/2}^{\omega}(\varphi)} = e^{\int(\varphi(\omega)/\sqrt{\omega})d\omega} = e^{\int\left(\omega^{\frac{-1}{2}} - 2\omega^{\frac{-3}{2}}\right)d\omega} = e^{(2\sqrt{\omega} + \frac{4}{\sqrt{\omega}})}$

Then the integrating factor is:

$$\mu = e^{2\sqrt{xy}} e^{\frac{4}{\sqrt{xy}}}.$$

Example 4.2: Consider

$$(x^\alpha + x^{4\alpha} + 2x^{2\alpha} y^{2\alpha} + y^{4\alpha})dx + y^\alpha dy = 0$$

This is equation not α - exact nor separable. But

$$M_y^\alpha = 4\alpha x^{2\alpha} y^\alpha + 4\alpha y^{3\alpha} \quad , N_x^\alpha = 0$$

$M_y^\alpha - N_x^\alpha = 4\alpha x^{2\alpha} y^\alpha + 4\alpha y^{3\alpha}$, $x^\alpha N - y^\alpha M = -y^\alpha (x^{4\alpha} + 2x^{2\alpha} y^{2\alpha} + y^{4\alpha}) = -y^\alpha (x^{2\alpha} + y^{2\alpha})^2$

$$\frac{M_y^\alpha - N_x^\alpha}{2\alpha(x^\alpha N - y^\alpha M)} = \frac{-2}{(x^{2\alpha} + y^{2\alpha})} = \varphi(x^{2\alpha} + y^{2\alpha})$$

$$\text{Let} \quad \varphi(x^{2\alpha} + y^{2\alpha}) = \varphi(\omega) = \frac{-2}{\omega}$$

The integrating factor is:

$$\mu = e^{I_\alpha^\omega(\varphi)} = e^{\int(-2/\omega^{2-\alpha})d\omega} = e^{\frac{-2}{\alpha-1}\omega^{\alpha-1}} = e^{\frac{-2}{\alpha-1}(x^{2\alpha}+y^{2\alpha})^{\alpha-1}}$$

Example 4.3: Consider

$$\left(3x^{\frac{1}{3}} y^{\frac{1}{3}} + y^{\frac{2}{3}}\right) dy - \left(3x^{\frac{1}{3}} y^{\frac{1}{3}} + x^{\frac{2}{3}}\right) dx = 0.$$

This is equation not $\frac{1}{3}-$ exact nor separable. But

$$M_y^{\frac{1}{3}} = x^{\frac{1}{3}} + \frac{2}{3} y^{\frac{1}{3}} \quad , N_x^{\frac{1}{3}} = -y^{\frac{1}{3}} - \frac{2}{3} x^{\frac{1}{3}}$$

$$3\left[\frac{M_y^{\frac{1}{3}} - N_x^{\frac{1}{3}}}{(N+M)}\right] = \frac{-5}{x^{\frac{1}{3}} - y^{\frac{1}{3}}} = \varphi(x^{\frac{1}{3}} - y^{\frac{1}{3}}) = \varphi(\omega) = \frac{-5}{\omega}$$

The integrating factor is:

$$\mu = e^{I_{\frac{1}{3}}^\omega(\varphi)} = e^{\int\left(\varphi(\omega)/\omega^{1-\frac{1}{3}}\right)d\omega} = e^{\int\left(-5\omega^{\frac{-5}{3}}\right)d\omega} = e^{7.5\omega^{\frac{-2}{3}}}$$

Example 4.4: Consider

$$(3x^\alpha y^\alpha + y^{2\alpha})dy + (3x^\alpha y^\alpha + x^{2\alpha})dx = 0.$$

This is equation not α −exact nor separable, $0 < \alpha < 1$. But

$$M_y^\alpha = 3\alpha x^\alpha + 2\alpha y^\alpha \quad , N_x^\alpha = 3\alpha y^\alpha + 2\alpha x^\alpha$$

$$\frac{M_y^\alpha - N_x^\alpha}{\alpha(N-M)} = \frac{1}{x^\alpha + y^\alpha} = \varphi(x^\alpha + y^\alpha) = \varphi(\omega) = \frac{1}{\omega}$$

The integrating factor is :

$$\mu = e^{\int \omega^{2-\alpha} d\omega} = e^{\frac{-2}{\alpha-1}\omega^{\alpha-1}} = e^{((x^\alpha+y^\alpha)^{1-\alpha}/1-\alpha)}$$

Example 4.5: Consider

$$y^\alpha(x^\alpha y^\alpha + 1)dx - x^\alpha(x^\alpha y^\alpha - 1)dy = 0$$

You can note that:

$$M = y^\alpha f(xy)^\alpha = y^\alpha(x^\alpha y^\alpha + 1) \; and \; N = x^\alpha g(xy)^\alpha = x^\alpha(x^\alpha y^\alpha - 1)$$

the integrating factor is:

$$\mu = \frac{1}{(xy)^{\alpha}(f-g)} = \frac{1}{x^{\alpha}y^{\alpha}((x^{\alpha}y^{\alpha}+1)-(x^{\alpha}y^{\alpha}-1))} = \frac{1}{2x^{\alpha}y^{\alpha}}$$

To solve this equation:

$$\frac{1}{2x^{\alpha}y^{\alpha}}[y^{\alpha}(x^{\alpha}y^{\alpha}+1)dx - x^{\alpha}(x^{\alpha}y^{\alpha}-1)dy = 0]$$

$$\frac{1}{x^{\alpha}y^{\alpha}}[\frac{1}{2x^{\alpha}}(x^{\alpha}y^{\alpha}+1)dx - \frac{1}{2y^{\alpha}}(x^{\alpha}y^{\alpha}-1)dy = 0]$$

$$\frac{y^{\alpha}dx - x^{\alpha}dy}{2x^{\alpha}y^{\alpha}} + \frac{y^{\alpha}dx + x^{\alpha}dy}{2(x^{\alpha}y^{\alpha})^2} = 0$$

$$\frac{1}{2\alpha}d^{\alpha}\ln\left(\frac{x^{\alpha}}{y^{\alpha}}\right) - \frac{1}{2\alpha}d^{\alpha}\left(\frac{1}{x^{\alpha}y^{\alpha}}\right) = 0$$

Now take $\alpha - integral$ to get the general solution:

$$\frac{1}{2\alpha}\ln\left(\frac{x^{\alpha}}{y^{\alpha}}\right) - \frac{1}{2\alpha}\left(\frac{1}{x^{\alpha}y^{\alpha}}\right) = c$$

where c is an arbitrary constant.

If $\alpha = 0.9$, the general solution is given by: $u(x^{0.9}, y^{0.9}) = \frac{1}{1.8}\ln\left(\frac{x^{0.9}}{y^{0.9}}\right) - \frac{1}{2\alpha}\left(\frac{1}{x^{0.9}y^{0.9}}\right) = c$

If $\alpha = 0.7$, the general solution is given by: $u(x^{0.7}, y^{0.7}) = \frac{1}{1.4}\ln\left(\frac{x^{0.7}}{y^{0.7}}\right) - \frac{1}{1.4.}\left(\frac{1}{x^{0.7}y^{0.7}}\right) = c$

The solutions are completely different, see figures 1 and 2.

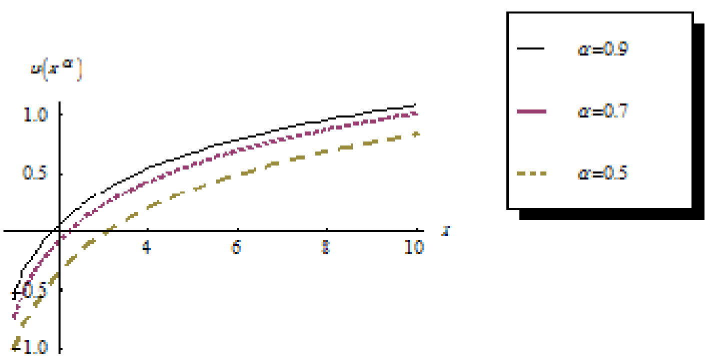

Fig. 1 The graph of the solution $u(x^{\alpha}, y^{\alpha}) = \frac{1}{2\alpha}\ln\left(\frac{x^{\alpha}}{y^{\alpha}}\right) - \frac{1}{2\alpha}\left(\frac{1}{x^{\alpha}y^{\alpha}}\right)$ versus x when $y = 1$

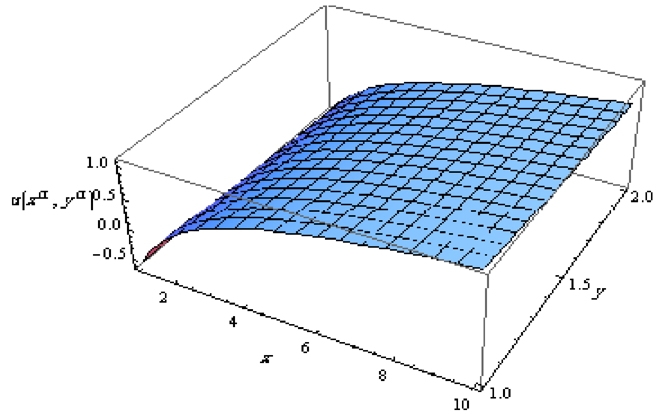

Fig. 2 The graph of $u(x^{\alpha}, y^{\alpha}) = \frac{1}{2\alpha}\ln\left(\frac{x^{\alpha}}{y^{\alpha}}\right) - \frac{1}{2\alpha}\left(\frac{1}{x^{\alpha}y^{\alpha}}\right)$ at $\alpha = 0.9$ for example 4.5.

5. Closing Remarks

1-If the equation contains the expression: $y^{\alpha}dx + x^{\alpha}dy$ then multiplying the equation by the function $u(x^{\alpha}, y^{\alpha}) = \frac{1}{x^{\alpha}y^{\alpha}}$ transforms $y^{\alpha}dx + x^{\alpha}dy$ to:

$$\frac{y^{\alpha}dx + x^{\alpha}dy}{x^{\alpha}y^{\alpha}} = \frac{1}{\alpha}d^{\alpha}(\ln(x^{\alpha}y^{\alpha}))$$

2-If the equation contains the expression: $x^{\alpha}dx + y^{\alpha}dy$ then multiplying the equation by the function $u(x^{\alpha}, y^{\alpha}) = \frac{1}{x^{2\alpha}+y^{2\alpha}}$ transforms $x^{\alpha}dx + y^{\alpha}dy$ to:

$$\frac{x^{\alpha}dx + y^{\alpha}dy}{x^{2\alpha} + y^{2\alpha}} = \frac{1}{2\alpha}d^{\alpha}(\ln(x^{2\alpha} + y^{2\alpha}))$$

3-If the equation contains the expression: $y^{\alpha}dx - x^{\alpha}dy$ then multiplying the equation by the function $u(x^{\alpha}, y^{\alpha}) = \frac{1}{y^{2\alpha}}$ transforms $y^{\alpha}dx - x^{\alpha}dy$ to:

$$\frac{y^{\alpha}dx - x^{\alpha}dy}{y^{2\alpha}} = \frac{1}{\alpha}d^{\alpha}\left(\frac{x^{\alpha}}{y^{\alpha}}\right)$$

Note that in remark 3 there are other choices for the function $u(x^{\alpha}, y^{\alpha})$:

i. $u(x^{\alpha}, y^{\alpha}) = \frac{1}{x^{2\alpha}}$ to get $\frac{y^{\alpha}dx - x^{\alpha}dy}{x^{2\alpha}} = -\frac{1}{\alpha}d^{\alpha}\left(\frac{y^{\alpha}}{x^{\alpha}}\right)$.

ii. $u(x^{\alpha}, y^{\alpha}) = \frac{1}{x^{\alpha}y^{\alpha}}$ to get $\frac{y^{\alpha}dx - x^{\alpha}dy}{x^{\alpha}y^{\alpha}} = \frac{1}{\alpha}d^{\alpha}(\ln\left(\frac{x^{\alpha}}{y^{\alpha}}\right))$.

iii. $u(x^{\alpha}, y^{\alpha}) = \frac{1}{x^{2\alpha}+y^{2\alpha}}$ to get $\frac{y^{\alpha}dx - x^{\alpha}dy}{x^{2\alpha}+y^{2\alpha}} = \frac{1}{\alpha}d^{\alpha}(tan^{-1}(\frac{x^{\alpha}}{y^{\alpha}}))$

Example 5.1: Solve $x^{\alpha}dy - y^{\alpha}dx - (1 - x^{2\alpha})dx = 0$

Solution: The equation contains the expression $y^\alpha dx - x^\alpha dy$. According to the third remark the function $u(x^\alpha, y^\alpha) = \frac{1}{x^{2\alpha}}$ is suitable. Now multiply the equation by it to get:

$$\frac{1}{x^{2\alpha}}[x^\alpha dy - y^\alpha dx - (1 - x^{2\alpha})dx = 0]$$

$$\frac{y^\alpha dx - x^\alpha dy}{x^{2\alpha}} + \frac{1 - x^{2\alpha}}{x^{2\alpha}}dx = 0$$

$$-\frac{1}{\alpha}d^\alpha\left(\frac{y^\alpha}{x^\alpha}\right) + x^{-2\alpha}dx - dx = 0$$

$$-\frac{1}{\alpha}d^\alpha\left(\frac{y^\alpha}{x^\alpha}\right) - \frac{1}{\alpha}d^\alpha x^{-\alpha} - \frac{1}{\alpha}d^\alpha x^\alpha = 0$$

Now take $\alpha - integral$ to get the general solution:

$$\frac{1}{\alpha}\left(\frac{y^\alpha}{x^\alpha}\right) + \frac{1}{\alpha}x^{-\alpha} + \frac{1}{\alpha}x^\alpha = c$$

where c is an arbitrary constant.

Example 5.2: Solve $\big(x^\alpha - y^\alpha(x^{2\alpha} + y^{2\alpha})\big)dx + \big(y^\alpha + x^\alpha(x^{2\alpha} + y^{2\alpha})\big)dy = 0$

Solution: The equation contains the expression $x^\alpha dx + y^\alpha dy$. According to the second remark the function $u(x^\alpha, y^\alpha) = \frac{1}{x^{2\alpha}+y^{2\alpha}}$ is suitable. Now multiply the equation by it to get:

$$\frac{1}{(x^{2\alpha}+y^{2\alpha})}[\big(x^\alpha - y^\alpha(x^{2\alpha} + y^{2\alpha})\big)dx + \big(y^\alpha + x^\alpha(x^{2\alpha} + y^{2\alpha})\big)dy = 0]$$

$$\frac{1}{2\alpha}d^\alpha(\ln(x^{2\alpha} + y^{2\alpha})) - (y^\alpha dx - x^\alpha dy \ ) = 0$$

This equation contains: $y^\alpha dx - x^\alpha dy$, by remark 3 one can choose: $u(x^\alpha, y^\alpha) = \frac{1}{x^{2\alpha}+y^{2\alpha}}$

$$\frac{1}{(x^{2\alpha}+y^{2\alpha})}\ [\frac{1}{2\alpha}d^\alpha(\ln(x^{2\alpha} + y^{2\alpha})) - (y^\alpha dx - x^\alpha dy \ ) = 0]$$

$$\frac{1}{2\alpha(x^{2\alpha} + y^{2\alpha})}d^\alpha(\ln(x^{2\alpha} + y^{2\alpha})) - \frac{1}{\alpha}d^\alpha(tan^{-1}(\frac{x^\alpha}{y^\alpha})) = 0$$

$$\frac{-1}{2\alpha}d^\alpha\left(\frac{1}{x^{2\alpha} + y^{2\alpha}}\right) - \frac{1}{\alpha}d^\alpha(tan^{-1}(\frac{x^\alpha}{y^\alpha})) = 0$$

Now take $\alpha - integral$ to get the general solution:

$$\frac{-1}{2\alpha}\left(\frac{1}{x^{2\alpha} + y^{2\alpha}}\right) - \frac{1}{\alpha}tan^{-1}(\frac{x^\alpha}{y^\alpha})) = c$$

where c is an arbitrary constant.

Example 5.3: Solve $3x^{4\alpha}y^{2\alpha}dx + y^{\alpha}dx + x^{\alpha}dy = 0$

Solution: The equation contains the expression $y^{\alpha}dx + x^{\alpha}dy$. According to the first remark the function $u(x^{\alpha}, y^{\alpha}) = \frac{1}{x^{\alpha}y^{\alpha}}$ is suitable. Now multiply the equation by it twice to get:

$$\frac{1}{(x^{\alpha}y^{\alpha})^2}[3x^{4\alpha}y^{2\alpha}dx + y^{\alpha}dx + x^{\alpha}dy = 0]$$

$$3x^{2\alpha}dx + \frac{y^{\alpha}dx + x^{\alpha}dy}{(x^{\alpha}y^{\alpha})^2} = 0$$

$$\frac{1}{\alpha}d^{\alpha}(x^{3\alpha}) - \frac{1}{\alpha}d^{\alpha}\left(\frac{1}{x^{\alpha}y^{\alpha}}\right) = 0$$

Now take $\alpha - integral$ to get the general solution:

$$\frac{1}{\alpha}(x^{3\alpha}) - \frac{1}{\alpha}\left(\frac{1}{x^{\alpha}y^{\alpha}}\right) = c$$

where c is an arbitrary constant and $x, y \neq 0$.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] M. Abu Hammad, and R. Khalil, Abel's formula and wronskian for conformable fractional differential equations, Int. J. Differ. Equ. Appl. 13 (3) (2014), 177–183.

[2] M. Abu Hammad, and R. Khalil, Conformable fractional heat differential equation, Int. J. Pure. Appl. Math. 94 (2) (2014), 215–221.

[3] O. Acan, O. Firat, A. Kurnaz, and Y. Keskin, Applications for New Technique Conformable Fractional Reduced Differential Transform Method, J. Comput. Theor. Nanosci. in press.

[4] A. ALHabees, Exact solutions of conformable fractional Harry Dym equation. J Comput Anal. Appl. 29 (4) (2021), 727-736.

[5] M. Alhorani, and R. Khalil, Total fractional differentials with applications to exact fractional differential equations. Int. J. Comput. Math. 95 (6-7) (2018), 1444-1452.

[6] Y. Cenesiz, and A. Kurt, The solutions of time and space conformable fractional heat equations with conformable Fourier transform, Acta Univ. Sapientiae, Math. 7 (2015), 130-140.

[7] O.S. Iyiola, and G.O. Ojo, On the analytical solution of Fornberg – Whitham equation with the new fractional derivative, Pramana, 85 (2015), 567–575.

[8] R. Khalil, M. Al Horani M. Abu Hammad, Geometric meaning of conformable derivative via fractional cords, J. Math. Computer Sci. 19 (2019), 241–245.

[9] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, J. Comput. Appl. Math. 264 (2014), 65–70.

[10] A. Kilbas, H. Srivastava, and J. Trujillo, Theory and Applications of Fractional Differential Equations, Math. Studies, North-Holland, New York, 2006.

[11] A. Korkmaz, Exact Solutions to Some Conformable Time Fractional Equations in Benjamin-Bona-Mohany Family. arXiv:1611.07086 [nlin.SI], 2016.

[12] Y. Luchko, R.Gorneflo, The initial value problem for some fractional differential equa-tions with the Caputo derivative, Preprint series A08-98, Fachbereich Mathematik und Informatik, Freie Universitat, Berlin, 1998.

[13] K.S. Miller, B.Ross, An introduction to the fractional calculus and fractional differential equations, John Wiley and Sons Inc., New York, 1993.

[14] I. Podlubny. Fractional differential equations. An introduction to fractional derivatives frac-tional differential equations some methods of their Solution and some of their applications. Academic Press, San Diego, 1999.

[15] J. Sabatier, O. P. Agrwal and J. A. T. Machado, Advances in Fractional Calculus, Springer, 2007.

[16] P. J. Torvik, R.L.Bagley, Onthe appearance of the fractional derivative in the behavior of real materials, J. Appl. Math. 51 (2) (1984), 294–298.