Available online at http://scik.org
J. Math. Comput. Sci. 10 (2020), No. 4, 1066-1082
https://doi.org/10.28919/jmcs/4548
ISSN: 1927-5307

# NUMERICAL SOLUTION OF TIME FRACTIONAL ORDER PARTIAL DIFFERENTIAL EQUATIONS

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**Abstract.** In this work, we have presented analysis of time fractional order linear and non-linear partial differential equations with initial value and boundary conditions by applying Riemann - Leivoulli fractional integral. As the explained differential equations are related to natural phenomenon, it may be observed under various circumstances for which the possible outcome may vary. The properties and nature of physical states of these equations have been emphasised more precisely by taking fractional order. Fractional order homotopy perturbation method has tackled the approximate solutions in the series form of well known time fractional order linear and non linear differential equations. Numerical simulations are demonstrated prominently in graphical format by using Matlab.

**Keywords:** time fractional order partial differential equations [TFOPDE]; time fractional order Emden-Fowler type differential equations [TFEFDE]; time fractional order evolution type differential equations[TFEDE]; time fractional order Klein- Gordon type differential equations [TFKGDE]; fractional order homotopy perturbation method [FOHPM].

**2010 AMS Subject Classification:** 35R11, 65H20, 65L05.

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Received February 28, 2020

## 1. INTRODUCTION

Fractional order differential equations and fractional integrals are becoming useful tool for describing the natural phenomenon of science and engineering models as well as to analyse the theory of complex systems. Models can be better analysed in the form of fractional order where we apply basic principles of calculus. These models are indeed becoming ubiquitous with scientists and researchers looking forward to get appropriate and fruitful results. It would be more appropriate to state that fractional calculus is the generalised form of traditional calculus [1], [2]. Many mathematical models of physical and chemical states are able to analyse better the form of fractional order [3]. In the recent developments in inter disciplinary fields of science and technology like Economics and finance [3], Physics [4], Hydraulics, Geology and fluid dynamics [5], Biology [6], Biomedical and biotechnology [7], Control systems [8], Signals and systems, Communication theory [9], Image processing [10] and so on, the scientists have fruitfully utilised fractional calculus to enhance the applications in the day to day human life. The extensive literature in the field of fractional calculus has created the challenges for researchers. The solution of fractional differential equations is an emanating area of present day research owing to its various practical applications. We can freely handle mathematical model in the form of fractional order of the derivative or integration and their solutions can be obtained by using various perturbative and non-perturbative mathematical methods. Ji Huan He [11] proposed homotopy perturbation method [HPM] and successfully applied on Lighthill equation [12], Duffing equation [13], and Blasius equation [14] to get the approximate series solutions. HPM has been applied to solve non linear wave equations [15], Initial and boundary value problems [16], Voltera integro differential equations [17], Lane Emden type differential equations [18], Emden Fowler type differential equations [19]-[22], Evolution type differential equations [23], [24], Klein -Gordon type differential [25], [26] and Lane Emden Fowler type equations [27].

In this paper, we have employed the technique of time fractional order homotopy perturbation method to assay some of the time dependent fractional order linear and non linear partial differential equations particularly Emden-Fowler type time fractional order differential equations, evolution type time fractional order differential equation, Klein- Gordon type time fractional order differential equations and analysed them appropriately. The numerical simulation of these partial differential equations significantly determines the impact of fractional order. The graphical results have been systematically represented by taking various fractional orders.

**1.1. Fractional Order Homotopy Perturbation Method.** We have briefly described basic techniques of fractional order homotopy perturbation method which is the extension of homotopy perturbation method [12], [13]. The present fractional order homotopy perturbation method has been implemented to solve general linear and non linear non homogeneous time fractional partial differential equations with boundary value conditions to get the approximate series solution. We have first defined the given differential equation in the following form

$$D_t^{\alpha}\left[f\left(x_1,x_2,...,x_n t\right)\right] = F\left(x_1,x_2,...,x_n t\right) \tag{1}$$

with boundary conditions

$f(x_1,x_2,...,x_n,0) = h(x_1,x_2,...,x_n)$ and $\frac{\partial}{\partial t} f(x_1,x_2,...,x_n,0) = g(x_1,x_2,...,x_n)$

Where '$\alpha$' is a positive real number which represents the fractional order of differential equation and $F\left(x_1,x_2,...,x_n,t\right)$ is a function of $x_1,x_2,...,x_n$ be 'n' number of space co-ordinates and 't' be a time domain.

Let's define the equation as

$$D^{\alpha}\left[f\left(x_1,x_2,...,x_n,t\right)\right] - F\left(x_1,x_2,...,x_n,t\right) = 0 \tag{2}$$

By homotopy technique one can construct a homotopy as

$$H\left(\tau, p\right) : \Omega \times [0, 1] \to R \tag{3}$$

Where $p \in [0, 1]$ , p is called as homotopy parameter.

We guess $f_0\left(x_1,x_2,...,x_n,t\right)$ as an initial approximation for the solution of equation, which satisfies

$$\begin{aligned} H\left(f,p\right) &= (1-p)\left[D^{\alpha}\left[f\left(x_1,x_2,...,x_n t\right) - f_0\left(x_1,x_2,...,x_n t\right)\right]\right] \\ &+ p\left[D^{\alpha}\left[f\left(x_1,x_2,...,x_n,t\right)\right] - F\left(x_1,x_2,...,x_n,t\right)\right] = 0 \end{aligned} \tag{4}$$

The above equation must satisfy the given boundary conditions which are defined in the problem and gives rise to

$$H\ (f,0) = [D^{\alpha} f\,(x_1,x_2,...,x_n\,,t) - D^{\alpha} f_0\,(x_1,x_2,...,x_n\,,t)] = 0$$

and

$$H\,(f,1) = [D^{\alpha}\,[f\,(x_1,x_2,...,x_n\,,t)] - F\,(x_1,x_2,...,x_n,\,t)] = 0$$

Assuming the solution of the equation,

$$f\,(x_1,x_2,...,x_n\,,t) = f_0\,(x_1,x_2,...,x_n\,,t) + p f_1\,(x_1,x_2,...,x_n\,,t) + ....$$

Taking limit $p \to 1$, we get the approximate series solution.

(5) $$f\,(x_1,x_2,...,x_n\,,t) = f_0\,(x_1,x_2,...,x_n\,,t) + f_1\,(x_1,x_2,...,x_n\,,t) + ....$$

It is necessary to note that the major advantage of fractional order homotopy perturbation method is that perturbation series can freely give the approximate solution and it is convergent in all sense.

## 1.2. Basic Definitions And Some Properties Of Fractional Calculus.

In this segment, we have presented some definitions of fractional derivatives and integrals for basic understanding and further use [1]-[3].

**Theorem 1.1.** [2] *A real function $g(t),\ t > 0$, is said to be in the space $C_\nu$, $\nu \in R$ if there exist a real number $s > \nu$ such that $g(t) = t^s g_1(t)$ where $g_1(t) \in C[0,\infty)$ and it is said to be in the space $C_\nu^n$ if and only if $g^n(t) \in C_\nu$, $n \in N$*

**Definition 1.1.** *(Riemann-Liouville fractional integral of order $\alpha$)* [2]
*Riemann-Liouville fractional integral operator $(J_t^{\alpha})$ of order $\alpha \geq 0$ of a function $g(t) \in C_\nu$, $\nu \geq -1$ is defined as*

$$._aJ_t^{\alpha}\, g(t) = \frac{1}{\Gamma\alpha} \int_a^t (t-\tau)^{\alpha-1}.g\,(\tau)\,d\tau$$

*Where $t \geq a \geq 0$ and $\Gamma(.)$ is a well known Gamma function.*

Some of the properties of Riemann -Liouville fractional integral operator have been explained.

For $g(t) \in C_\nu$, $\nu \in R$, $\nu > -1$,

$a, \alpha, \beta \geq 0$ and $\nu \geq -1$

1. ${}_{\cdot a}J_t^{\alpha}\, g(t)\, {}_{\cdot a}J_t^{\beta}\, g(t) = {}_{\cdot a}J_t^{\alpha+\beta}\, g(t)$

2. ${}_{\cdot a}J_t^{\alpha}\, g(t)\, {}_{\cdot a}J_t^{\beta}\, g(t) = {}_{\cdot a}J_t^{\beta}\, g(t)\, {}_{\cdot a}J_t^{\alpha}\, g(t)$

3. ${}_{\cdot a}J_t^{\alpha}\, (t-a)^{\nu} = \frac{\Gamma \nu+1}{\Gamma \alpha+\nu+1}(t-a)^{(\alpha+\nu)}$

4. ${}_{\cdot a}J_t^{0}\, g(t) = g(t)$

**Definition 1.2.** *(Riemann-Liouville fractional derivative of order $\alpha$)* [2]

*If $g(t) \in C[a,b]$ and $a < t < b$ then*

$$ {}_{\cdot a}^{RL}D_t^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{g(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau $$

*for $t \geq a \geq 0$, is called as Riemann-Liouville fractional derivative of order $\alpha$.*

*where $n$ is a positive integer and $\alpha$ is a positive real number such that $n-1 < \alpha < n$*

**Definition 1.3.** *Caputo fractional derivative of order $\alpha$* [2]

*Caputo fractional derivative $\left({}_{\cdot a}^{C}D_t^{\alpha}\right)$ of $g(t) \in C\,[a,b]$ and $a < t < b$ is defined as*

$$ {}_{\cdot a}^{C}D_t^{\alpha}\, g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{g^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau $$

*For $t \geq a \geq 0$ and $n$ is a positive integer and $\alpha$ is a positive real number such that $n-1 < \alpha < n$.*

## 2. Working Examples:

**2.1. Time fractional order Emden-Fowler type differential equations.** In view of wide application of Emden -Fowler type singular differential equation through the concept of thermal behaviour of a spherical cloud of gas and thermionic currents acting under mutual attraction of it's molecules subject to the laws of thermodynamics [19]- [22]. The model also defines the diffusion and reaction in a slab. Emden- Fowler type time fractional singular differential equations with initial conditions have been formulated and solved by fractional order homotopy perturbation method [FOHPM] [12], [13].

**Ex 2.1.1.** *Consider time fractional order Emden-Fowler type non-linear differential equation*

$$D_t^{\alpha} f(x,t) = f_{xx}(x,t) + \frac{2}{x} f_x(x,t) - \left(4+4x^2\right) f(x,t) \tag{6}$$

*where* $0 \leq \alpha \leq 1$
*with initial condition* $f(x,0) = e^{x^2}$
*and boundary conditions* $f(0,t) = 1$ *and* $f_x(0,t) = 0$.

*Solution:* By applying fractional order homotopy perturbation method [FOHPM], we may construct homotopy for above boundary value problem as

$$\begin{aligned} H(f,p) &= (1-p)D^{\alpha}[f(x,t) + f_0(x,t)] \\ &+ p\left(D_t^{\alpha} f(x,t) - f_{xx}(x,t) - \frac{2}{x} f_x(x,t) + \left(4+4x^2\right) f(x,t)\right) = 0 \end{aligned} \tag{7}$$

where $p \in [0, 1]$
Taking initial guess

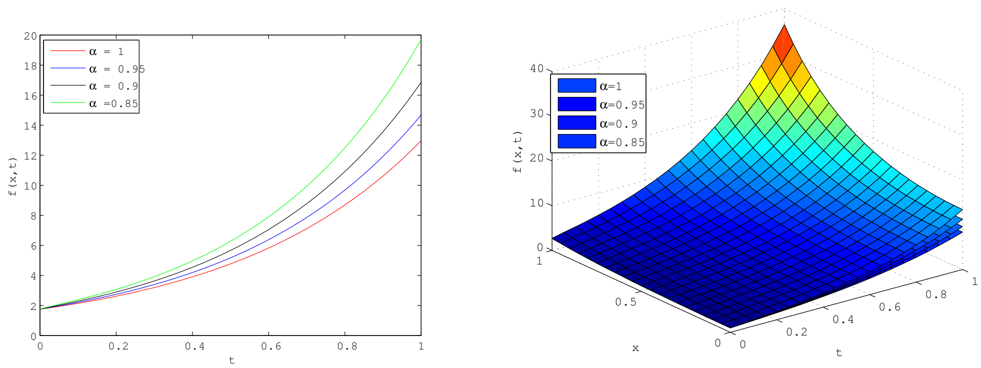
$f_o(x,t) = e^{x^2}$
Equating coefficients of 'p' in equation 7 and by putting in equation 5, the approximate solution in the form of series is given by putting $p = 1$ as

$$\begin{aligned} f(x,t) &= e^{x^2} + 2e^{x^2}\frac{t^{\alpha}}{\Gamma\alpha+1} + 2^2 e^{x^2}\frac{t^{2\alpha}}{\Gamma 2\alpha+1} + 2^3 e^{x^2}\frac{t^{3\alpha}}{\Gamma 3\alpha+1} + \ldots \\ &= e^{x^2}\left[1 + 2\frac{t^{\alpha}}{\Gamma\alpha+1} + 2^2\frac{t^{2\alpha}}{\Gamma 2\alpha+1} + 2^3\frac{t^{3\alpha}}{\Gamma 3\alpha+1} + \ldots\right] \end{aligned} \tag{8}$$

It gives exact solution for integer order by putting $\alpha = 1$, which yields

$$f(x,t) = e^{x^2} e^{2t}$$

The analysis for various fractional order have been represented graphically as 1(a) and 1(b).




(a) 2D plot $f(x,t)$ verses t for $\alpha = 1$, $\alpha = 0.95$, $\alpha = 0.9$, $\alpha = 0.85$ at $x = 0.75$

(b) 3D plot of $f(x,t)$ verses t and x for $\alpha = 1$, $\alpha = 0.95$, $\alpha = 0.9$, $\alpha = 0.85$

FIGURE 1. Numerical simulation for time fractional order Emden -Fowler type differential equation,in problem 2.1.1

**Ex 2.1.2.** *Let's take another form of time fractional order Emden-Fowler type non-linear differential equation*

$$D_t^\alpha f(x,t) = f_{xx}(x,t) + \frac{2}{x} f_x(x,t) - \left(5+4x^2\right) f(x,t) - \left(6-5x^2-4x^4\right) \tag{9}$$

*where* $0 \leq \alpha \leq 1$

*with initial condition* $f(x,0) = x^2 + e^{x^2}$

*and boundary conditions* $f(0,t) = 1$ *and* $f_x(0,t) = 0.$

*Solution:* By applying fractional order homotopy perturbation method [FOHPM], we may construct homotopy for above boundary value problem as

$$\begin{aligned} H(f,p) &= (1-p) D^\alpha [f(x,t) + f_0(x,t)] \\ &+ p\left(D_t^\alpha f(x,t) - f_{xx}(x,t) - \frac{2}{x} f_x(x,t) + \left(5+4x^2\right) f(x,t) + \left(6-5x^2-4x^4\right)\right) = 0 \end{aligned} \tag{10}$$

where $p \in [0,\, 1]$

Taking initial guess $f_0(x,t) = x^2 + e^{x^2}$

Equating coefficients of 'p' in equation 10 and by putting in the equation 5, the approximate solution in the form of series is given by putting $p = 1$ as

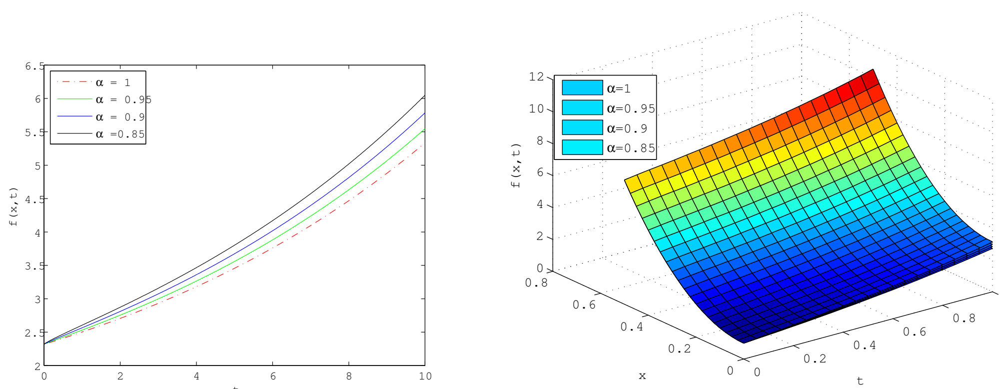
$$f(x,t) = x^2 + e^{x^2} + e^{x^2}\frac{t^{\alpha}}{\Gamma\alpha+1} + e^{x^2}\frac{t^{2\alpha}}{\Gamma 2\alpha+1} + e^{x^2}\frac{t^{3\alpha}}{\Gamma 3\alpha+1} + \ldots$$

$$= x^2 + e^{x^2}\left[1 + \frac{t^{\alpha}}{\Gamma\alpha+1} + \frac{t^{2\alpha}}{\Gamma 2\alpha+1} + \frac{t^{3\alpha}}{\Gamma 3\alpha+1} + \ldots\right] \tag{11}$$

It gives exact solution by putting $\alpha = 1$, which yields

$$f(x,t) = x^2 + e^{x^2} e^t$$

The analysis for various fractional order have been represented graphically as 2(a) and 2(b).




(a) 2D plot of $f(x,t)$ verses x for $\alpha = 1$, $\alpha = 0.95$, $\alpha = 0.9$, $\alpha = 0.8$ at $x = 0.85$

(b) 3D plot of $f(x,t)$ verses t and x for $\alpha = 1$, $\alpha = 0.95$, $\alpha = 0.9$, $\alpha = 0.85$

FIGURE 2. Numerical simulation for time fractional order Emden -Fowler type differential equation in problem 2.1.2

## 2.2. Time fractional order Evolution type differential equations.

A wide range of phenomenon in classical mechanics can be formulated in the form of evolution type differential equations. The concept of evolution type of differential equations also handles various financial aspects like risk, prices of commodities, inflation etc [23], [24]. In this section, some time fractional evolution type differential equations with initial conditions have been solved. Some of the time fractional order evolution type differential equations have been solved by taking initial conditions as a initial guess.

**Ex 2.2.1.** *Let's take time fractional order evolution type differential equation with initial conditions as*

$$D_t^{\alpha} f(x,t) = f_{xxt}(x,t) - \frac{\left(f^2(x,t)\right)_x}{2}$$

*where* $0 \leq \alpha \leq 1$

*with initial condition* $f(x,0) = x$ *the boundary conditions* $f(0,t) = 0$ *and* $f_x(0,t) = 1$.

*Solution:* By applying fractional order homotopy perturbation method [FOHPM], we may construct homotopy for above boundary value problem as

$$H(f,p) = (1-p)D_t^{\alpha}[f(x,t) + f_0(x,t)] + p\left(D_t^{\alpha} f(x,t) - f_{xxt}(x,t) + \frac{\left(f^2(x,t)\right)_x}{2}\right) = 0 \tag{12}$$

where $\quad p \in [0,1]$

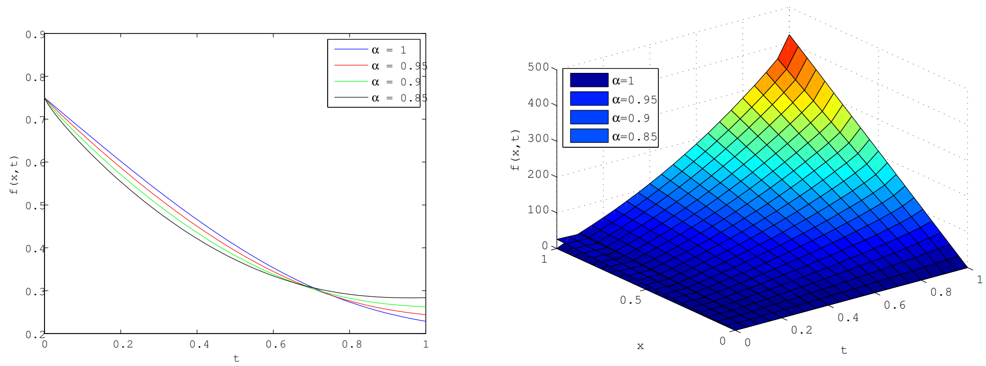
Taking initial guess $f_0(x,t) = x$

Equating coefficients of 'p' in equation 12 and by putting in the equation 5,the approximate solution in the form of series is given by putting $p = 1$ as

(13)

$$f(x,t) = x\left(1 - \frac{t^{\alpha}}{\Gamma\alpha+1} + \frac{\Gamma 2\alpha+1}{(\Gamma\alpha+1)^2}\frac{t^{3\alpha}}{\Gamma 3\alpha+1} - \left(\frac{\Gamma 2\alpha+1}{(\Gamma\alpha+1)^2.\Gamma 3\alpha+1}\right)^2 \frac{\Gamma(6\alpha+1)t^{7\alpha}}{\Gamma 7\alpha+1} + \ldots\right)$$

It gives exact solution of integer order evolution type differential equations by putting $\alpha = 1$. The analysis for fractional orders have been represented graphically as 3(a) and 3(b).




(a) 2D plot of $f(x,t)$ verses t for $\alpha = 1$, $\alpha = 0.95$, $\alpha = 0.9$, $\alpha = 0.85$ and $x = 0.75$

(b) 3D plot of $f(x,t)$ verses t and x for $\alpha = 1$, $\alpha = 0.95$, $\alpha = 0.9$, $\alpha = 0.85$

FIGURE 3. Numerical simulation for time fractional order evolution type differential equation of problem 2.2.1

**Ex 2.2.2.** *Let us take time fractional order evolution type differential equation with initial conditions as*

$$D_t^{\alpha} f(x,t) = f_{xxt}(x,t) - f_x(x,t) \tag{14}$$

*where* $0 \le \alpha \le 1$
*with initial condition* $f(x,0) = e^{-x}$

*Solution:* By applying fractional order homotopy perturbation method [FOHPM], we may construct homotopy for above boundary value problem as

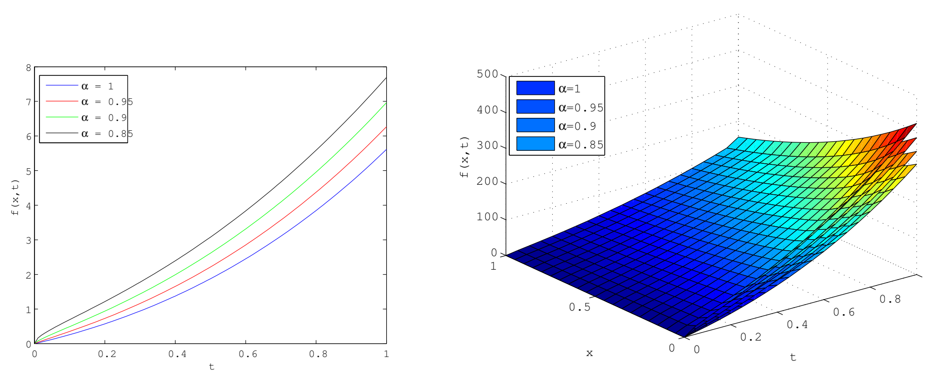
$$H(f,p) = (1-p)\,D_t^{\alpha}[f(x,t) + f_0(x,t)] + p\,(D_t^{\alpha} f(x,t) - f_{xxt}(x,t) + f_x(x,t)) = 0 \tag{15}$$

where $p \in [0, 1]$
Let's take initial guess as $f_0(x,t) = te^{-x}$
Equating coefficients of 'p' in equation 15 and by putting in the equation 5, the approximate

solution in the form of series is given by putting $p=1$ as

$$
\begin{aligned}
f(x,t) =& e^{-x}t+e^{-x}\left[\frac{t^{\alpha}}{\Gamma\alpha+1}+\frac{t^{\alpha+1}}{\Gamma\alpha+2}\right]+e^{-x}\left[\frac{t^{2\alpha+1}}{\Gamma 2\alpha+2}+\frac{2t^{2\alpha}}{\Gamma 2\alpha+1}+\frac{t^{2\alpha-1}}{\Gamma 2\alpha}\right]\\
&+e^{-x}\left[\frac{t^{3\alpha-2}}{\Gamma 3\alpha-1}+3\frac{t^{3\alpha-1}}{\Gamma 3\alpha}+3\frac{t^{3\alpha}}{\Gamma 3\alpha+1}+\frac{t^{3\alpha+1}}{\Gamma 3\alpha+2}\right]\\
&+\left[\frac{t^{4\alpha-3}}{\Gamma 4\alpha-2}+4\frac{t^{4\alpha-2}}{\Gamma 4\alpha-1}+6\frac{t^{4\alpha-1}}{\Gamma 4\alpha}+4\frac{t^{4\alpha}}{\Gamma 4\alpha+1}+\frac{t^{4\alpha+1}}{\Gamma 4\alpha+2}\right]e^{-x}+\ldots
\end{aligned}
\tag{16}
$$

The analysis for various fractional order have been represented graphically as 4(a) and 4(b).




(a) 2D plot of $f(x,t)$ verses t for $\alpha=1$, $\alpha=0.95$, $\alpha=0.9$, $\alpha=0.85$ at $x=0.75$

(b) 3D plot of $f(x,t)$ verses t for $\alpha=1$, $\alpha=0.95$, $\alpha=0.9$, $\alpha=0.85$

FIGURE 4. Numerical simulation for time fractional order evolution type differential equation in problem 2.2.2

**Ex 2.2.3.** *Let's take another type of time fractional order linear Evolution type differential equation*

$$D_t^{\alpha}u(f,t)=-f_{xxxx}(x,t) \tag{17}$$

*Where* $0\leq\alpha\leq 1$.

*with initial condition* $f(x,0)=\sin x$ .

*Solution:* By applying fractional order homotopy perturbation method [FOHPM], we may construct homotopy for above boundary value problem as

$$H(f,p)=(1-p)D_t^{\alpha}[f(x,t)-f_0(x,t)]+p(D_t^{\alpha}f(x,t)+f_{xxxx}(x,t))=0 \tag{18}$$

where $\quad p \in [0,1]$

Taking initial guess $f_0(x,0) = \sin x$

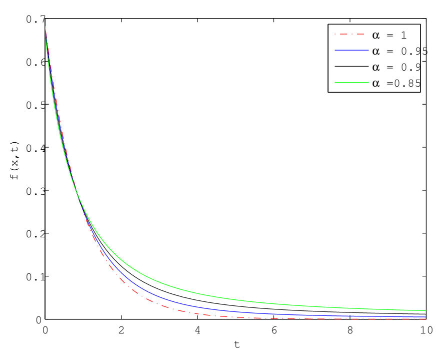
Equating coefficients of 'p' in equation 18 and by putting in the equation 5,the approximate solution in the form of series is given by putting $p = 1$ as

$$f(x,t) = \sin x \left[1 - \frac{t^{\alpha}}{\Gamma\alpha+1} + \frac{t^{2\alpha}}{\Gamma 2\alpha+1} - \frac{t^{3\alpha}}{\Gamma 3\alpha+1} + \ldots\right] \tag{19}$$

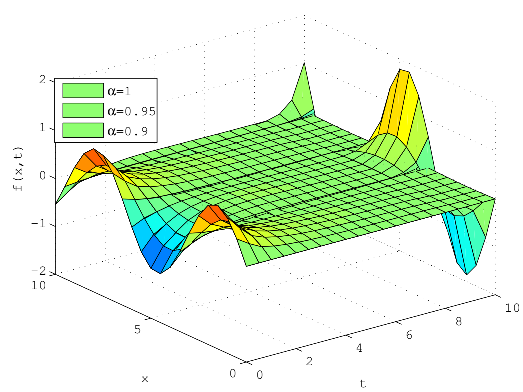
It gives exact solution by putting $\alpha = 1$, which yields

$$f(x,t) = \sin xe^{-t}$$

The analysis for various fractional order have been represented graphically as 5(a) and 5(b).




(a) 2D plot of $f(x,t)$ verses x for $\alpha = 1$, $\alpha = 0.95$, $\alpha = 0.9$, $\alpha = 0.85$ at $x = 0.75$




(b) 3D plot of $u(x,t)$ verses t and x for $\alpha = 1$, $\alpha = 0.95$, $\alpha = 0.9$

FIGURE 5. Numerical simulation for time fractional order evolution type differential equation of problem 2.2.3

## 2.3. Time fractional order Klein- Gordon type differential equations.

Klein -Gordon type differential equation is known to be a relativistic wave equation in complex quantum mechanics. Einstein's energy equation where the energy and momentum terms are replaced with quantum mechanical parameters derives Klein -Gordon type differential equation [25], [26]. we have two cases of Klein -Gordon type fractional order differential equations which have been solved by using fractional order homotopy perturbation method (FOHPM) and they have given the approximate solution. Here we are solving time fractional Klein -Gordon differential equations by taking the initial guess properly to get better approximations.

**Ex 2.3.1.** *Let us take time fractional order Klein -Gordon type differential equation*

$$D_t^{\alpha} f(x,t) = f_{xx}(x,t) + f(x,t)$$

*Where* $0 \leq \alpha \leq 2$.
*with initial condition* $u(x,0) = 1 + \sin x$ *and* $u_t(x,0) = 0$.

*Solution:* By applying fractional order homotopy perturbation method [FOHPM], we may construct homotopy for above boundary value problem as

$$H\ (f,\, p) = (1-p)\, D_t^{\alpha}[f(x,t) - f_0(x,t)] + p\,(D_t^{\alpha} f(x,t) - f_{xx}(x,t) - f(x,t)) = 0 \tag{20}$$

where $p \in [0,\, 1]$
Taking initial guess
$f_0(x,t) = 1 + \sin x$

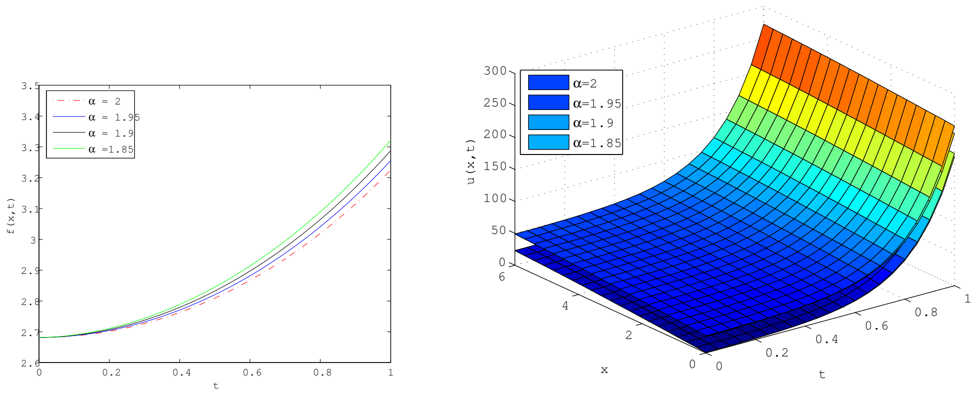
Equating coefficients of 'p' in equation 20 and by putting in the equation 5, the approximate solution in the form of series is given by putting $p = 1$ as

$$f(x,t) = \sin x + \left[1 + \frac{t^{\alpha}}{\Gamma \alpha + 1} + \frac{t^{2\alpha}}{\Gamma 2\alpha + 1} + \frac{t^{3\alpha}}{\Gamma 3\alpha + 1} + \ldots\right] \tag{21}$$

It gives exact solution by putting $\alpha = 2$, which yields

$$f(x,\, t) = \sin x + \cosh t$$

The analysis for various fractional order have been represented graphically as 6(a) and 6(b).




(a) 2D plot of $f(x,t)$ verses t for $\alpha = 2$, $\alpha = 1.95$, $\alpha = 1.9$,$\alpha = 1.85$ at $x = 0.75$

(b) 3D plot of $f(x,t)$ verses t and x at $\alpha = 2$, $\alpha = 1.95$, $\alpha = 1.9$, $\alpha = 1.85$

FIGURE 6. Numerical simulation for time fractional order Klein-Gordon type differential equation of problem 2.3.1

**Ex 2.3.2.** *Let us take time fractional order non linear Klein -Gordon type differential equation* [23]

$$D_t^\alpha f(x,t) = f_x^2(x,t) + f^2(x,t)$$

*Where* $0 \leq \alpha \leq 2$*.*
*with initial condition* $f(x,0) = 1 + \sin x$ *and* $f_t(x,0) = 0$*.*

*Solution:* By applying fractional order homotopy perturbation method [FOHPM], we may construct homotopy for above boundary value problem as

$$H(f, p) = (1-p) D_t^\alpha [f(x,t) - f_0(x,t)] + p \left( D_t^\alpha f(x,t) - f_x^2(x,t) - f^2(x,t) \right) = 0 \tag{22}$$

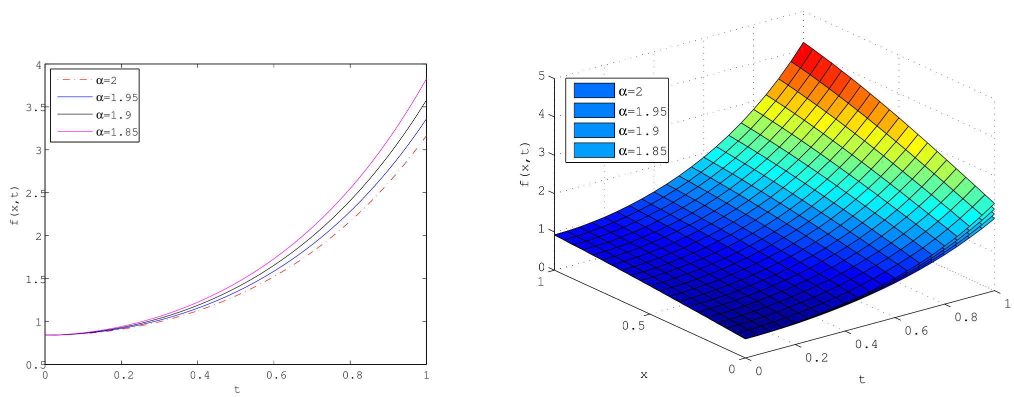
where $p \in [0, 1]$
Taking initial guess $f_0(x,t) = 1 + \sin x$

Equating coefficients of 'p' in equation 22 and by putting in the equation 5, the approximate solution in the form of series is given by putting $p = 1$ as

$$f(x,t) = (1+\sin x) + 2(1+\sin x)\left[\frac{t^{\alpha}}{\Gamma\alpha+1}\right] + 2^{3}(1+\sin x)\left[\frac{t^{2\alpha}}{\Gamma 2\alpha+1}\right] +$$

$$2^{7}(1+\sin x)\left[\frac{t^{3\alpha}}{\Gamma 3\alpha+1}\right] + \dots \quad (23)$$

The analysis for various fractional order have been represented graphically as 7(a) and 7(b).




(a) 2D plot of $f(x,t)$ verses t at $\alpha = 2$, $\alpha = 1.95$, $\alpha = 1.9$, $\alpha = 1.8$ and $x = 0.75$

(b) 3D plot of $f(x,t)$ verses t and x at $\alpha = 2$, $\alpha = 1.95$, $\alpha = 1.9$, $\alpha = 1.85$

FIGURE 7. Numerical simulation for time fractional order Klein Gordon type differential equation of problem 2.3.2

## 3. CONCLUSIONS

In this paper, we have proposed few guidelines to analyse the solutions of some types of time fractional differential equations. Fractional differential equations express the more generalised results of the physical models. Subsequently graphical representation deals with the results for various fractional orders of differential equations which emphasizes the possible outcomes. Fractional order homotopy perturbation method [FOHPM] has been applied to get the solution of the time fractional differential equations and the solution matches suitably to the exact solution. It is necessary to state that the series solution obtained by homotopy perturbation method precisely converges. We conclude that fractional order differential equations show significant changes and memory effects as compared to integer order differentiation.

## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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