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APPROXIMATION OF CONJUGATE OF A FUNCTION IN GENERALIZED HÖLDER CLASS

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Abstract. In this work, we study the error estimation of the function $\tilde{g} \in H_r^{(\eta)}$ $(r \geq 1)$, where $\tilde{g}$ is a conjugate function of a 2π-periodic function g using Matrix-Hausdorff $(T\Delta_H)$ product means of its conjugate Fourier series. Our Theorem 1 generalizes four earlier known results. Several useful results in the form of corollaries have also deduced from the main theorem.

Keywords: error estimation; generalized Hölder class; matrix (T) means; Hausdorff (Δ_H) means; matrix-Hausdorff $(T\Delta_H)$ product means; conjugate Fourier series.

2010 AMS Subject Classification: 41A10, 41A25, 42B05, 42A50, 40G05, 40C05.

1. Introduction

In the past few decades, several researchers like [2, 4, 6, 8, 9, 10, 11, 14, 15] etc. have been interested in obtaining the results on degree of approximation of the function belonging to $Lip\alpha$ and $Lip(\alpha,r)$ classes of using summability operators of conjugate Fourier series due to their variety of applications in science and engineering.

In the present work, we obtain an error estimate of function $\tilde{g}$, conjugate to a function g (2π-periodic) in generalized Hölder class $H_r^{(\eta)}$ $(r \geq 1)$ by Matrix-Hausdorff $(T\Delta_H)$ product means

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of its conjugate Fourier series. In fact, we establish a theorem on the best error approximation of conjugate function $\tilde{g}$ of a function g (2π-periodic) in generalized Hölder class $H_r^{(\eta)}$ $(r \geq 1)$ using Matrix-Hausdorff $(T\Delta_H)$ product means of its conjugate Fourier series. Thus, Theorem 1 of the present paper generalizes the results of [4, 6, 14, 15] in view of Remark 6.

Note 1. *The conjugate Fourier series is not necessarily a Fourier series for example: The series* $\sum_{l=2}^{\infty} \frac{\sin(lx)}{\log l}$ *conjugate to the Fourier series* $\sum_{l=2}^{\infty} \frac{\cos(lx)}{\log l}$ *is not a Fourier series* [1].

In view of the above example, a separate study, of conjugate series of Fourier series in the present direction of work is very much required.

2. Preliminaries

Let $\sum_{l=0}^{\infty} c_l$ be an infinite series having l^{th} partial sum $s_l = \sum_{\nu=0}^{l} c_\nu$.

Let $T \equiv (b_{l,j})$ be an infinite triangular matrix satisfying the conditions of regularity [12], i.e.

$$(1) \qquad \begin{cases} \sum_{j=0}^{l} b_{l,j} = 1 \quad \text{as} \quad l \to \infty; \\ \forall \quad j \geq 0, \quad b_{l,j} = 0 \quad \text{as} \quad l \to \infty; \\ \exists \quad M > 0 \quad \forall \quad l \geq 0, \quad \sum_{j=0}^{\infty} |b_{l,j}| < M. \end{cases}$$

The sequence-to-sequence transformation

$$t_l^T := \sum_{j=0}^{l} b_{l,j} s_j = \sum_{j=0}^{l} b_{l,l-j} s_{l-j}$$

defines the sequence t_l^T of triangular matrix means of the sequence $\{s_l\}$ generated by the sequence of coefficients $(b_{l,j})$.

If $t_l^T \to s$ as $l \to \infty$, then the infinite series $\sum_{l=0}^{\infty} c_l$ or the sequence $\{s_l\}$ is summable to s by triangular matrix (T) [1].

A Hausdorff matrix $H \equiv (h_{l,j})$ is an infinite lower triangular matrix [5] defined by

$$h_{l,j} \equiv \begin{cases} \binom{l}{j} \Delta^{l-j} \mu_j, & 0 \leq j \leq l; \\ 0, & j > l, \end{cases}$$

where the operator Δ is defined $\Delta\mu_j \equiv \mu_j - \mu_{j+1}$ and $\Delta^{l+1}\mu_j \equiv \Delta^l(\Delta\mu_j)$.
If $t_l^{\Delta_H} = \sum_{m=0}^{l} h_{l,m}s_m$ as $l \to \infty$ then the series or the sequence $\{s_l\}$ is summable to the sum s by the Hausdorff method (Δ_H method).

A Hausdorff matrix H is regular, i.e., H preserves the limit of each convergent sequence iff

$$\int_0^1 |d\xi(z)| < \infty,$$

where the mass function $\xi \in BV[0,1]$, $\xi(0+) = \xi(0) = 0$, and $\xi(1) = 1$. In this case, μ_l has the representation

$$\mu_l = \int_0^1 z^l d\xi(z)$$

[7].

Superimposing T- method on Δ_H method, $(T\Delta_H)$ is obtained. $T\Delta_H$ mean of the sequence $\{s_l\}$ is given by

$$\begin{aligned} t_l^{T\Delta_H} &:= \sum_{j=0}^{l} b_{l,j} t_j^{\Delta_H} \\ &= \sum_{j=0}^{l} b_{l,j} \sum_{v=0}^{j} h_{j,v} s_v. \end{aligned}$$

If $t_l^{T\Delta_H} \to s$ as $l \to \infty$, then $\{s_l\}$ is summable by the $T\Delta_H$ means to the limit s.

Since T and Δ_H method are regular, then $T\Delta_H$ method is also regular. This can be shown as

$$\begin{aligned} s_l \to s \quad &\Rightarrow \quad t_l^{\Delta_H} \to s, \text{ as } l \to \infty, \text{ since the } \Delta_H \text{ method is regular,} \\ &\Rightarrow \quad T(t_l^{\Delta_H}) = t_l^{T\Delta_H} \to s, \text{ as } l \to \infty, \text{ since the } T \text{ method is regular,} \\ &\Rightarrow \quad T\Delta_H \text{ method is regular.} \end{aligned}$$

Remark 1. *$T\Delta_H$ means reduces to*

(i) $(C,\alpha)\Delta_H$ *or* $C_\alpha\Delta_H$ *means when* $b_{l,j} = \frac{\binom{l-j+\alpha-1}{\alpha-1}}{\binom{l+\alpha}{\alpha}}$ *for all* $\alpha \geq -1$.

(ii) $\left(H, \frac{1}{l+1}\right)\Delta_H$ *or* $H_{1/l+1}\Delta_H$ *means if* $b_{l,j} = \frac{1}{(l-j+1)\log(l+1)}$.

(iii) $(N, p_l, q_l)\Delta_H$ *or* $N_{p,q}\Delta_H$ *means if* $b_{l,j} = \frac{p_{l-j}q_j}{R_l}$, $R_l = \sum_{j=0}^{l} p_j q_{l-j}$.

(iv) $(N, p_l)\Delta_H$ *or* $N_p\Delta_H$ *means if* $b_{l,j} = \frac{p_{l-j}}{P_l}$ *where* $P_l = \sum_{j=0}^{l} p_j$, $q_l = 1$.

(v) $(\tilde{N}, p_l)\Delta_H$ *or* $\tilde{N}_p\Delta_H$ *means if* $b_{l,j} = \frac{p_j}{P_l}$, $q_l = 1\ \forall\ l$.

(vi) $(E,q_l)\Delta_H$ *or* $E_q\Delta_H$ *means if* $b_{l,j} = \frac{1}{(1+q)^l}\binom{l}{j}q^{l-j}$.

(vii) $T(C,\alpha)$ *or* TC_α *means if* $\xi(z) = \prod_{j=1}^{\alpha} z^j$, $\alpha \geq 1$.

(viii) $T(E,q_l)$ *or* TE_q *means if* $h_{l,j} = \binom{l}{j}\frac{q^{l-j}}{(1+q)^l}$, $0 \leq j \leq l$.

In above Remark 1 (iii), (iv) and (v), $\{p_l\}$ *and* $\{q_l\}$ *are two non-negative monotonic non-decreasing sequence of real constants.*

Remark 2.

(i) $(C,\alpha)\Delta_H$ *or* $C_\alpha\Delta_H$ *means further reduces to*

(a) $(C,\alpha)(C,\alpha)$ *or* $C_\alpha C_\alpha$ *means if* $\xi(z) = \prod_{j=1}^{\alpha} z^j$, $\alpha \geq 1$.

(b) $(C,\alpha)(E,q_l)$ *or* $C_\alpha E_q$ *means if* $h_{l,j} = \binom{l}{j}\frac{q^{l-j}}{(1+q)^l}$, $0 \leq j \leq l$.

(c) $(C,1)\Delta_H$ *or* $C_1\Delta_H$ *means if* $\alpha = 1$.

(ii) $\left(H,\frac{1}{l+1}\right)\Delta_H$ *or* $H_{1/l+1}\Delta_H$ *means further reduces to*

(a) $\left(H,\frac{1}{l+1}\right)(C,\alpha)$ *or* $H_{1/l+1}C_\alpha$ *means if* $\xi(z) = \prod_{j=1}^{\alpha} z^j$, $\alpha \geq 1$.

(b) $\left(H,\frac{1}{l+1}\right)(E,q_l)$ *or* $H_{1/l+1}E_q$ *if* $h_{l,j} = \binom{l}{j}\frac{q^{l-j}}{(1+q)^l}$, $0 \leq j \leq l$.

(iii) $(N,p_l,q_l)\Delta_H$ *or* $N_{p,q}\Delta_H$ *means further reduces to*

(a) $(N,p_l,q_l)(C,\alpha)$ *or* $N_{p,q}C_\alpha$ *means if* $\xi(z) = \prod_{j=1}^{\alpha} z^j$, $\alpha \geq 1$.

(b) $(N,p_l,q_l)(E,q_l)$ *or* $N_{p,q}E_q$ *means if* $h_{l,j} = \binom{l}{j}\frac{q^{l-j}}{(1+q)^l}$, $0 \leq j \leq l$.

(iv) $(N,p_l)\Delta_H$ *or* $N_p\Delta_H$ *means further reduces to*

(a) $(N,p_l)(C,\alpha)$ *or* N_pC_α *means if* $\xi(z) = \prod_{j=1}^{\alpha} z^j$, $\alpha \geq 1$.

(b) $(N,p_l)(E,q_l)$ *or* N_pE_q *means if* $h_{l,j} = \binom{l}{j}\frac{q^{l-j}}{(1+q)^l}$, $0 \leq j \leq l$.

(v) $(\tilde{N},p_l)\Delta_H$ *or* $\tilde{N}_p\Delta_H$ *means further reduces to*

(a) $(\tilde{N},p_l)(C,\alpha)$ *or* $\tilde{N}_pC_\alpha$ *means if* $\xi(z) = \prod_{j=1}^{\alpha} z^j$, $\alpha \geq 1$.

(b) $(\tilde{N},p_l)(E,q_l)$ *or* $\tilde{N}_pE_q$ *means if* $h_{l,j} = \binom{l}{j}\frac{q^{l-j}}{(1+q)^l}$, $0 \leq j \leq l$.

(vi) $(E,q_l)\Delta_H$ *or* $E_q\Delta_H$ *means further reduces to*

(a) $(E,q_l)(C,\alpha)$ *or* E_qC_α *means if* $\xi(z) = \prod_{j=1}^{\alpha} z^j$, $\alpha \geq 1$.

(b) $(E,q_l)(E,q_l)$ *or* E_qE_q *means if* $h_{l,j} = \binom{l}{j}\frac{q^{l-j}}{(1+q)^l}$, $0 \leq j \leq l$.

(vii) $T(C,\alpha)$ *or* TC^α *means further reduces to*

(a) $T(C,1)$ *or* TC^1 *means if* $\alpha = 1$.

(viii) $T(E,q_l)$ *or* TE_q *means further reduces to*

(a) $T(E,1)$ *or* TE_1 *means if* $q_l = 1\ \forall\ l$.

Remark 3.

(i) *Above particular case (i)(b) in Remark 2 is further reduced to* C_1E_q, $C_\alpha E_1$ *and* C_1E_1 *means for* $\alpha = 1$, $q_l = 1\ \forall\ l$ *and* $\alpha = 1$, $q_l = 1\ \forall\ l$ *respectively.*

(ii) *Above particular cases (ii)(a) and (b) in Remark 2 are further reduced to* $H_{1/l+1}C_1$ *and* $H_{1/l+1}E_1$ *means for* $\alpha = 1$ *and* $q_l = 1\ \forall\ l$ *respectively.*

(iii) *Above particular cases (iii)(a) and (b) in Remark 2 are further reduced to* $(N, p_l, q_l)(C,1)$ *and* $(N, p_l, q_l)(E,1)$ *means for* $\alpha = 1$ *and* $q_l = 1\ \forall\ l$ *respectively.*

(iv) *Above particular cases (iv)(a) and (b) in Remark 2 are further reduced to* $(N, p_l)(C,1)$ *and* $(N, p_l)(E,1)$ *means for* $\alpha = 1$ *and* $q_l = 1\ \forall\ l$ *respectively.*

(v) *Above particular cases (v)(a) and (b) in Remark 2 are further reduced to* $(\tilde{N}, p_l)(C,1)$ *and* $(\tilde{N}, p_l)(E,1)$ *means for* $\alpha = 1$ *and* $q_l = 1\ \forall\ l$ *respectively.*

(vi) *Above particular cases (vi)(a) in Remark 2 is further reduced to* E_qC_1, E_1C_α *and* E_1C_1 *means for* $\alpha = 1$, $q_l = 1\ \forall\ l$ *and* $q_l = 1\ \forall\ l$, $\alpha = 1$ *respectively.*

The space of the functions L^r is given by

$$L^r[0,2\pi] = \left\{ g : [0,2\pi] \mapsto \mathbb{R} : \int_0^{2\pi} |g(x)|^r dx < \infty, r \geq 1 \right\}.$$

The norm $\|\cdot\|_{(r)}$ by

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} |g(x)|^r dx \right\}^{1/r},\ r \geq 1.$$

As defined in [1], $\eta : [0,2\pi] \mapsto \mathbb{R}$ is an arbitrary function with $\eta(s) > 0$ for $0 < s \leq 2\pi$ and $\lim_{s\to 0^+} \eta(s) = \eta(0) = 0$.

Now, we define

$$H_r^{(\eta)} := \left\{ g \in L^r[0,2\pi] : \sup_{s\neq 0} \frac{\|g(\cdot,+s) - g(\cdot)\|_r}{\eta(s)} < \infty, r \geq 1 \right\}$$

and

$$\|\cdot\|_r^{(\eta)} = \|g\|_r^{(\eta)} = \|g\|_r + \sup_{s\neq 0} \frac{\|g(\cdot,+s) - g(\cdot)\|_r}{\eta(s)}; r \geq 1.$$

Clearly, $\|\cdot\|_r^{(\eta)}$ is a norm on $H_r^{(\eta)}$.

Note 2. *$\eta(s)$ and $\chi(s)$ denote moduli of continuity of order two such that $\frac{\eta(s)}{\chi(s)}$ is positive, non-decreasing and*

$$\|g\|_r^{(\chi)} \leq \max\left(1, \frac{\eta(2\pi)}{\chi(2\pi)}\right)\|g\|_r^{(\eta)} < \infty.$$

Thus,

$$H_r^{(\eta)} \subset H_r^{(\chi)} \subset L^r;\ r \geq 1$$

[1].

Remark 4.

(i) If $\eta(s) = s^\alpha$ in $H^{(\eta)}$, $H^{(\eta)}$ implies $H^{(\alpha)}$ class.
(ii) If $\eta(s) = s^\alpha$ in $H_r^{(\eta)}$, $H^{(\eta)}$ implies $H_{\alpha,r}$ class.
(iii) If $r \to \infty$ in $H_r^{(\eta)}$, $H_r^{(\eta)}$ implies $H^{(\eta)}$ class and $H_{\alpha,r}$ implies H_α class.

The j^{th} partial sum of conjugate Fourier series is defined as

$$s_l(\tilde{g};x) - \tilde{g}(x) = \frac{1}{2\pi}\int_0^\pi \psi(x,s)\frac{\cos\left(l+\frac{1}{2}\right)s}{\sin\frac{s}{2}}ds.$$

The l-order error estimation of function g is given by

$$E_l(g) = \min\|g - t_l\|_r,$$

where t_l is a trigonometric polynomial of degree l [1].

If $E_l(g) \to 0$ as $l \to \infty$, the $E_l(g)$ is said to be the best approximation of g [1].

We write

$$\psi(x,s) = g(x+s) - g(x-s);$$

$$\Delta b_{l,j} = b_{l,j} - b_{l,j+1};$$

$$\tilde{K}_l^{T\Delta_H}(s) = \frac{1}{2\pi}\sum_{j=0}^{l} b_{l,j}\sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}d\xi(z)\frac{\cos\left(a+\frac{1}{2}\right)s}{\sin\frac{s}{2}}.$$

3. Main Theorems

Theorem 1. *If $\tilde{g} \in H_r^{(\eta)}$ class; $r \geq 1$, then the error estimation of $\tilde{g}$ by $T\Delta_H$ means of its conjugate Fourier series is*

$$\|\tilde{t}_l^{T\Delta_H} - \tilde{g}\|_r^{(\chi)} = O\left(\frac{\log(l+1)+1}{l+1}\int_{\frac{1}{l+1}}^{\pi} \frac{\eta(s)}{s^2\chi(s)}ds\right),$$

where $T \equiv (b_{l,j})$ is an infinite triangular matrix satisfying (1) *and η, χ are as defined in Note 2 provided*

$$\sum_{j=0}^{l-1} |\Delta b_{l,j}| = O\left(\frac{1}{l+1}\right) \tag{2}$$

and

$$(l+1)b_{l,l} = O(1). \tag{3}$$

4. Lemmas

Lemma 1. *Under the condition of regularity of a matrix $T \equiv (b_{l,j})$,*

$$\tilde{K}_l^{T\Delta_H}(s) = O\left(\frac{1}{s}\right) \text{ for } 0 < s < \frac{1}{l+1}.$$

Proof. For $0 < s \leq \frac{1}{l+1}$, using $\sin\frac{s}{2} \geq \frac{s}{\pi}$ and $|\cos ls| \leq 1$, $\sup_{0\leq z\leq 1} |\xi'(z)| = N$, we obtain

$$\begin{aligned}
\tilde{K}_l^{T\Delta_H}(s) &= \sum_{j=0}^{l} b_{l,j} \sum_{a=0}^{j} \int_0^1 \binom{j}{a} z^a(1-z)^{j-a} d\xi(z) \frac{\cos\left(a+\frac{1}{2}\right)s}{\sin\frac{s}{2}} \\
&= \frac{1}{2\pi} \sum_{j=0}^{l} b_{l,j} \sum_{a=0}^{j} \int_0^1 \binom{j}{a} z^a(1-z)^{j-a} d\xi(z) \frac{\cos\left(a+\frac{1}{2}\right)s}{\frac{s}{\pi}} \\
&= \frac{1}{2s} \sum_{j=0}^{l} b_{l,j} \sum_{a=0}^{j} \int_0^1 \binom{j}{a} z^a(1-z)^{j-a} d\xi(z) \cos\left(a+\frac{1}{2}\right)s \\
&= \frac{1}{2s} \sum_{j=0}^{l} b_{l,j} \sum_{a=0}^{j} \int_0^1 \binom{j}{a} z^a(1-z)^{j-a} \left|d\xi(z)\right| \left|\cos\left(a+\frac{1}{2}\right)s\right| \\
&= \frac{1}{2s} \sum_{j=0}^{l} b_{l,j} \sum_{a=0}^{j} \int_0^1 \binom{j}{a} z^a(1-z)^{j-a} d\xi(z)
\end{aligned}$$

$$\leq \frac{N}{2s}\sum_{j=0}^{l} b_{l,j}\sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}dz$$

$$= \frac{N}{2s}\sum_{j=0}^{l} b_{l,j} \text{ since } \sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}dz = 1$$

$$= O\left(\frac{1}{s}\right).$$

□

Lemma 2. *Under the condition of regularity of a matrix $T \equiv (b_{l,j})$,*

$$\tilde{K}_l^{T\Delta_H}(s) = O\left(\frac{1}{s^2(l+1)}\right), \textit{ for } \frac{1}{l+1} \leq s \leq \pi.$$

Proof. For $\frac{1}{l+1} \leq s \leq \pi$, $\sin(l+1)s \leq 1$, $\sin\frac{s}{2} \geq \frac{s}{\pi}$, $\sup_{0\leq z\leq 1}|\xi'(z)| = N$ and by Abel's lemma, we get

$$\tilde{K}_l^{T\Delta_H}(s) = \frac{1}{2\pi}\left|\sum_{j=0}^{l} b_{l,j}\sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}d\xi(z)\frac{\cos\left(a+\frac{1}{2}\right)s}{\sin\frac{s}{2}}\right|$$

$$\leq \frac{N}{2s}\left|\sum_{j=0}^{l} b_{l,j}\sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}d\xi(z)\cos\left(a+\frac{1}{2}\right)s\right|. \tag{4}$$

First we solve,

$$\sum_{a=0}^{j}\int_0^1 \binom{j}{a} z^a(1-z)^{j-a}d\xi(z)\cos\left(a+\frac{1}{2}\right)s$$

$$= (1-z)^j\mathrm{Re}\left[\sum_{a=0}^{j}\int_0^1 \binom{j}{a}\left(\frac{z}{1-z}\right)^a d\xi(z)e^{i\left(a+\frac{1}{2}\right)s}\right]$$

$$= (1-z)^j\mathrm{Re}\left[\sum_{a=0}^{j}\int_0^1 \binom{j}{a}\left(\frac{z}{1-z}\right)^a d\xi(z)e^{ias}e^{\frac{is}{2}}\right]$$

$$= (1-z)^j\mathrm{Re}\left[e^{\frac{is}{2}}\sum_{a=0}^{j}\int_0^1 \binom{j}{a}\left(\frac{ze^{is}}{1-z}\right)^a d\xi(z)\right]$$

$$= \mathrm{Re}\left[e^{\frac{is}{2}}\int_0^1 (1-z+ze^{is})^j dz\right]$$

$$= \mathrm{Re}\left[e^{\frac{is}{2}}\int_0^1 \left\{1+(e^{is}-1)\right\}dz\right]$$

$$
\begin{aligned}
&= \mathrm{Re}\left[\frac{e^{i(j+1)s}-1}{(j+1)\left(e^{\frac{is}{2}}-e^{\frac{-is}{2}}\right)}\right]\\
&= \mathrm{Re}\left[\frac{e^{i(j+1)s}-1}{(j+1)2i\sin\frac{s}{2}}\right]\\
&= \mathrm{Re}\left[\frac{\cos(j+1)s+i\sin(j+1)s-1}{2i(j+1)\sin\frac{s}{2}}\right]\\
&= \frac{\sin(j+1)s}{2(j+1)\sin\frac{s}{2}}. \qquad (5)
\end{aligned}
$$

From equation (4) and (5), we get

$$
\begin{aligned}
\tilde{K}_l^{T\Delta_H}(s) &\le \frac{N}{2\pi}\left|\sum_{j=0}^{l} b_{l,j}\frac{\sin(j+1)s}{2(j+1)\sin\frac{s}{2}}\right|\\
&= \frac{N}{2\pi}\left|\sum_{j=0}^{l} b_{l,j}\frac{1}{2(j+1)\frac{s}{\pi}}\right|\\
&= \frac{N\pi}{4s^2}\left|\sum_{j=0}^{l} b_{l,j}\frac{1}{j+1}\right|\\
&= \frac{N\pi}{4s^2}\left|\sum_{j=0}^{l-1}\left(b_{l,j}-b_{l,j+1}\right)\sum_{a=0}^{j}\frac{1}{a+1}+b_{l,l}\sum_{j=0}^{l}\frac{1}{j+1}\right|\\
&\le \frac{N\pi}{4s^2}\left|\sum_{j=0}^{l-1}\Delta b_{l,j}\sum_{a=0}^{j}\frac{1}{a+1}\right|+b_{l,l}\left|\sum_{j=0}^{l}\frac{1}{j+1}\right|\\
&\le \frac{N\pi}{4s^2}\left[\sum_{j=0}^{l-1}|\Delta b_{l,j}|+b_{l,l}\right]\max_{0\le j\le p}\left|\sum_{j=0}^{p}\frac{1}{j+1}\right|\\
&= \frac{N\pi}{4s^2}\left[\sum_{j=0}^{l-1}|\Delta b_{l,j}|+b_{l,l}\right]\\
&= \frac{N\pi}{4s^2}\left[O\left(\frac{1}{l+1}\right)+O\left(\frac{1}{l+1}\right)\right]\\
&= O\left(\frac{1}{s^2(l+1)}\right).
\end{aligned}
$$

□

Lemma 3. *Let* $\tilde{g}\in H_r^{(\eta)}$, *then for* $0<s\le\pi$:

(i) $\|\psi(\cdot,s)\|_r = O(\eta(s))$;

(ii) $\|\psi(\cdot+z,s)-\psi(\cdot,s)\|_r = \begin{cases} O(\eta(s)) \\ O(\eta(z)). \end{cases}$

(iii) If $\eta(s)$ *and* $\chi(s)$ *are as defined in Note 2, then* $\|\psi(\cdot+z,s)-\psi(\cdot,s)\|_r = O\left(\chi(|z|)\left(\frac{\eta(s)}{\chi(s)}\right)\right)$

([13]*, p. 93).*

5. Proof of the Main Theorems

5.1. Proof of Theorem 1.

Proof. The integral representation of $s_l(\tilde{g};x)$ is given by

$$s_l(\tilde{g};x)-\tilde{g}(x) = \frac{1}{2\pi}\int_0^\pi \psi(x,s)\frac{\cos\left(j+\frac{1}{2}\right)s}{\sin\frac{s}{2}}ds.$$

The $T\Delta_H$ transform of $s_l(\tilde{g}:x)$, denoted by $\tilde{t}_l^{T\Delta_H}$, we get

$$\begin{aligned}
\tilde{t}_l^{T\Delta_H}(x)-g(x) &= \sum_{j=0}^{l} b_{l,j}\left(H_l(x)-\tilde{g}(x)\right)\\
&= \sum_{j=0}^{l} b_{l,j}\left\{\sum_{v=0}^{j}\binom{j}{v}\Delta^{j-v}\mu_v\left(\frac{1}{2\pi}\int_0^\pi \psi(x,s)\frac{\cos\left(v+\frac{1}{2}\right)s}{\sin\frac{s}{2}}ds\right)\right\}\\
&= \frac{1}{2\pi}\int_0^\pi \psi(x,s)\sum_{j=0}^{l} b_{l,j}\left\{\sum_{v=0}^{j}\binom{j}{v}\Delta^{j-v}\left(\int_0^1 z^v d\xi(z)\right)\frac{\cos\left(v+\frac{1}{2}\right)s}{\sin\frac{s}{2}}ds\right\}\\
&= \frac{1}{2\pi}\int_0^\pi \psi(x,s)\sum_{j=0}^{l} b_{l,j}\left\{\sum_{v=0}^{j}\int_0^1\binom{j}{v}z^v(1-z)^{j-v}d\xi(z)\frac{\cos\left(v+\frac{1}{2}\right)s}{\sin\frac{s}{2}}ds\right\}\\
&= \int_0^\pi \psi(x,s)\tilde{K}_l^{T\Delta_H}(s)ds.
\end{aligned}$$

Let

$$\tilde{T}_l(x) = \tilde{t}_l^{T\Delta_H}(x)-\tilde{g}(x) = \int_0^\pi \psi(x,s)\tilde{K}_l^{T\Delta_H}(s)ds.$$

Then,

$$\tilde{T}_l(x+y)-\tilde{T}_l(x) = \int_0^\pi \left\{\psi(x+z,s)-\psi(x,s)\right\}\tilde{K}_l^{T\Delta_H}(s)ds.$$

Using generalized Minkowski's inequality Chui [3], we get

$$\begin{aligned}\|\tilde{T}_l(\cdot+z)-\tilde{T}_l(\cdot)\|_r &\leq \int_0^{\pi}\|\psi(\cdot+z,s)\|_r\tilde{K}_l^{T\Delta_H}(s)ds\\ &\left(\int_0^{\frac{1}{l+1}}+\int_{\frac{1}{l+1}}^{\pi}\right)\|\psi(\cdot+z,s)-\psi(\cdot,s)\|_r|\tilde{K}_l^{T\Delta_H}(s)|ds\\ &=J_1+J_2. \end{aligned}\tag{6}$$

Using Lemmas 1 and 3 (iii), we have

$$\begin{aligned}J_1 &= \int_0^{\frac{1}{l+1}}\|\psi(\cdot+z,s)-\psi(\cdot,s)\|_r|\tilde{K}_l^{T\Delta_H}(s)|ds\\ &=O\left(\chi(|z|)\frac{\eta\left(\frac{1}{l+1}\right)}{\chi\left(\frac{1}{l+1}\right)}\int_0^{\frac{1}{l+1}}\frac{1}{s}ds\right)\\ &=O\left(\chi(|z|)\frac{\eta\left(\frac{1}{l+1}\right)}{\chi\left(\frac{1}{l+1}\right)}\log(l+1)\right).\end{aligned}\tag{7}$$

Again, using Lemmas 2 and 3 (iii), we have

$$\begin{aligned}J_2 &= \int_{\frac{1}{l+1}}^{\pi}\|\psi(\cdot+z,s)-\psi(\cdot,s)\|_r|\tilde{K}_l^{T\Delta_H}(s)|ds\\ &=O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\chi(|z|)\frac{\eta(s)}{s^2\chi(s)}ds\right).\end{aligned}\tag{8}$$

Using (6), (7) and (8), we have

$$\sup_{z\neq 0}\frac{\|\tilde{T}_l(\cdot,+z)-\tilde{T}_l(\cdot)\|_r}{\chi(|z|)}=O\left(\frac{\eta\left(\frac{1}{l+1}\right)}{\chi\left(\frac{1}{l+1}\right)}\log(l+1)\right)+O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right).\tag{9}$$

Again applying Minkowski's inequality, Lemmas 1, 2 and 3 (i), we have

$$\begin{aligned}\|\tilde{T}_l(\cdot)\|_r &= \|\tilde{t}_l^{T\Delta_H}-\tilde{g}\|_r\\ &\leq\left(\int_0^{\frac{1}{l+1}}+\int_{\frac{1}{l+1}}^{\pi}\right)\|\psi(\cdot,s)\|_r|\tilde{K}_l^{T\Delta_H}(s)|ds\\ &=O\left(\int_0^{\frac{1}{l+1}}\frac{\eta(s)}{s}ds\right)+O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2}ds\right)\\ &=O\left(\eta\left(\frac{1}{l+1}\right)\log(l+1)\right)+O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2}ds\right).\end{aligned}\tag{10}$$

Now, we have

$$\|\tilde{T}(\cdot)\|_r^{(\chi)} = \|\tilde{T}_l(\cdot)\|_r + \sup_{z\neq 0}\frac{\|\tilde{T}_l(\cdot,+z)-\tilde{T}_l(\cdot)\|_r}{\chi(|z|)}. \tag{11}$$

Using (9), (10) and (11), we get

$$\begin{aligned}\|\tilde{T}_l(\cdot)\|_r^{(\chi)} &= O\left(\log(l+1)\eta\left(\frac{1}{l+1}\right)\right)+O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2}ds\right)\\ &\quad +O\left(\log(l+1)\frac{\eta\left(\frac{1}{l+1}\right)}{\chi\left(\frac{1}{l+1}\right)}\right)+O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right).\end{aligned}$$

By the monotonicity of $\chi(s)$, we have $\eta(s)=\frac{\eta(s)}{\chi(s)}\chi(s)\leq\chi(\pi)\frac{\eta(s)}{\chi(s)}$, $0<s\leq\pi$, we get

$$\|\tilde{T}_l(\cdot)\|_r^{(\chi)} = O\left(\log(l+1)\frac{\eta\left(\frac{1}{l+1}\right)}{\chi\left(\frac{1}{l+1}\right)}\right)+O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right). \tag{12}$$

Since η and ξ as defined in Note 2, therefore

$$\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds \geq \frac{1}{l+1}\frac{\eta\left(\frac{1}{l+1}\right)}{\chi\left(\frac{1}{l+1}\right)}\int_{\frac{1}{l+1}}^{\pi}\frac{1}{s^2}ds \geq \frac{\eta\left(\frac{1}{l+1}\right)}{2\chi\left(\frac{1}{l+1}\right)}.$$

Then

$$\frac{\eta\left(\frac{1}{l+1}\right)}{\chi\left(\frac{1}{l+1}\right)} = O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right). \tag{13}$$

From (12) and (13), we get

$$\|\tilde{T}_l(\cdot)\|_r^{(\chi)} = O\left(\frac{\log(l+1)}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right)+O\left(\frac{1}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right)$$

$$\therefore \|\tilde{t}_l^{T\Delta_H}-\tilde{g}\|_r^{(\chi)} = O\left(\frac{1+\log(l+1)}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right). \tag{14}$$

□

6. Corollaries

Corollary 1. *Let* $\tilde{g}\in H_{(\alpha),r}$; $r\geq 1$ *and* $0\leq\beta<\alpha\leq 1$.

Then

$$\|\tilde{t}^{T\Delta_H}-\tilde{g}\|_{(\beta),r} = \begin{cases} O\left[\log(l+1)e(l+1)^{\beta-\alpha}\right] & \text{if} \quad 0\leq\beta<\alpha<1,\\ O\left[\frac{(\log(l+1)e)(\log\pi(l+1))}{l+1}\right] & \text{if} \quad \beta=0,\alpha=1.\end{cases}$$

Corollary 2. *Following the Remark 1 (i), we obtain*

$$\|\tilde{t}_l^{C_\alpha \Delta_H} - \tilde{g}\|_r^{(\chi)} = O\left(\frac{1+\log(l+1)}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right).$$

Corollary 3. *Following the Remark 1 (ii), we obtain*

$$\|\tilde{t}_l^{H_{1/l+1}\Delta_H} - \tilde{g}\|_r^{(\chi)} = O\left(\frac{1+\log(l+1)}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right).$$

Corollary 4. *Following the Remark 1 (iii), we obtain*

$$\|\tilde{t}_l^{N_{p,q}\Delta_H} - \tilde{g}\|_r^{(\chi)} = O\left(\frac{1+\log(l+1)}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right).$$

Corollary 5. *Following the Remark 1 (iv), we obtain*

$$\|\tilde{t}_l^{N_p\Delta_H} - \tilde{g}\|_r^{(\chi)} = O\left(\frac{1+\log(l+1)}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right).$$

Corollary 6. *Following the Remark 1 (v), we obtain*

$$\|\tilde{t}_l^{\tilde{N}_p\Delta_H} - \tilde{g}\|_r^{(\chi)} = O\left(\frac{1+\log(l+1)}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right).$$

Corollary 7. *Following the Remark 1 (vi), we obtain*

$$\|\tilde{t}_l^{E_q\Delta_H} - \tilde{g}\|_r^{(\chi)} = O\left(\frac{1+\log(l+1)}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right).$$

Corollary 8. *Following the Remark 1 (vii), we obtain*

$$\|\tilde{t}_l^{TC_\alpha} - \tilde{g}\|_r^{(\chi)} = O\left(\frac{1+\log(l+1)}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right).$$

Corollary 9. *Following the Remark 1 (viii), we obtain*

$$\|\tilde{t}_l^{TE_q} - \tilde{g}\|_r^{(\chi)} = O\left(\frac{1+\log(l+1)}{l+1}\int_{\frac{1}{l+1}}^{\pi}\frac{\eta(s)}{s^2\chi(s)}ds\right).$$

Remark 5.

(i) Corollary 2 can be further reduced as $C_\alpha E_q$ and $C_1\Delta_H$ means in view of Remark 2 (i)(b) and (c) respectively.

(ii) Corollary 3 can be further reduced as $H_{1/l+1}C_\alpha$ and $H_{1/l+1}E_q$ means in view of Remark 2 (ii) (a) and (b) respectively.

(iii) *Corollary 4 can be further reduced as* $N_{p,q}C_\alpha$ *and* $N_{p,q}E_q$ *in view of Remark 2 (iii) (a) and (b) respectively.*

(iv) *Corollary 5 can be further reduced as* N_pC_α *and* N_pE_q *means in view of Remark 2 (iv) (a) and (b) respectively.*

(v) *Corollary 6 can be further reduced as* $\tilde{N}_pC_\alpha$ *and* $\tilde{N}_pE_q$ *means in view of Remark 2 (v) (a) and (b) respectively.*

(vi) *Corollary 7 can be further reduced as* E_qC_α *means in view of Remark 2 (vi) (a).*

(vii) *Corollaries 8 can be further reduced as* TC_1 *means in view of Remark 2 (vii) (a).*

(viii) *Corollaries 9 can be further reduced as* TE_1 *means in view of Remark 2 (viii) (a).*

Remark 6.

(i) *If* $r \to \infty$ *in* $H_r^{(\eta)}$ *class, then* $H_r^{(\eta)}$ *class turns down to* $H^{(\eta)}$ *class. Also putting* $\eta(s) = s^\alpha$ *and* $\chi(s) = s^\beta$ *in Theorem 1,* $H^{(\eta)}$ *class turns down to* H_α *class. Then, by putting* $\beta = 0$ *in* H_α *class,* H_α *class turns down to Lip*α *class.*

(ii) *In our Theorem 1, by putting* $\eta(s) = s^\alpha$, $\chi(s) = s^\beta$ *in* $H_r^{(\eta)}$ *class,* $H_r^{(\eta)}$ *class turns down to* $H_{\alpha,r}$ *class. Then, by putting* $\beta = 0$ *in* $H_{\alpha,r}$ *class,* $H_{\alpha,r}$ *class turns down to Lip*(α, r) *class.*

Remark 7.

(i) *Using Remark 6 (i), putting* $b_{l,j} = \frac{1}{(1+q)^l}\binom{l}{j}q^{l-j}$ *and* $\xi(z) = \prod_{j=1}^{\alpha} z^j$, $\alpha \geq 1$ *in our Theorem 1, then the result of Tiwari and Sharma* [4] *follows.*

(ii) *Using Remark 6 (i), putting* $b_{l,j} = \frac{1}{(1+q)^l}\binom{l}{j}q^{l-j}$ *and* $h_{l,j} = \frac{1}{l+1}$, $0 \leq j \leq l$ *and in our Theorem 1, then the result of Nigam* [6] *follows.*

(iii) *Using Remark 6 (ii), putting* $b_{l,j} = \frac{1}{2^l}\binom{l}{j}$ *and* $h_{l,j} = \frac{1}{l+1}$, $0 \leq j \leq l$ *in our Theorem 1, then the result of Lal and Singh* [14] *follows.*

(iv) *Using Remark 6 (ii), putting* $b_{l,j} = \frac{1}{l+1}$, $0 \leq j \leq l$ *and* $h_{l,j} = \frac{1}{(1+q)^l}\binom{l}{j}q^{l-j}$, *in our Theorem 1, then the result of Sonker and Singh* [15] *follows.*

7. Conclusion

In this paper, we obtain the error estimation of the function $\tilde{g}$, the conjugate of a function g in the Hölder space $H_r^{(\eta)}$ $(r \geq 1)$ by Matrix-Hausdorff $(T\Delta_H)$ product means of its conjugate

Fourier series. Since, in view of Remark 1, the product summability means $C_\alpha\Delta_H$, $H_{1/l+1}\Delta_H$, $N_{p,q}\Delta_H$, $N_p\Delta_H$, $\tilde{N}_p\Delta_H$, $E_q\Delta_H$, TC_α and TE_q are the particular cases of $T\Delta_H$ product means. Some useful results are also deduced in the form of corollaries from our theorem.

Some other studies regarding modulus of continuity (smoothness) of functions using more generalized functional spaces may be the future interest of a few investigators in the direction of this work.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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