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COMPUTATIONAL TECHNIQUE FOR TWO PARAMETER SINGULARLY PERTURBED PARABOLIC CONVECTION-DIFFUSION PROBLEM

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Abstract: We study the singularly perturbed parabolic differential equation of convection-diffusion type with two small parameters affecting the derivatives. Using backward Euler process, time discretization is achieved. Problem is discretized in space using two fitting factors on uniform mesh where these factors take care of the two small parameters. Numerical scheme is constructed using two parameter fitting method. Tridiagonal solver is used to solve the resulting system of equations. Numerical results justify the parameter-uniform convergence of the scheme. We also mull numerical examples in comparing with remaining methods in the literature to uphold the method.

Keywords: parabolic convection-diffusion equation; two parameter fitting; parameter-uniform convergence.

2010 AMS Subject Classification: 65L10, 65L11, 65L12.

1. INTRODUCTION

The highest derivative affected by a small parameter in the governance equation of a acknowledged singular perturbation problems (SPP) results the conspicuous gradients in the set over constrictive regions of the field. Multi parameter singularly perturbed boundary value problems (SPBVP) are identifiable in the discipline of engineering and science. In general, these problems arise in diverse area of practical mathematics specified as aerodynamics, elasticity, reaction-diffusion affect, fluid

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and quantum fluid mechanics. Especially, such problems fall in transport phenomena of chemical setup theory, lubrication and bioscience theory [1], [2], [5], [11], [14].

Clavero et al., [3] constructed a scheme for a time dependent one-dimensional diffusion-convection problem on a variable mesh with dominating convection term. An adaptive finite difference method was proposed by Das & Mehrmann [4] for one dimensional parabolic reaction-convection-diffusion initial BVPs with two small parameters. An upwind finite difference scheme is developed by Gowrisankar & Natesan [6] on layer-adapted non-uniform meshes for parabolic diffusion-convection problems displaying regular boundary layers. Kadalbajoo & Yadaw [7] studied the numerical methods which provide parameter uniform for a class of two parameter SPBVPs. Munyakaji [12] provided a robust finite difference method to solve a class of time-dependent singularly perturbed parabolic differential equations in which the diffusion and convection terms affected by the two parameters. One dimensional time dependent singularly perturbed diffusion-reaction problem is treated by Munyakaji & Patidar [13], then the problem is solved using a novel fitted operator finite difference method. Patidar [17] approached a fitted operator method to solve singularly perturbed two parameter BVPs on uniform mesh. For the same family of problems, a streamline diffusion FEM is designed by Roos & Uzelac [18].

With this motivation, in this paper, we aimed to solve a singularly perturbed time dependent diffusion-convection problems using two parameter fitting method on uniform mesh where the diffusion and convection terms are affected by the two small parameters.

We organized the rest of the paper as follows. Section 1 Problem is described precisely in Section 1. We are concerned with the discretization of the time and space variables through proposed scheme in section 2. In section 3, convergence of the method is shown. Numerical examples with results confirming our findings on comparison of some existing methods are provided in section 4. Finally, some discussions and conclusions are drawn in last section.

2. PROBLEM DESCRIPTION

Consider a class of two parameters convection-diffusion problem

$$L_{x,t}y \equiv \varepsilon_1 \frac{\partial^2 y}{\partial x^2} + \varepsilon_2 a(x)\frac{\partial y}{\partial x} - b(x)y - \frac{\partial y}{\partial t} = f(x,t) \tag{1}$$

where $(x,t) \in [0,1] \times [0,T]$, subject to the initial and boundary conditions

$$y(x,0) = y_0(x),\ \ x \in (0,1) \tag{1a}$$

$$y(0,t) = y(1,t) = 0,\ \ t \in [0,T] \tag{1b}$$

with two small parameters $0 < \varepsilon_1, \varepsilon_2 \leq 1$ and the functions $a(x), b(x), f(x,t)$ are smooth enough and satisfy $b(x) \geq \beta > 0$ and $a(x) \geq \alpha > 0$.

3. NUMERICAL METHOD

The time variable is discretised using the implicit Euler method with an unvarying step size τ such that the interval $[0,T]$ can be partitioned as $t_0 = 0, t_K = T,\ t_k = t_0 + k\tau,\ k = 1,2,\ldots,K$. By this time discretization, we can write the Equation 1 at each time level as

$$L_x z \equiv \varepsilon_1 z_{xx}(x,t_k) + \varepsilon_2 a(x) z_x(x,t_k) - \left(b(x) + \frac{1}{\tau}\right) z(x,t_k) = f(x,t_k) - \frac{1}{\tau} z(x,t_{k-1}) \qquad (2)$$

with the conditions $z(x,0) = y_0,\ x \in (0,1);\ \ z(0,t_k) = z(1,t_k) = 0$

The characteristic equation whose roots describe the solution of Eq. (2) is

$$\varepsilon_1 \lambda^2(x) + \varepsilon_2 a(x) \lambda(x) - \left(\frac{1}{\tau} + b(x)\right) = 0 \qquad (3)$$

Two continuous functions produced by Eq. (3) are

$$\lambda_1(x) = -\frac{\varepsilon_2 a(x)}{2\varepsilon_1} - \sqrt{\frac{\left(b(x)+\frac{1}{\tau}\right)}{\varepsilon_1} + \left(\frac{\varepsilon_2 a(x)}{2\varepsilon_1}\right)^2} \qquad (4)$$

$$\lambda_2(x) = -\frac{\varepsilon_2 a(x)}{2\varepsilon_1} + \sqrt{\frac{\left(b(x)+\frac{1}{\tau}\right)}{\varepsilon_1} + \left(\frac{\varepsilon_2 a(x)}{2\varepsilon_1}\right)^2} \qquad (5)$$

The boundary layers at *x=0* and *x=1* are characterized by these two real solutions respectively. Let

$$\theta_0 = -\max_{x\in[0,1]} \lambda_1(x) \quad and \quad \theta_1 = \max_{x\in[0,1]} \lambda_2(x).$$

Introducing the two fitting factors in order to control the two parameters in the Eq. (2), we have

$$L_x z_j \equiv \varepsilon_1 \sigma_j z_{xx}(x_j,t_k) + \varepsilon_2 \eta_j a(x_j) z_x(x_j,t_k) - \left(b(x_j) + \frac{1}{\tau}\right) z(x_j,t_k) = f(x_j,t_k) - \frac{1}{\tau} z(x_j,t_{k-1}) \qquad (6)$$

with the conditions $z(x_j,0) = y_0,\ x_j \in (0,1);\ \ z(0,t_k) = z(1,t_k) = 0.$

Lemma 1. *Let* $\kappa(x,t) \in C^2(R) \cap C^0(\bar{R})$ be a smooth function such that $\kappa(x,t)|_{\partial R} \geq 0$ and $L_{x,t}\kappa(x,t)|_R \leq 0$ then $\kappa(x,t)|_{\bar{R}} \geq 0$.

Proof. See Munyakazi [12].

Lemma 2. For any $0 < p < 1$ we have, up to a certain order q that it depends on the smoothness of the data

$$|z^k(x)| \leq C \begin{pmatrix} 1 + \theta_0^k\, exp(-p\theta_0 x) + \\ \theta_1^k\, exp(-p\theta_1 x) \end{pmatrix} \text{ for } 0 \leq k \leq q$$

Proof. See Kadalbajoo & Yadaw [9].

Now, let the space interval $[0,1]$ be partitioned into N subintervals such that $x_0 = 0, x_N = 1,\ x_i = x_0 + ih, i = 1,2,\dots,N$ with $h = x_i - x_{i-1}$.

To handle the two small parameters ε_1 and ε_2, we introduce two fitting parameters σ_j *and* η_j at the respective positions and construct the following difference scheme for space variable

$$L_\varepsilon^{N,K} y(x_j,t_k) \equiv \varepsilon_1\sigma_j \frac{y(x_{j+1},t_k) - 2y(x_j,t_k) + y(x_{j-1},t_k)}{h^2} + \varepsilon_2\eta_j a(x_j)\frac{y(x_{j+1},t_k) - y(x_j,t_k)}{h}$$
$$-\left(b(x_j) + \tfrac{1}{\tau}\right) y(x_j,t_k) = f(x_j,t_k) - \tfrac{1}{\tau} y(x_j,t_{k-1}) \qquad (7)$$

with the help of discrete conditions $y(x_j,0) = y_0(x_j),\ x_j \in (0,1)$

$$y(0,t_k) = y(1,t_k) = 0,\ t_k \in [0,T].$$

Using Eq. (4) and Eq. (5) in Eq. (7), the two fitting factors can be determined as

$$\sigma_j = \frac{-\left(b(x_j)+\frac{1}{\tau}\right)\rho h}{4}\left(\frac{e^{\left(\frac{-\varepsilon_2 a(x_j)h}{2\varepsilon_1}\right)}}{sinh\left(\frac{\lambda_1(x_j)h}{2}\right) sinh\left(\frac{\lambda_2(x_j)h}{2}\right)}\right)$$

$$\eta_j = \frac{\left(b(x_j)+\frac{1}{\tau}\right)h}{2\varepsilon_2 a(x_j)}\left(coth\left(\frac{\lambda_1(x_j)h}{2}\right) + coth\left(\frac{\lambda_2(x_j)h}{2}\right)\right) \quad \text{for } j = 1,2,\dots,N.$$

Here $\rho = \frac{h}{\varepsilon_1}$.

The above scheme produces a linear system as

$$AY = F \qquad (8)$$

Entries of the co-efficient matrix A and column vector F are

$A_{ij} = \dfrac{\varepsilon_1\sigma_j(x_j)}{h^2}$ *for* $i = j+1$ where $j = 1,2,\dots,N-2,$

$A_{ij} = -\left(\dfrac{2\varepsilon_1\sigma_j(x_j)}{h^2} + \dfrac{\varepsilon_2\eta_j(x_j)a(x_j)}{h} + \left(b(x_j) + \dfrac{1}{\tau}\right)\right)$ *for* $i = j$ *where* $j = 1,2,\dots,N-1$

$A_{ij} = \dfrac{\varepsilon_1\sigma_j(x_j)}{h^2} + \dfrac{\varepsilon_2\eta_j(x_j)a(x_j)}{h}$ *for* $i = j-1$ *where* $j = 2,3,\dots,N-1,$

$F_j = f(x_j,t_k) - \dfrac{1}{\tau} y(x_j,t_{k-1})$ *for* $j = 1,2,\dots,N-1.$

Lemma 3. At any time level t_i, if U_j^i is any mesh function such that $U_0^i = U_N^i = 0$ then $|U_k^i| \le \frac{1}{\alpha}\max_{1\le j\le N-1}|L^{N,K}U_j^i|$, *for* $0 < k < N$.

Proof: See Munyakazi [12].

Lemma 4. The global error estimate of the time discretization is given by $\|E_i\|_\infty \le C\tau$.

Proof: See Munyakazi & Patidar [13].

4. CONVERGENCE ANALYSIS

The local truncation error of the proposed scheme is

$$L^{N,K}(y_j - z_j) = L_x z_j - L^{N,K} z_j$$

$$= \varepsilon_1 \sigma_j z'' + \varepsilon_2 \eta_j a_j z_j' - \varepsilon_1 \sigma_j \frac{z_{j+1} - 2z_j + z_{j-1}}{h^2} - \varepsilon_2 \eta_j \frac{z_{j+1} - z_j}{h}$$

Using expansion of Taylor's series and considering the truncated Taylor expansion, we obtain

$$L^{N,K}(y_j - z_j) = -\frac{\varepsilon_2 \eta_j a_j h}{2} z_j'' - \frac{\varepsilon_2 \eta_j a_j h^2}{6} z_j''' - \frac{\varepsilon_1 \sigma_j h^2}{24} z_j^{iv}(\xi_1) - \frac{\varepsilon_2 \eta_j a_j h^3}{24} z_j^{iv}(\xi_2) \tag{9}$$

where $\xi_1 \in (x_j, x_{j+1})$ and $\xi_2 \in (x_{j-1}, x_j)$.

Using Lemma 2, for small h, noticing that both $\theta_0^k\, exp(-p\theta_0 x_j)$ and $\theta_1^k\, exp(-p\theta_1(1 - x_j))$ approach to zero as $\varepsilon_1 \to 0$ for all $k = 0,1,2,\ldots$, we obtain

$$\left|L^{N,K}(y_j - z_j)\right| \le Mh \tag{10}$$

Now applying Lemma 3,

$$\max_{0\le j\le N} \left|y_j^k - z_j^k\right| \le Mh \tag{11}$$

Also by Lemma 4, we have $\max\limits_{0\le k\le K} \left|y_j^k - z_j^k\right| \le M\tau$ (12)

From Eq. (11) and Eq. (12), we get

$$\max_{0\le j\le N, 0\le k\le K} \left|y_j^k - z_j^k\right| \le M(h + \tau) \tag{13}$$

Equation 13 shows that our method is first order convergent.

5. NUMERICAL RESULTS

We consider the two parameter SPBVPs to demonstrate the proposed method computationally. The maximum point-wise errors at all the mesh points are calculated using $E_{\varepsilon_1,\varepsilon_2}^{N,K} = \max\limits_{0\le j\le N; 0\le k\le K} \left|\left(y_{\varepsilon_1,\varepsilon_2}^{N,K}\right)_{j,k} - \left(y_{\varepsilon_1,\varepsilon_2}^{2N,2K}\right)_{j,k}\right|$ when exact solution is unknown.

Example 1. $\varepsilon_1 \frac{\partial^2 y}{\partial x^2} + \varepsilon_2 (1 + x) \frac{\partial y}{\partial x} - y(x) - \frac{\partial y}{\partial t} = 16x^2(1 - x)^2$

where $(x,t) \in (0,1) \times (0,1]$; $y(x,0) = 0$, for $x \in (0,1)$; $y(0,t) = y(1,t) = 0$, for $t \in [0,1]$. Comparison of results shown in Tables 1and 2 with those in Tables 1 and 5 of Munyakazi [12] for the various values $\varepsilon_1$and ε_2 is presented. The result profile at the prescribed values of *N* and *K* is represented in Fig. 1.

Example 2. $\varepsilon_1 \frac{\partial^2 y}{\partial x^2} - (2 - x^2)\frac{\partial y}{\partial x} - xy - \frac{\partial y}{\partial t} = -10t^2 e^{-t} x(1-x)$

where $y(x,0) = 0$, for all $x \in (0,1)$; $y(0,t) = y(1,t) = 0$, for all $t \in (0,3)$.

Table 3 represents the maximum point wise errors for various values of ε_1 and fixed ε_2 in comparison with the results of Kadalbajoo & Puneet [8]. Fig. 2 represents the solution profile at the prescribed values of *N* and *K*.

Example 3. $\varepsilon_1 \frac{\partial^2 y}{\partial x^2} + \varepsilon_2(1 + x(1-x) + t^2)\frac{\partial y}{\partial x} - \frac{\partial y}{\partial t} - (1 + 5xt)y = x(1-x)(e^t - 1)$

where $y(x,0) = 0$, for $x \in (0,1)$; $y(0,t) = y(1,t) = 0$, for $t \in [0,1]$.

The comparison of the maximum absolute errors is presented in Tables 4 for diverse values ε_1 and fixed ε_2 with the results of Das & Mehrmann [4]. The solution contour of this example is shown graphically in Fig. 3.

6. Discussions and Conclusion

We considered a singularly perturbed parabolic diffusion-convection problem with two miniature parameters. A temporal discretization using backward Euler's method and the spatial discretization on an unvarying mesh using the standard finite difference method with the help of two fitting factors led to a fully discrete problem. Convergence analysis of the proposed scheme had shown that the method is parameter uniform convergent. We have tested our method on the numerical examples and also compared with some existing methods available in literature. It is observed that the proposed method gives better results. We also observe that, the maximum absolute error in Table 1, Table 3 and Table 4 is constant as $\varepsilon_1$approaches to zero for fixed ε_2. It shows that the implemented method is ε_1-uniformly convergent. Similarly, the ε_2-uniform convergence is confirmed in Table 2 for fixed ε_1, h and τ.

Table 1. Comparison of maximum point wise error for $\varepsilon_2 = 2^{-2}$ and *N (=K)* in Example 1

$\varepsilon_1 \backslash N$	8	16	32	64	128	256	512
				Our results			
2^{-4}	2.88E-4	6.28E-4	5.33E-4	3.27E-4	1.79E-4	9.36E-5	4.78E-5
2^{-6}	2.18E-3	4.48E-4	1.83E-5	5.07E-5	4.10E-5	2.45E-5	1.33E-5
2^{-8}	2.80E-3	7.70E-4	1.84E-4	3.42E-5	1.50E-6	3.35E-6	2.71E-6
2^{-10}	3.01E-3	8.55E-4	2.24E-4	5.53E-5	1.25E-5	2.25E-6	1.01E-7
2^{-12}	3.07E-3	8.84E-4	2.36E-4	6.04E-5	1.49E-5	3.58E-6	7.97E-7
2^{-14}	3.09E-3	8.92E-4	2.40E-4	6.22E-5	1.57E-5	3.90E-6	9.52E-7
2^{-16}	3.09E-3	8.95E-4	2.41E-4	6.27E-5	1.59E-5	4.01E-6	9.99E-7
2^{-40}	3.09E-3	8.95E-4	2.41E-4	6.29E-5	1.60E-5	4.05E-6	1.02E-6
				Results by Munyakazi [12]			
2^{-4}	0.301E-1	0.116E-1	0.450E-2	0.190E-2	0.866E-3	0.412E-3	0.201E-3
2^{-6}	0.511E-1	0.195E-1	0.729E-2	0.291E-2	0.127E-2	0.589E-3	0.284E-3
2^{-8}	0.626E-1	0.321E-1	0.131E-1	0.491E-2	0.192E-2	0.819E-3	0.375E-3
2^{-10}	0.628E-1	0.343E-1	0.175E-1	0.820E-2	0.328E-2	0.124E-2	0.499E-3
2^{-12}	0.628E-1	0.343E-1	0.176E-1	0.885E-2	0.440E-2	0.204E-2	0.818E-3
2^{-14}	0.628E-1	0.343E-1	0.176E-1	0.885E-2	0.442E-2	0.220E-2	0.110E-2
2^{-40}	0.628E-1	0.343E-1	0.176E-1	0.885E-2	0.442E-2	0.220E-2	0.110E-2

Table 2. Comparison of point wise errors in Example 1 for $\varepsilon_1 = 2^{-5}$ and *N (=2K)*

$\varepsilon_2 \backslash N$	32	64	128	256	512	1024
				Our results		
2^{-4}	0.175E-2	0.108E-2	0.594E-3	0.311E-3	0.159E-3	0.804E-4
2^{-6}	0.179E-2	0.109E-2	0.598E-3	0.312E-3	0.159E-3	0.805E-4
2^{-8}	0.180E-2	0.109E-2	0.598E-3	0.312E-3	0.159E-3	0.805E-4
2^{-10}	0.180E-2	0.109E-2	0.599E-3	0.312E-3	0.159E-3	0.805E-4
2^{-40}	0.180E-2	0.109E-2	0.599E-3	0.312E-3	0.159E-3	0.805E-4
				Results by Munyakazi [12]		
2^{-4}	0.752E-2	0.329E-2	0.153E-2	0.735E-3	0.361E-3	0.178E-2
2^{-6}	0.744E-2	0.323E-2	0.149E-2	0.718E-3	0.351E-3	0.174E-3
2^{-8}	0.743E-2	0.323E-2	0.149E-2	0.716E-3	0.351E-3	0.173E-3
2^{-10}	0.743E-2	0.322E-2	0.149E-2	0.716E-3	0.350E-3	0.173E-3
2^{-40}	0.743E-2	0.322E-2	0.149E-2	0.716E-3	0.350E-3	0.173E-3

Table 3. Similitude of maximum point-wise errors for $\varepsilon_2 = 2^{-0}$ and N in Example 2

$\varepsilon_1 \downarrow$ $N \rightarrow$ / $\tau \rightarrow$	16 / 0.1	32 / 0.05	64 / 0.025	128 / 0.0125	256 / 0.00625
			Our Results		
2^{-4}	7.6407E-4	1.0512E-4	1.3620E-5	1.7282E-6	2.1745E-7
2^{-6}	7.5197E-4	1.0629E-4	1.3991E-5	1.7836E-6	2.2466E-7
2^{-8}	7.5023E-4	1.0546E-4	1.3927E-5	1.7890E-6	2.2634E-7
2^{-10}	7.5049E-4	1.0547E-4	1.3916E-5	1.7853E-6	2.2607E-7
2^{-12}	7.5056E-4	1.0548E-4	1.3916E-5	1.7853E-6	2.2602E-7
2^{-14}	7.5058E-4	1.0548E-4	1.3916E-5	1.7853E-6	2.2602E-7
2^{-16}	7.5058E-4	1.0548E-4	1.3916E-5	1.7853E-6	2.2602E-7
			Results by Kadalbajoo & Puneet [8]		
2^{-4}	3.1459E-3	8.7403E-4	2.2491E-4	5.6694E-5	1.4210E-5
2^{-6}	7.5329E-3	3.2399E-3	1.0098E-3	2.6972E-4	6.8631E-5
2^{-8}	7.8633E-3	4.3662E-3	2.2026E-3	8.7969E-4	2.6709E-4
2^{-10}	7.8633E-3	4.3700E-3	2.2954E-3	1.1741E-3	5.7065E-4
2^{-12}	7.8633E-3	4.3700E-3	2.2954E-3	1.1751E-3	5.9444E-4
2^{-14}	7.8633E-3	4.3700E-3	2.2954E-3	1.1751E-3	5.9444E-4
2^{-16}	7.8633E-3	4.3700E-3	2.2954E-3	1.1751E-3	5.9444E-4

Table 4. Similitude of maximum point wise errors for $\varepsilon_2 = 10^{-7}$ in Example 3

$N \rightarrow$ / $\varepsilon_1 \downarrow$ $\tau \rightarrow$	64 / 1/16	128 / 1/32	256 / 1/64	512 / 1/128
		Our Results		
10^{-6}	4.6780E-4	1.1982E-4	3.0257E-5	7.5982E-6
10^{-7}	4.6780E-4	1.1982E-4	3.0257E-5	7.5982E-6
10^{-8}	4.6780E-4	1.1982E-4	3.0257E-5	7.5982E-6
10^{-9}	4.6780E-4	1.1982E-4	3.0257E-5	7.5982E-6
		Results by Das & Mehrmann [4]		
10^{-6}	9.6949E-4	4.9906E-4	2.5231E-4	1.2824E-4
10^{-7}	9.8712E-4	5.0049E-4	2.5485E-4	1.2853E-4
10^{-8}	9.5128E-4	5.0026E-4	2.5237E-4	1.2781E-4
10^{-9}	9.6746E-4	5.0012E-4	2.5237E-4	1.2803E-4

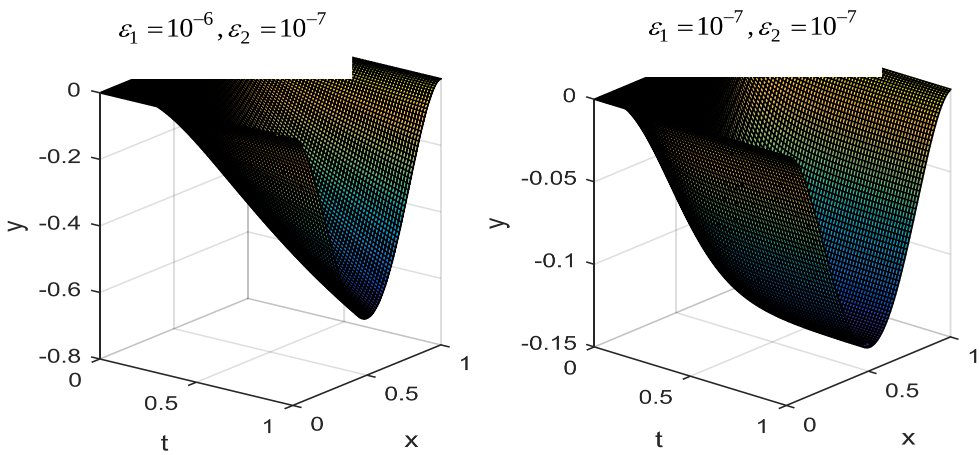

Fig 1. Solution profile for Example 1 with $K = 64, N = 128$ and various values of ε_1 and ε_2

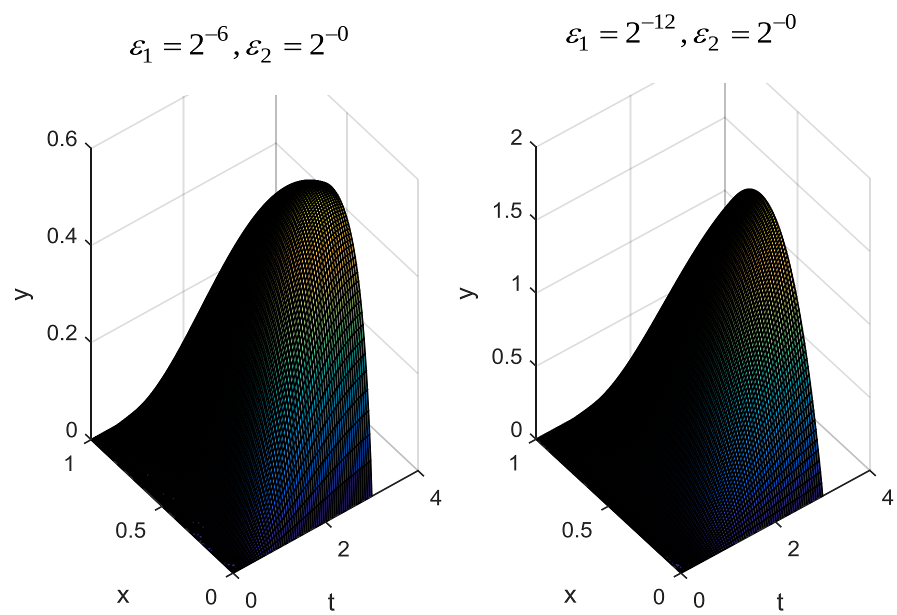

Fig 2. Solution profile for Example 2 with $K = 64, N = 128$ and diverse values of ε_1 and ε_2

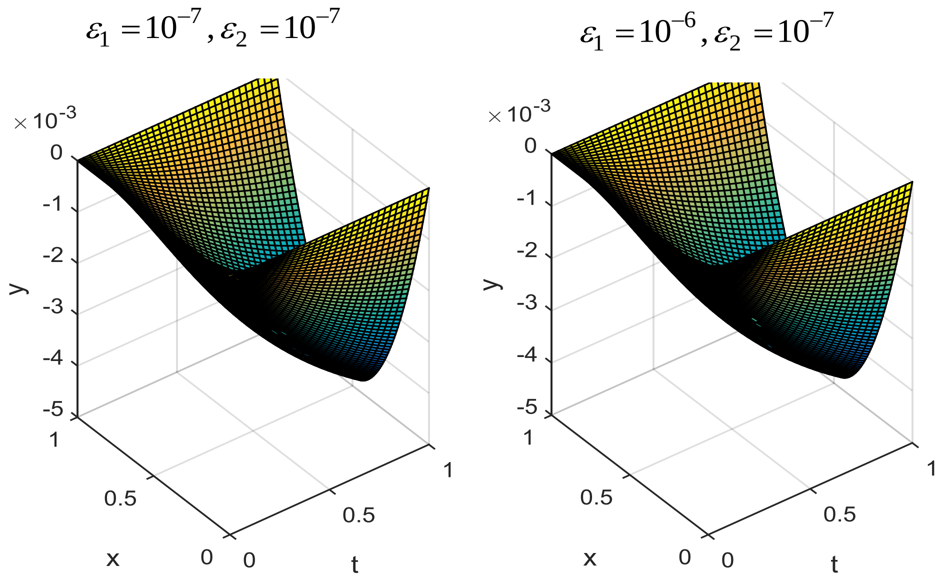

Fig 3. Solution profile for Example 3 with $K = 32, N = 128$ and diverse values of ε_1 and ε_2

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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