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SOME FIXED POINT THEOREMS IN M_A-METRIC SPACE

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Abstract. In this note, we introduce the concept of M_A-metric space as a generalisation of partial A-metric space. We also, prove some fixed point theorems satisfying fundamental contraction principles in the setting of M_A-metric space.

Keywords: A-metric space; partial A-metric space; M_A-metric space; fixed point.

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1. Introduction

The generalisations of metric fixed point have been an important research area for the last many years and many researchers had contributed a lot in this area. The results on generalization of metric space can be seen in the research papers [1–14] and references therein. These generalisations were then also used to extend the scope of the study of fixed point theory.

Mujahid Abbas, Bashir Ali and Yusuf I Suleiman [15] inroduced the concept of $n-$tuple metric space $A : X^n \to [0, \infty)$ and also generalised coupled common fixed point theorems for mixed weakly monotone maps in partially ordered $A-$ metric spaces.

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Using the concept of partially $A-$metric space, we extend fixed point results in M_A-metric space.

Definition 1.1. *[11] Let X be a nonempty set and $p : X \times X \longrightarrow [0,+\infty)$. We say that (X, p) is an ordinary partial metric space if for all $x,y,z \in X$ we have:*

(1) $x = y$ *if and only if* $p(x,y) = p(x,x) = p(y,y)$;

(2) $p(x,x) \leq p(x,y)$;

(3) $p(x,y) = p(y,x)$;

(4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X, p) is called partial metric space.

Definition 1.2. *[16] Let X be a nonempty set. A function $m : X \times X \to \mathbb{R}$ is called an m-metric space if the following conditions are satisfied:*

(m1) $m(x,x) = m(y,y) = m(x,y) \Leftrightarrow x = y$,

(m2) $m_{xy} \leq m(x,y)$,

(m3) $m(x,y) = m(y,x)$,

(m4) $(m(x,y) - m_{xy}) \leq (m(x,z) - m_{xz}) + (m(z,y) + m_{zy})$.

Then the pair (X, m) is called an M-metric space.

Definition 1.3. *[1] Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions,*

1. $S(x,y,z) \geq 0$,
2. $S(x,y,z) = 0$ *if and only if* $x = y = z$
3. $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$

for each $x,y,z,a \in X$.
The pair (X,S) is called S-metric space.

Definition 1.4. *[15] Let X be a nonempty set. A function $A : X^n \to [0,\infty)$ is called an A-metric on X if for any $x_i, a \in X, i = 1,2,...,n$, the following conditions hold:*

(A1) $A(x_1,x_2,x_3,...,x_{n-1},x_n) \geq 0$, (A2) $A(x_1,x_2,x_3,...,x_{n-1},x_n)=0$ *if and only if* $x_1=x_2=x_3=...=x_{n-1}=x_n$,

(A3)

$$\begin{aligned}A(x_1,x_2,x_3,...,x_{n-1},x_n) \leq\ & [A(x_1,x_1,x_1,...,(x_1)_{n-1},a)\\ & +A(x_2,x_2,x_2,...,(x_2)_{n-1},a)\\ & +A(x_3,x_3,x_3,...,(x_3)_{n-1},a)\\ \vdots\ & \\ & +A(x_{n-1},x_{n-1},x_{n-1},...,(x_{n-1})_{n-1},a)\\ & +A(x_n,x_n,x_n,...,(x_n)_{n-1},a).]\end{aligned}$$

The pair (X,A) is called an A-metric space.

Definition 1.5. *[15] Let X be a nonempty set. A partial A-metric space is a function $A_P : X^n \to [0,\infty)$ that satisfies the following conditions, for all $x_1,x_2,\dots,x_n,t \in X$;*

(i) $A_P(x_1,x_2,\dots,x_n) \geq 0$,

(ii) $x_1 = x_2 = \cdots = x_n$ *if and only if* $A_P(x_1,x_1,\dots,x_1) = A_P(x_2,x_2,\dots,x_2) = \cdots = A_P(x_n,x_n,\dots,x_n)$,

(iii)

$$\begin{aligned}A_P(x_1,x_2,\dots,x_n) \leq\ & A_P(x_1,x_1,\dots,(x_1)_{n-1},t)+A_P(x_2,x_2,\dots,(x_2)_{n-1},t)\\ & +\cdots+A_P(x_n,x_n,\dots,(x_n)_{n-1},t)-A_P(t,t,\dots,t),\end{aligned}$$

(iv) $A_P(x_1,x_1,\dots,x_1) \leq A_P(x_1,x_2,\dots,x_n)$,

(v) $A_P(x_1,x_1,\dots,x_1,x_2) = A_P(x_2,x_2,\dots,x_2,x_1)$.

The pair (X,A_P) is called a partial A-metric space.

Definition 1.6. *[15] Let X be a nonempty set. A partial A-metric on X is a function $A_p : X^n \to [0,\infty)$ that satisfies the following conditions for all $x_1,x_2,\dots,x_n,t \in X$,*

(i) $x_1 = x_2$ *if and only if* $A_p(x_1,x_1,\dots,x_1) = A_p(x_2,x_2,\dots,x_2) = A_p(x_1,x_1,\dots,x_1,x_2)$.

(ii)

$$A_p(x_1,x_2,\dots,x_n) \leq A_p(x_1,x_1,\dots,x_1,t)+A_p(x_2,x_2,\dots,x_2,t) + \cdots + A_p(x_n,x_n,\dots,x_n,t)+A_p(t,t,\dots,t).$$

(iii) $A_p(x_1,x_1,\dots,x_1) \leq A_p(x_1,x_2,\dots,x_n)$.

(iv) $A_p(x_1,x_1,\dots,x_1,x_2) = A_p(x_2,x_2,\dots,x_2,x_1)$.

The pair (X,A_p) is called a partial A-metric space.

Next, we give the definition of an M_A-metric space, but first we introduce the following notations.

Notation 1.

1. $m_{A_{x_1,x_2,\dots,x_n}} := \min\{m_A(x_1,x_1,\dots,x_1), m_A(x_2,x_2,\dots,x_2),\dots,m_A(x_n,x_n,\dots,x_n)\}$.
2. $M_{A_{x_1,x_2,\dots,x_n}} := \max\{m_A(x_1,x_1,\dots,x_1), m_A(x_2,x_2,\dots,x_2),\dots,m_A(x_n,x_n,\dots,x_n)\}$.

Definition 1.7. *An M_A-metric on a nonempty set X is a function $m_A : X^n \to \mathbb{R}^+$ such that for all $x_1,x_2,\dots,x_n,t \in X$, the following conditions are satisfied:*

1. $m_A(x_1,x_1,\dots,x_1) = m_A(x_2,x_2,\dots,x_2) = m_A(x_1,x_1,\dots,x_1,x_2)$ *if and only if* $x_1 = x_2$.
2. $m_{A_{x_1,x_2,\dots,x_n}} \leq m_A(x_1,x_2,\dots,x_n)$.
3. $m_A(x_1,x_1,\dots,x_1,x_2) = m_A(x_2,x_2,\dots,x_2,x_1)$.
4.

$$\left(m_A(x_1,x_2,\dots,x_n) - m_{A_{x_1,x_2,\dots,x_n}}\right) \leq \left(m_A(x_1,x_1,\dots,x_1,t) - m_{A_{x_1,x_1,\dots,x_1,t}}\right) + \left(m_A(x_2,x_2,\dots,x_2,t) - m_{A_{x_2,x_2,\dots,x_2,t}}\right) + \dots + \left(m_A(x_n,x_n,\dots,x_n,t) - m_{A_{x_n,x_n,\dots,x_n,t}}\right)$$

The pair (X, m_A) is called an M_A-metric space. Notice that the condition $m_A(x_1, x_1, \ldots, x_1) = m_A(x_2, x_2, \ldots, x_2) = \cdots = m_A(x_n, x_n, \ldots, x_n = m_A(x_1, x_2, \ldots, x_n) \Leftrightarrow x_1 = x_2 = \cdots = x_n$ implies that (1) above.

It is straightforward to verify that every partial A-metric space is an M_A-metric space but the converse is not true. The following example is an M_A-metric which is not a partial A-metric space.

Example 1. *Let $X = \{1, 2, \ldots, n\}$ and define*

Definition 1.8. *Let (X, m_A) be an M_A-metric space. Then*

1. *a sequence $\{x_p\}$ in X converges to a point x if and only if $\lim_{p\to\infty} \left(m_A(x_p, x_p, \ldots, x_p, x) - m_{A_{x_p, x_p, \ldots, x_p, x}}\right) = 0$.*
2. *a sequence $\{x_p\}$ in X is said to be M_A-Cauchy sequence if and only if*

$$\lim_{p,q\to\infty} \left(m_A(x_p, x_p, \ldots, x_p, x_q) - m_{A_{x_p, x_p, \ldots, x_p, x_q}}\right)$$

and

$$\lim_{p,q\to\infty} \left(M_{A_{x_p, x_p, \ldots, x_p, x_q}} - m_{A_{x_p, x_p, \ldots, x_p, x_q}}\right)$$

exists and finite.

3. *an M_A-metric space is said to be complete if every M_A-Cauchy sequence $\{x_p\}$ converges to a point x such that*

$$\lim_{p\to\infty} \left(m_A(x_p, x_p, \ldots, x_p, x) - m_{A_{x_p, x_p, \ldots, x_p, x}}\right) = 0$$

and

$$\lim_{p\to\infty} \left(M_{A_{x_p, x_p, \ldots, x_p, x}} - m_{A_{x_p, x_p, \ldots, x_p, x}}\right) = 0.$$

A ball in the M_A-metric (X, m_A) space with centre $x \in X$ and radius $\eta > 0$ is defined by

$$B_A[x, \eta] = \{x_2 \in X : m_A(x_1, x_1, \ldots, x_1, x_2) - m_{A_{x_1, x_1, \ldots, x_1, x_2}}\} \leq \eta.$$

The topology of (X, M_A) is generated by means of the basis $\beta = \{B_A[x, \eta] : \eta > 0\}$.

Lemma 1.1. *Assume $x_p \to x$ and $y_p \to y$ as $p \to \infty$ in an M_A-matric space (X, m_A). Then,*

$$\lim_{p\to\infty} \left(m_A(x_p, x_p, \ldots, x_p, y_p) - m_{A_{x_p, x_p, \ldots, x_p, y_p}}\right) = m_A(x, x, \ldots, x, y) - m_{A_{x, x, \ldots, x, y}}.$$

Proof. The proof follows by the inequality (4) in definition (1.7). Indeed, we have

$$\begin{aligned} & |(m_A(x_p,x_p,\ldots,x_p,y_p) - m_{A_{x_p,x_p,\ldots,x_p,y_p}}) - (m_A(x,x,\ldots,x,y) - m_{A_{x,x,\ldots,x,y}})| \\ \leq \; & (n-1)|(m_A(x_p,x_p,\ldots,x_p,x) - m_{A_{x_p,x_p,\ldots,x_p,x}}) + (m_A(y_p,y_p,\ldots,y_p,y) - m_{A_{y_p,y_p,\ldots,y_p,y}})| \end{aligned}$$

□

2. Main Results

In this section, we consider some results about the existence and uniqueness of fixed point for self-mappings on an M_A-metric space, under different contraction principles.

Theorem 2.1. *Let (X, m_A) be a complete M_A-metric space and T be a self-mapping on X satisfying the following condition:*

$$(1) \qquad m_A(Tx,Tx,\ldots,Tx,Ty) \leq km_A(x,x,\ldots,x,y)$$

for all $x,y \in X$, where $k \in [0,1)$. Then T has a unique fixed point u. Moreover, $m_A(u,u,\ldots,u) = 0$.

Proof. Since $k \in [0,1)$, we can choose a natural number n_0 such that for a given $0 < \varepsilon < 1$, we have $k^{n_0} < \frac{\varepsilon}{4(n-1)}$. Let $T^{n_0} = F$ and $F^i x_0 = x_i$ for all natural number i, where x_0 is arbitrary. Hence, for all $x,y \in X$, we have

$$\begin{aligned} m_A(Fx,\ldots,Fx,Fy) &= m_A(T^{n_0}x,\ldots,T^{n_0}x,T^{n_0 y}) \\ &\leq k^{n_0} m_A(x,x,\ldots,x,y) \end{aligned}$$

For any i, we have

$$\begin{aligned} m_A(x_{i+1},\ldots,x_{i+1},x_i) &= m_A(Fx_i,\ldots,Fx_i,Fx_{i-1}) \\ &\leq k^{n_0} m_A(x_i,\ldots,x_i,x_{i-1}) \\ &\leq k^{n_0+i} m_A(x_1,\ldots,x_1,x_0) \to 0 \text{ as } i \to \infty. \end{aligned}$$

Similarly, by (1) we have $m_A(x_i,\ldots,x_i,x_i) \to 0$ as $i \to \infty$. Thus, we choose l such that

$$m_A(x_{l+1},\ldots,x_{l+1},x_l) < \frac{\varepsilon}{4(n-1)}$$

and

$$m_A(x_l,\ldots,x_l,x_l) < \frac{\varepsilon}{2(n-1)}.$$

Now, let $\eta = \frac{\varepsilon}{2} + m_A(x_l,\ldots,x_l,x_l)$. Define the set

$$B_A[x_l,\eta] = \{y \in X | m_A(x_l,\ldots,x_l,y) - m_{A_{x_l,x_l,\ldots,x_l},y\leq \eta}\}.$$

Note that, $x_1 \in B_A[x_l,\eta]$. Therefore $B_A[x_l,\eta] \neq \phi$. Let $z \in B_A[x_l,\eta]$ be arbitrary. Hence,

$$\begin{aligned}
m_A(Fz,\ldots,Fz,Fx_l) &\leq k^{n_0} m_A(z,\ldots,z,x_l) \\
&\leq k^{n_0}[(n-1)\{m_A(z,z,\ldots,z) - m_{A_{z,z,\ldots,z}}\} \\
&\quad + m_A(x_l,x_l,\ldots,x_l) - m_{A_{x_l,x_l,\ldots,x_l}} + m_{A_{z,z,\ldots,z,x_l}}] \\
&\leq \frac{\varepsilon}{4(n-1)}[(n-1)\frac{\varepsilon}{2(n-1)} + m_{A_{z,z,\ldots,z,x_l}} + m_A(x_l,x_l,\ldots,x_l)] \\
&\leq \frac{\varepsilon}{4(n-1)}[\frac{\varepsilon}{2} + m_{A_{z,z,\ldots,z,x_l}} + m_A(x_l,x_l,\ldots,x_l)] \\
&\leq \frac{\varepsilon}{4(n-1)}[1 + 2m_A(x_l,x_l,\ldots,x_l)].
\end{aligned}$$

Also, we know that

$$m_A(Fx_l,Fx_l,\ldots,Fx_l,x_l) = m_A(x_{l+1},x_{l+1},\ldots,x_{l+1},x_l) < \frac{\varepsilon}{4(n-1)}.$$

Therefore,

$$\begin{aligned}
m_A(Fz,Fz,\ldots,Fz,x_l) - m_{A_{Fz,\ldots,Fz,x_l}} &\leq (n-1)[m_A(Fz,Fz,\ldots,Fx_l) - m_{A_{Fz,\ldots,Fz,Fx_l}}] \\
&\quad + m_A(Fx_l,\ldots,Fx_l,x_l) - m_{A_{Fx_l\ldots,Fx_l,x_l}} \\
&\leq (n-1)m_A(Fz,Fz,\ldots,Fx_l) + m_{A_{Fx_l\ldots,Fx_l,x_l}} \\
&\leq (n-1)\frac{\varepsilon}{4(n-1)}[1 + 2m_A(x_l,x_l,\ldots,x_l)] + \frac{\varepsilon}{4(n-1)} \\
&= \frac{\varepsilon}{4} + \frac{\varepsilon}{4(n-1)} + \frac{\varepsilon}{2}m_A(x_l,x_l,\ldots,x_l) \\
&= \frac{n\varepsilon}{4(n-1)} + \frac{\varepsilon}{2}m_A(x_l,x_l,\ldots,x_l) \\
&< \frac{\varepsilon}{2} + m_A(x_l,x_l,\ldots,x_l).
\end{aligned}$$

Thus, $Fz \in B_b[x_l,\eta]$ which implies that F maps $B_b[x_l,\eta]$ into itself. Thus by repeating the process we deduce that for all $n \geq 1$, we have $F^n x_l \in B_b[x_l,\eta]$ and that is $x_m \in B_b[x_l,\eta]$ for all $m \geq l$. Therefore, for all $m > n \geq l$ where $n = l+i$ for some i.

$$\begin{aligned}
m_A(x_n,\ldots,x_n,x_m) &= m_A(Fx_{n-1},\ldots,Fx_{n-1},Fx_{m-1}) \\
&\leq k^{n_0} m_A(x_{n-1},\ldots,x_{n-1},x_{m-1}) \\
&\leq k^{2n_0} m_A(x_{n-2},\ldots,x_{n-2},x_{m-2}) \\
&\vdots \\
&\leq k^{in_0} m_A(x_l,\ldots,x_l,x_{m-i}) \\
&\leq m_A(x_l,\ldots,x_l,x_{m-i}) \\
&\leq \frac{\varepsilon}{2} + m_{A_{x_l,\ldots,x_l,x_{m-i}}} + m_A(x_l,\ldots,x_l,x_l) \\
&\leq \frac{\varepsilon}{2} + 2m_A(x_l,\ldots,x_l,x_l)
\end{aligned}$$

Also, we have $m_A(x_l,\ldots,x_l,x_l) < \frac{\varepsilon}{4}$, which implies that $m_A(x_n,\ldots,x_n,x_m) < \varepsilon$ for all $m > n > l$, and thus $m_A(x_n,\ldots,x_n,x_m) - m_{A_{x_n,\ldots,x_n,x_m}} < \varepsilon$ for all $m > n > l$. By the contraction condition (1), we see that the sequence $\{m_A(x_n,\ldots,x_n,x_l)\}$ is decreasing and hence, for all $m > n > l$, we have

$$\begin{aligned}
M_{A_{x_n,\ldots,x_n,x_m}} - m_{A_{x_n,\ldots,x_n,x_m}} &\leq M_{A_{x_n,\ldots,x_n,x_m}} \\
&= m_A(x_n,\ldots,x_n,x_n) \\
&\leq k m_A(x_{n-1},x_{n-1},\ldots,x_{n-1}) \\
&\vdots \\
&\leq k^n m_A(x_0,x_0,\ldots,x_0) \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus, we deduce that

$$\lim_{n,m\to\infty} [m_A(x_n,\ldots,x_n,x_m) - m_{A_{x_n,\ldots,x_n,x_m}}] = 0$$

and

$$\lim_{n,m\to\infty} [M_{A_{x_n,\ldots,x_n,x_m}} - m_{A_{x_n,\ldots,x_n,x_m}}] = 0.$$

Hence, the sequence $\{x_n\}$ is an M_A-Cauchy. Since X is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty}[m_A(x_n,\dots,x_n,u) - m_{A_{x_n,\dots,x_n,u}}] = 0$$

and

$$\lim_{n\to\infty}[M_A(x_n,\dots,x_n,u) - m_{A_{x_n,\dots,x_n,u}}] = 0.$$

The contraction condition (1) implies that $m_A(x_n,x_n,\dots,x_n) \to 0$ as $n \to \infty$. Moreover, notice that

$$\lim_{n\to\infty}[M_A(x_n,\dots,x_n,u) - m_{A_{x_n,\dots,x_n,u}}] \;=\; \lim_{n\to\infty}|m_A(x_n,x_n,\dots,x_n) - m_A(u,u\dots,u)| = 0,$$

and hence $m_A(u,u\dots,u) = 0$. Since $x_n \to u$, $m_A(u,u\dots,u) = 0$ and $m_A(x_n,x_n,\dots,x_n) \to 0$ as $n \to \infty$, then

$$\lim_{n\to\infty} m_A(x_n,\dots,x_n,u) = \lim_{n\to\infty} m_{A_{x_n,\dots,x_n,u}} = 0.$$

Since $m_A(Tx_n,\dots,Tx_n,Tu) \le k m_A(x_n,\dots,x_n,u) \to 0$ as $n \to \infty$, then $Tx_n \to Tu$.

Now, we show that $Tu = u$. By Lemma (1.1) and that $Tx_n \to Tu$ and $x_{n+1} = Tx_n \to u$, we have

$$\begin{aligned}
\lim_{n\to\infty} m_A(x_n,\dots,x_n,u) &= m_{A_{x_n,\dots,x_n,u}} = 0\\
&= \lim_{n\to\infty} m_A(x_{n+1},\dots,x_{n+1},u) - m_{A_{x_{n+1},\dots,x_{n+1},u}}\\
&= \lim_{n\to\infty} m_A(Tx_n,\dots,Tx_n,u) - m_{A_{Tx_n,\dots,Tx_n,u}}\\
&= m_A(u,\dots,u,u) - m_{A_{Tu,\dots,Tu,u}}\\
&= m_A(Tu,\dots,Tu,u) - m_{A_{Tu,\dots,Tu,u}}
\end{aligned}$$

Hence, $m_A(Tu,\dots,Tu,u) = m_{A_{Tu,\dots,Tu,u}} = m_A(u,u,\dots,u)$, but also by the contraction condition (1) we see that $m_{A_{Tu,\dots,Tu,u}} = m_A(Tu,Tu,\dots,Tu)$. Therefore, (2) in definition (1.7) implies that $Tu = u$.

To prove the uniqueness of the fixed point u, assume that T has two fixed points $u,v \in X$; that is $Tu = u$ and $Tv = v$. Thus,

$$m_A(u,\dots,u,v) = m_A(Tu,\dots,Tu,Tv) \le km_A(u,\dots,u,v) < m_A(u,u,\dots,u,v),$$

$$m_A(u,\dots,u,u) = m_A(Tu,Tu,\dots,Tu) \le km_A(u,\dots,u,u) < m_A(u,u,\dots,u,u),$$

and

$$m_A(v,v,\dots,v) = m_A(Tv,Tv,\dots,Tv) \le km_A(v,v,\dots,v) < m_A(v,v,\dots,v),$$

which implies that $m_A(u,u,\dots,u,v) = 0 = m_A(u,u,\dots,u) = m_A(v,v,\dots,v)$, and hence $u = v$ as disered. Finally, assume that u is a fixed point of T. Then applying the contraction condition (1) with $k \in [0,1)$, implies that

$$\begin{aligned} m_A(u,u,\dots,u) &= m_A(Tu,Tu,\dots,Tu) \\ &\le km_A(u,u,\dots,u) \\ &\vdots \\ &\le k^n m_A(u,u,\dots,u). \end{aligned}$$

Taking the limit as $n \to \infty$, implies that $m_A(u,u,\dots,u) = 0$.

In the following result, we prove the existence and uniqueness of a fixed point for a self-mapping in M_A-metric space, but under a more general contraction. □

Theorem 2.2. *Let (X,m_A) be a complete M_A-metric space and T be a self-mapping on X satisfying the following condition*

$$m_A(Tx,\dots,Tx,Ty) \le \lambda[m_A(x,\dots,x,Tx) + m_A(y,\dots,y,Ty)] \tag{2}$$

for all $x,y \in X$, where $\lambda \in [0,\frac{1}{2})$. Then T has a unique fixed point u, where $m_A(u,u,\dots,u) = 0$.

Proof. Let $x_0 \in X$ be arbitrary. Consider the sequence $\{x_n\}$ defined by $x_n = T^n x_0$ and $m_{A_n} = m_A(x_n,\dots,x_n,x_{n+1})$. Note that if there exists a natural number n such that $m_{A_n} = 0$, then x_n is a fixed point of T and we are done. So, we may assume that $m_A > 0$ for $n \geq 0$. By (2), we obtain for any $n \geq 0$,

$$\begin{aligned}
m_{A_n} &= m_A(x_n, \ldots, x_n, x_{n+1}) \\
&= m_A(Tx_{n-1}, \ldots, Tx_{n-1}, Tx_n) \\
&\leq \lambda[m_A(x_{n-1}, \ldots, x_{n-1}, Tx_{n-1}) + m_A(x_n, \ldots, x_n, Tx_n)] \\
&= \lambda[m_A(x_{n-1}, \ldots, x_{n-1}, x_n) + m_A(x_n, \ldots, x_n, x_{n-1})] \\
&= \lambda[m_{A_{n-1}} + m_{A_n}] \\
\Rightarrow m_{A_n} &\leq \lambda m_{A_{n-1}} + \lambda m_{A_n} \\
\Rightarrow m_{A_n} &\leq \mu m_{A_{n-1}}
\end{aligned}$$

where $\mu = \frac{\lambda}{1-\lambda} < 1$ as $\lambda \in [0, \frac{1}{2})$.
By repeating this process, we get

$$m_{A_n} \leq \mu^n m_{A_0}.$$

Thus, $\lim_{n\to\infty} m_{A_n} = 0$. By (2), for all natural number n, m, we have

$$\begin{aligned}
m_A(x_n, \ldots, x_n, x_m) &= m_A(T^n x_0, \ldots, T^n x_0, T^m x_0) \\
&= m_A(Tx_{n-1}, \ldots, Tx_{n-1}, Tx_{m-1}) \\
&\leq \lambda[m_A(x_{n-1}, \ldots, x_{n-1}, Tx_{n-1}) + m_A(x_{m-1}, \ldots, x_{m-1}, Tx_{m-1})] \\
&= \lambda[m_A(x_{n-1}, \ldots, x_{n-1}, x_n) + m_A(x_{m-1}, \ldots, x_{m-1}, x_m)] \\
&\leq \lambda[m_{A_{n-1}} + m_{A_{m-1}}].
\end{aligned}$$

Since $\lim_{n\to\infty} m_{A_n} = 0$, for every $\varepsilon > 0$, we can find a natural number n_0 such that $m_{A_n} < \frac{\varepsilon}{2}$ and $m_{A_m} < \frac{\varepsilon}{2}$ for all $m, n > n_0$. Therefore, it follows that

$$\begin{aligned}
m_A(x_n, \ldots, x_n, x_m) &\leq \lambda[m_{A_{n-1}} + m_{A_{m-1}}] \\
&< \lambda[\frac{\varepsilon}{2} + \frac{\varepsilon}{2}] \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = 0 \text{ for all } n, m > n_0.
\end{aligned}$$

This implies that

$$m_A(x_n,\ldots,x_n,x_m) - m_{A_{x_n,\ldots,x_n,x_m}} < \varepsilon$$

for all $n,m > n_0$.

Now, for all natural numbers n,m, we have

$$\begin{aligned}
M_{A_{x_n,\ldots,x_n,x_m}} &= m_A(Tx_{n-1},\ldots,Tx_{n-1},Tx_{n-1}) \\
&\leq \lambda[m_A(x_{n-1},\ldots,x_{n-1},Tx_{n-1}) + m_A(x_{n-1},\ldots,x_{n-1},Tx_{n-1})] \\
&= \lambda[m_A(x_{n-1},\ldots,x_{n-1},x_n) + m_A(x_{n-1},\ldots,x_{n-1},x_n)] \\
&= \lambda[m_{A_{n-1}} + m_{A_{n-1}}] \\
&= 2\lambda m_{A_{n-1}}.
\end{aligned}$$

As $\lim_{n\to\infty} m_{A_{n-1}} = 0$, for every $\varepsilon > 0$ we can find a natural number n_0 such that $m_{A_n} < \frac{\varepsilon}{2}$ and for all $m,n > n_0$. Therefore, it follows that

$$\begin{aligned}
M_{A_{x_n,\ldots,x_n,x_m}} &\leq \lambda[m_{A_{n-1}} + m_{A_{n-1}}] \\
&< \lambda[\frac{\varepsilon}{2} + \frac{\varepsilon}{2}] \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = 0 \text{ for all } n,m > n_0,
\end{aligned}$$

which implies that

$$M_{A_{x_n,\ldots,x_n,x_m}} - m_{A_{x_n,\ldots,x_n,x_m}} < \varepsilon \text{ for all } n,m > n_0.$$

Thus, $\{x_n\}$ is an M_A-Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} m_A(x_n,\ldots,x_n,u) - m_{A_{x_n,\ldots,x_n,u}} = 0.$$

Now, we show that u is a fixed point of T in X. For any natural number n, we have,

$$\begin{aligned}
&\lim_{n\to\infty} m_A(x_n,\ldots,x_n,u) - m_{A_{x_n,\ldots,x_n,u}} = 0 \\
&= \lim_{n\to\infty} m_A(x_{n+1},\ldots,x_{n+1},u) - m_{A_{x_{n+1},\ldots,x_{n+1},u}} \\
&= \lim_{n\to\infty} m_A(Tx_n,\ldots,Tx_n,u) - m_{A_{Tx_n,\ldots,Tx_n,u}} \\
&= m_A(Tu,\ldots,Tu,u) - m_{A_{Tu,\ldots,Tu,u}}.
\end{aligned}$$

This implies that $m_A(Tu,\ldots,Tu,u) = m_{A_{u,\ldots,u,Tu}} = 0$, and that is $m_A(Tu,\ldots,Tu,u) = m_{A_{u,\ldots,u,Tu}}$. Now, assume that

$$\begin{aligned} m_A(Tu,\ldots,Tu,u) &= m_A(Tu,\ldots,Tu,Tu) \\ &\leq 2\lambda m_A(u,\ldots,u,Tu) \\ &= 2\lambda m_A(Tu,\ldots,Tu,u) \\ &< m_A(u,\ldots,u,Tu) \end{aligned}$$

Thus,

$$\begin{aligned} m_A(Tu,\ldots,Tu,u) &= m_A(u,\ldots,u,u) \\ &\leq m_A(Tu,\ldots,Tu,Tu) \\ &\leq 2\lambda m_A(u,\ldots,u,Tu) \\ &< m_A(u,\ldots,u,Tu) \end{aligned}$$

Therefore, $Tu = u$ and thus u is a fixed point of T.

Next, we show that if u is a fixed point, then $m_A(u,\ldots,u,u) = 0$. Assume that u is a fixed point of T, then using the contraction (2), we have

$$\begin{aligned} m_A(u,u,\ldots,u) &= m_A(Tu,\ldots,Tu,Tu) \\ &\leq \lambda[m_A(u,u,\ldots,u,Tu) + m_A(u,u,\ldots,u,Tu)] \\ &= 2\lambda m_A(u,u,\ldots,u,Tu) \\ &= 2\lambda m_A(u,u,\ldots,u) \\ &< m_A(u,u,\ldots,u) \text{ since } \lambda \in [0,\frac{1}{2}), \end{aligned}$$

that is, $m_A(u,u,\ldots,u) = 0$.

Finally, to prove the uniqueness, assume that T has two fixed points, say $u, v \in X$. Hence,

$$\begin{aligned} m_A(u,\dots,u,v) &= m_A(Tu,\dots,Tu,Tv) \\ &\leq \lambda[m_A(u,u,\dots,u,Tu)+m_A(v,v,\dots,v,Tv)] \\ &= \lambda[m_A(u,u,\dots,u)+m_A(v,v,\dots,v)] = 0, \end{aligned}$$

which implies that

$$m_A(u,\dots,u,v) = 0 = m_A(u,u,\dots,u) = m_A(v,v,\dots,v),$$

and $u = v$ as required. □

In closing, the authors would like to bring to the reader's attention that in this interesting M_A-metric space, it is possible to add some control functions in both contractions of Theorems 1 and 2.

Theorem 2.3. *Let (X, m_A) be a complete M_A-metric space and T be a self-mapping on X satisfying the following condition: for all $x_1, x_2, \dots, x_n \in X$*

$$m_A(Tx_1,Tx_2,\dots,Tx_n) \leq m_A(x_1,x_2,\dots,x_n) - \phi(m_A(x_1,x_2,\dots,x_n)), \tag{3}$$

where $\phi : [0,\infty) \to [0,\infty)$ is a continuous and non-decreasing function and $\phi^{-1}(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}$ in X such that $x_n = T^{n-1}x_0 = Tx_{n-1}$ for all $n \in \mathbb{N}$. Note that if there exists an $n \in \mathbb{N}$ such that $x_{n+1} = x_n$, then x_n is a fixed point for T. Without loss of generality, assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Now

$$\begin{aligned} m_A(x_n,x_{n+1},\dots,x_{n+1}) &= m_A(Tx_{n-1},Tx_n,\dots,Tx_n) \\ &\leq m_A(x_{n-1},x_n,\dots,x_n) - \phi(m_A(x_{n-1},x_n,\dots,x_n)) \\ &\leq m_A(x_{n-1},x_n,\dots,x_n) \qquad (4) \end{aligned}$$

Similarly, we can prove that $m_A(x_{n-1},x_n,\dots,x_n) \leq m_A(x_{n-2},x_{n-1},\dots,x_{n-1})$. Hence, $m_A(x_n,x_{n+1},\dots,x_{n+1})$ is a nondecreasing sequence. Hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} m_A(x_n,x_{n+1},\dots,x_{n+1}) = r.$$

Now, by taking the limit as $n \to \infty$ in the inequality (4), we get $r \leq r - \phi(r)$ which leads to a contraction unless $r = 0$. Therefore,

$$\lim_{n\to\infty} m_A(x_n, x_{n+1}, \ldots, x_{n+1}) = 0.$$

Suppose that $\{x_n\}$ is not an M_A-Cauchy sequence. Then there exists an $\varepsilon > 0$ such that we can find subsequences x_{m_k} and x_{n_k} of $\{x_n\}$ such that

$$m_A(x_{n_k}, x_{m_k}, \ldots, x_{m_k}) - m_{A_{x_{n_k}, x_{m_k}, \ldots, x_{m_k}}} \quad \geq \quad \varepsilon \tag{5}$$

Choose n_k to be the smallest integer with $n_k > m_k$ and satisfies the inequality (5). Hence,

$$m_A(x_{n_k}, x_{m_{k-1}}, \ldots, x_{m_{k-1}}) - m_{A_{x_{n_k}, x_{m_{k-1}}, \ldots, x_{m_{k-1}}}} \quad < \quad \varepsilon.$$

Now,

$$\begin{aligned}
\varepsilon &\leq m_A(x_{m_k}, x_{n_k}, \ldots, x_{n_k}) - m_{A_{x_{m_k}, x_{n_k}, \ldots, x_{n_k}}} \\
&\leq m_A(x_{m_k}, x_{n_k-1}, \ldots, x_{n_k-1}) + (n-1) m_A(x_{n_k-1}, \ldots, x_{n_k-1}) - m_{A_{x_{m_k}, x_{n_k-1}, \ldots, x_{n_k-1}}} \\
&\geq \varepsilon + (n-1) m_A(x_{n_k-1}, \ldots, x_{n_k-1}) \\
&< \varepsilon,
\end{aligned}$$

as $n \to \infty$. Hence, we have contradiction. Without loss of generality, assume that $m_{A_{x_n, x_n, \ldots, x_n, x_m}} = m_A(x_n, x_n, \ldots, x_n, x_m)$. Then we have

$$\begin{aligned}
0 &\leq m_{A_{x_n, x_n, \ldots, x_n, x_m}} - m_A(x_n, x_n, \ldots, x_n, x_m) \\
&\leq M_{A_{x_n, x_n, \ldots, x_n, x_m}} \\
&= m_A(x_n, x_n, \ldots, x_n, x_m) \\
&= m_A(Tx_{n-1}, Tx_{n-1}, \ldots, Tx_{n-1}) \\
&\leq m_A(x_{n-1}, x_{n-1}, \ldots, x_{n-1}) - \phi(m_A(x_{n-1}, x_{n-1}, \ldots, x_{n-1})) \\
&\leq m_A(x_{n-1}, x_{n-1}, \ldots, x_{n-1}) \\
&\vdots \\
&\leq m_A(x_0, x_0, \ldots, x_0).
\end{aligned}$$

Hence, $\lim_{n\to\infty} m_{A_{x_n,x_n,\dots,x_n,x_m}} - m_{A_{x_n,x_n,\dots,x_n,x_m}}$ exists and finite. Therefore, $\{x_n\}$ is an M_A-Cauchy sequence. Since X is a complete, the sequence $\{x_n\}$ converges to an element $x \in X$; that is,

$$\begin{aligned} 0 &= \lim_{n\to\infty} m_A(x_n,x_n,\dots,x_n,x) - m_{A_{x_n,x_n,\dots,x_n,x}} \\ &= \lim_{n\to\infty} m_A(x_{n+1},x_{n+1},\dots,x_{n+1},x) - m_{A_{x_{n+1},x_{n+1},\dots,x_{n+1},x}} \\ &= \lim_{n\to\infty} m_A(Tx_n,Tx_n,\dots,Tx_n,x) - m_{A_{Tx_n,Tx_n,\dots,Tx_n,x}} \\ &= m_A(Tx,Tx,\dots,Tx,x) - m_{A_{Tx,Tx,\dots,Tx,x}} \end{aligned}$$

Similar to the proof of the Theorem 2, it is not difficult to show that this implies that, $Tx = x$ and so x is a fixed point.

Finally, we show that T has a unique fixed point. Assume that there are two fixed points $u, v \in X$ of T. If we have $m_A(u,u,\dots,u,v) > 0$, then condition (3) implies that

$$\begin{aligned} m_A(u,u,\dots,u,v) - m_A(Tu,Tu,\dots,Tu,v) &\le m_A(u,u,\dots,u,v) - \phi(m_A(u,u,\dots,u,v)) \\ &< m_A(u,u,\dots,u,v) \end{aligned}$$

and that is a contradiction. Therefore, $m_A(u,u,\dots,u,v) = 0$ and similarly $m_A(u,u,\dots,u) = M_A(v,v,\dots,v) = 0$ and thus $u = v$ as desired. □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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