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COMMON FIXED POINTS OF A PAIR OF ALMOST GERAGHTY-SUZUKI CONTRACTION TYPE MAPS IN b-METRIC SPACES

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Abstract. In this paper, we introduce almost Geraghty-Suzuki contraction type (I) maps, almost Geraghty-Suzuki contraction type (II) maps, for a pair of selfmaps in b-metric spaces and prove the existence and uniqueness of common fixed points. We draw some corollaries from our results and provide examples in support of our results.

Keywords: common fixed points; b-metric space; b-continuous; almost Geraghty-Suzuki contraction type maps.

2010 AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

The development of fixed point theory is based on the generalization of contraction conditions in one direction or/and generalization of ambient spaces of the operator under consideration on the other. Banach contraction principle plays an important role in solving non linear equations, and it is one of the most useful result in fixed point theory. In the direction of generalization of contraction conditions, in 1973, Geraghty [20] proved a fixed point theorem, generalizing

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Banach contraction principle. Later, several authors proved the existence of fixed points using Geraghty-type contraction maps. In continuation to the extensions of contraction maps, Berinde [9] introduced 'weak contractions' as a generalization of contraction maps. Berinde renamed 'weak contractions' as 'almost contractions' in his later work [10]. For more works on almost contractions and its generalizations, we refer Babu, Sandhya and Kameswari [6], Abbas, Babu and Alemayehu [1], Babu, Babu, Rao and Prasad [5] the related references cited in these papers. In 1975, Dass and Gupta [16] established fixed point results using contraction condition involving rational expressions. In 2008, Suzuki [31] proved two fixed point theorems, one of which is a new type of generalization of the Banach contraction principle and does characterize the metric completeness.

The main idea of b-metric was initiated from the works of Bourbaki [13] and Bakhtin [8]. The concept of b-metric space or metric type space was introduced by Czerwik [14] as a generalization of metric space. Afterwards, many authors studied fixed point theorems for single-valued and multi-valued mappings in b-metric spaces, for more information we refer [3, 7, 11, 12, 15, 22, 23, 24, 25, 29, 30].

2. Preliminaries

Definition 2.1. [14] Let X be a non-empty set. A function $d : X \times X \to [0,\infty)$ is said to be a b-metric if the following conditions are satisfied: for any $x,y,z \in X$

(i) $0 \leq d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) there exists $s \geq 1$ such that $d(x,z) \leq s[d(x,y)+d(y,z)]$.

In this case, the pair (X,d) is called a b-metric space with coefficient s.

Every metric space is a b-metric space with $s = 1$. In general, every b-metric space is not a metric space.

Definition 2.2. [11] Let (X,d) be a b-metric space.

(i) A sequence $\{x_n\}$ in X is called b-convergent if there exists $x \in X$ such that $d(x_n,x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$ and x is unique.

(ii) A sequence $\{x_n\}$ in X is called b-Cauchy if $d(x_n,x_m) \to 0$ as $n,m \to \infty$.

(iii) A b-metric space (X,d) is said to be a complete b-metric space if every b-Cauchy sequence in X is b-convergent in X.

(iv) A set $B \subset X$ is said to be b-closed if for any sequence $\{x_n\}$ in B such that $\{x_n\}$ is b-convergent to $z \in X$ then $z \in B$.

In general, a b-metric is not necessarily continuous.

In this paper, we denote $\mathbb{R}^+ = [0,\infty)$ and $\mathbb{N}$ is the set of all natural numbers.

Example 2.3. [24] Let $X = \mathbb{N} \cup \{\infty\}$. We define a mapping $d : X \times X \to [0,\infty)$ as follows:

$$d(m,n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m,n \text{ is even and the other is even or } \infty, \\ 5 & \text{if one of } m,n \text{ is odd and the other is odd or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

Then (X,d) is a b-metric space with coefficient $s = \frac{5}{2}$.

Definition 2.4. [12] Let (X,d_X) and (Y,d_Y) be two b-metric spaces. A function $f : X \to Y$ is a b-continuous at a point $x \in X$, if it is b-sequentially continuous at x. i.e., whenever $\{x_n\}$ is b-convergent to x, fx_n is b-convergent to fx.

In 1973,Geraghty [20] introduced a class of functions

$\mathfrak{S} = \{\beta : [0,\infty) \to [0,1) / \lim\limits_{n\to\infty} \beta(t_n) = 1 \implies \lim\limits_{n\to\infty} t_n = 0\}$.

Theorem 2.5 [20] Let (X,d) be a complete metric space. Let $T : X \to X$ be a selfmap satisfying the following: there exists $\beta \in \mathfrak{S}$ such that $d(Tx,Ty) \leq \beta(d(x,y))d(x,y)$ for all $x,y \in X$. Then T has a unique fixed point.

We denote $\mathfrak{B} = \{\alpha : [0,\infty) \to [0,\frac{1}{s}) / \lim\limits_{n\to\infty} \alpha(t_n) = \frac{1}{s} \implies \lim\limits_{n\to\infty} t_n = 0\}$.

In 2011, Dukic,Kadelburg and Radenovic [17] extended Theorem 2.5 to the case of b-metric space as follows.

Theorem 2.6 [17] Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and let $T : X \to X$ be a selfmap of X. Suppose that there exists $\alpha \in$ such that $d(tx,Ty) \leq \alpha(d(x,y))d(x,y)$ for all $x,y \in X$. Then T has a unique fixed point in X.

Throughout this paper, we denote

$\mathfrak{F} = \{\beta : [0,\infty) \to [0,\frac{1}{s}) / \limsup\limits_{n\to\infty} \beta(t_n) = 1 \implies \lim\limits_{n\to\infty} t_n = 0\}$.

The following lemmas are useful in proving our main results.

Lemma 2.7 [21] Let (X,d) is be a b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be selfmap. Suppose that $\{x_n\}$ is a sequence in X induced by $x_{n+1} = Tx_n$ such that
$d(x_n,x_{n+1}) \leq \lambda d(x_{n-1},x_n)$ for all $n \in \mathbb{N}$, where $\lambda \in [0,1)$ is a constant. Then $\{x_n\}$ is a b-cauchy sequence in X.

Lemma 2.8 [2] Suppose (X,d) is a b-metric space with coefficient $s \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y respectively, then we have
$\frac{1}{s^2}d(x,y) \leq \liminf\limits_{n\to\infty} d(x_n,y_n) \leq \limsup\limits_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y)$.
In particular, if $x = y$, then we have $\lim\limits_{n\to\infty} d(x_n,y_n) = 0$. Moreover for each $z \in X$ we have
$\frac{1}{s}d(x,z) \leq \liminf\limits_{n\to\infty} d(x_n,z) \leq \limsup\limits_{n\to\infty} d(x_n,z) \leq sd(x,z)$.

Lemma 2.9 [28] Suppose (X,d) is a b-metric space with coefficient $s \geq 1$ and $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} d(x_n,x_{n+1}) = 0$. If $\{x_n\}$ is not a cauchy sequence then there exist an $\varepsilon > 0$ and sequence of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k},x_{n_{k-1}}) \geq \varepsilon$.
For each $k > 0$, corresponding to m_k, we can close n_k to be the smallest positive integer such that $d(x_{m_k},x_{n_k}) \geq \varepsilon, d(x_{m_k},x_{n_{k-1}}) < \varepsilon$ and

(i) $\varepsilon \leq \liminf\limits_{k\to\infty} d(x_{m_k},x_{n_k}) \leq \limsup\limits_{k\to\infty} d(x_{m_k},x_{n_k}) \leq s\varepsilon$
(ii) $\frac{\varepsilon}{s} \leq \liminf\limits_{k\to\infty} d(x_{m_{k+1}},x_{n_k}) \leq \limsup\limits_{k\to\infty} d(x_{m_{k+1}},x_{n_k}) \leq s^2\varepsilon$
(iii) $\frac{\varepsilon}{s} \leq \liminf\limits_{k\to\infty} d(x_{m_k},x_{n_{k+1}}) \leq \limsup\limits_{k\to\infty} d(x_{m_k},x_{n_{k+1}}) \leq s^2\varepsilon$
(iv) $\frac{\varepsilon}{s^2} \leq \liminf\limits_{k\to\infty} d(x_{m_{k+1}},x_{n_{k+1}}) \leq \limsup\limits_{k\to\infty} d(x_{m_{k+1}},x_{n_{k+1}}) \leq s^3\varepsilon$.

In 2015, Latif, Parvanesh, Salimi and Al-Mazrooei [26] proved the existence and uniqueness of fixed points of a single selfmap satisfying Suzuki type contraction condition in b-metric spaces. In 2017, Leyew and Abbas [27] proved the existance and uniqueness of fixed points of generalized Suzuki-Geraghty contraction maps in complete b-metric spaces.

In 2019, Faraji, Savic and Radenovic [19] proved the following theorem.

Theorem 2.10 [19] Let (X,d) be a complete b-metric space with coefficient $s \geq 1$. Let $T,S : X \to X$ be selfmaps on X which satisfy: there exists $\beta \in \mathfrak{F}$ such that

$d(Tx,Sy) \leq \beta(M(x,y))M(x,y)$ for all $x,y \in X$, where $M(x,y) = max\{d(x,y),d(x,Tx),d(y,Sy)\}$. If T or S continuous, then T and S have a unique common fixed point.

The following theorem is due to Faraji [18] in the setting of partial metric spaces.

Theorem 2.11 [18] Let (X,p) be a partial metric space and let the mappings $f,g : X \to X$ satisfy the condition $p(gx,gy) \leq \beta(M(x,y))m(x,y)$, for all $x,y \in X$, where $\beta \in \mathfrak{S}$ and
$m(x,y) = \max\{p(fx,fy), p(fx,gx), p(fy,gy)\}$,
$M(x,y) = \max\{p(fx,fy), p(fx,gx), p(fy,gy), \frac{p(fx,gy)+p(fy,gx)}{2}\}$.
Suppose also that $g(X) \subset f(X)$ and $f(X)$ is a complete subspace of X.
Then f,g have a unique point of coincident in X. Moreover if f and g are weakly compatible, then f and g have a unique common fixed point in X.

In 2019, Babu and Babu [4] proved the following two theorems.

Theorem 2.12 [4] Let (X,d) be a complete b-metric space with coefficient $s \leq 1$ and let $f : X \to X$ be selfmap satisfies the following condition: there exist $\beta \in \mathfrak{F}$ and $L \geq 0$ such that
$d(fx,fy) \leq \beta(M(x,y))M(x,y) + LN(x,y)$ for all $x,y \in X$, where
$M(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), \frac{1}{2s}[d(x,fy)+d(y,fx)]\}$ and
$N(x,y) = \min\{d(x,fx), d(x,fy), d(y,fx)\}$.
Then f has a unique fixed point in X.

Theorem 2.13 [4] Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and let $f,g : X \to X$ be a self map on X satisfy the following condition: there exist $\beta \in \mathfrak{F}$ and $L \geq 0$ such that $sd(fx,gy) \leq \beta(M(x,y))M(x,y) + LN(x,y)$ for all $x,y \in X$, where
$M(x,y) = \max\{d(x,y), d(x,fx), d(y,gy)\}$ and
$N(x,y) = \min\{d(x,fx), d(x,gy), d(y,fx)\}$.
If either f or g is b-continuous then f and g have a unique common fixed point in X.

Motivated by works of Babu and Babu [4], Faraji [18], we introduce almost Geraghty-Suzuki contraction type (I), almost Geraghty-Suzuki contraction type (II) maps for a pair of selfmaps in b-metric spaces. We prove the existence and uniqueness of common fixed points. We draw some corollaries and provide examples in support of our results. We also give the importance of L in the inequalities (3.1) and (3.2) [Example 5.1 and Example 5.2].

3. Common Fixed Points of Almost Geraghty-Suzuki Contraction Type Maps

The following we introduce almost Geraghty-Suzuki contraction type (I) and almost Geraghty-Suzuki contraction type (II) maps for a pair of selfmaps in b-metric spaces.

Definition 3.1. Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $f,g : X \to X$ be two selfmaps on X. We say that (f,g) is the pair an almost Geraghty-Suzuki contraction type (I) maps, if there exist $L \geq 0$ and $\beta \in \mathfrak{F}$ such that

$$\frac{1}{2s}\min\{d(x,fx),d(y,gy)\} \leq d(x,y) \ \Rightarrow \ sd(fx,gy) \leq \beta(M_1(x,y))M_2(x,y)+LN_1(x,y) \tag{3.1}$$

where $M_1(x,y) = \max\{d(x,y),d(x,fx),d(y,gy),\frac{d(x,gy)+d(fx,y)}{2s}\}$,

$M_2(x,y) = \max\{d(x,y),d(x,fx),d(y,gy)$ and $N_1(x,y) = \min\{d(y,gy),d(y,fx)\}$.

Remark 3.2 If $L = 0$ in the inequality (3.1) then we say that the pair (f,g) is a Geraghty-Suzuki contraction type (I) maps.

Example 3.3. Let $X = (0,\infty)$ and let $d : X \times X \to [0,\infty)$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ (x+y)^2 & \text{if } x \neq y. \end{cases}$$

Then clearly (X,d) is a complete b-metric space with $s = 2$.

We define $f,g : X \to X$ by

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \in (0,1) \\ \frac{1}{2} & \text{if } x \in [1,\infty) \end{cases} \quad \text{and } g(x) = \begin{cases} \frac{x}{6} & \text{if } x \in (0,1) \\ \frac{1}{3} & \text{if } x \in [1,\infty). \end{cases}$$

Without loss of generality, we assume that $x \geq y$

We define $\beta : [0,\infty) \to [0,\frac{1}{s})$ by $\beta(t) = \frac{1}{2+t}$. Then we have $\beta \in \mathfrak{F}$

Case (i). $x,y \in (0,1)$.

Since $\frac{1}{2s}\min\{d(x,fx),d(y,gy)\} = \frac{1}{4}\min\{(x+\frac{x}{2})^2,(y+\frac{y}{6})^2\} = \frac{1}{4}(y+\frac{y}{6})^2 \leq (x+y)^2 = d(x,y)$.

$M_1(x,y) = \max\{(x+y)^2,(x+\frac{x}{2})^2,(y+\frac{y}{6})^2,\frac{(x+\frac{y}{6})^2+(y+\frac{x}{2})^2}{4}\}$

$M_2(x,y) = \max\{(x+y)^2,(x+\frac{x}{2})^2,(y+\frac{y}{6})^2\}$ and

$N_1(x,y) = \min\{(x+y)^2,(y+\frac{y}{6})^2,(y+\frac{x}{2})^2\} = (y+\frac{y}{6})^2$.

Subcase (a). $y \leq \frac{x}{2}$

$M_1(x,y) = M_2(x,y) = (x+\frac{x}{2})^2$

We now consider

$$sd(fx,gy) = 2(\tfrac{x}{2}+\tfrac{y}{6})^2 = \tfrac{1}{18}(3x+y)^2 \leq \tfrac{1}{2+(x+\frac{x}{2})^2}(x+\tfrac{x}{2})^2 + L(y+\tfrac{y}{6})^2$$
$$\leq \beta(M_1(x,y))M_2(x,y) + LN_1(x,y).$$

Subcase (b). $y > \frac{x}{2}$.

$M_1(x,y) = M_2(x,y) = (x+y)^2$

We now consider

$$sd(fx,gy) = 2(\tfrac{x}{2}+\tfrac{y}{6})^2 \leq \tfrac{1}{2+(x+y)^2}(x+y)^2 + L(y+\tfrac{y}{6})^2 \leq \beta(M_1(x,y))M_2(x,y) + LN_1(x,y).$$

Case (ii). $x,y \in [1,\infty)$.

$\frac{1}{2s}\min\{d(x,fx),d(y,gy)\} = \frac{1}{4}\min\{(x+\frac{1}{2})^2,(y+\frac{1}{3})^2\} = \frac{1}{4}(y+\frac{1}{3})^2 \leq (x+y)^2 = d(x,y).$

$M_1(x,y) = M_2(x,y) = (x+y)^2$ and $N_1(x,y) = (y+\frac{1}{3})^2$.

Now, we consider

$$sd(fx,gy) = 2(\tfrac{5}{6})^2 \leq \tfrac{1}{2+(x+y)^2}(x+y)^2 + L(y+\tfrac{1}{3})^2 \leq \beta(M_1(x,y))M_2(x,y) + LN_1(x,y).$$

Case (iii). $x \in [1,\infty), y \in [0,1)$.

We have

$\frac{1}{2s}\min\{d(x,fx),d(y,gy)\} = \frac{1}{4}\min\{(x+\frac{1}{2})^2,(y+\frac{y}{6})^2\} = \frac{1}{4}(y+\frac{y}{6})^2\} \leq (x+y)^2 = d(x,y).$

Subcase (a). $y \leq \frac{1}{2}$.

$M_1(x,y) = (x+\frac{1}{2})^2 = N_2(x,y)$ and $N_1(x,y) = (y+\frac{y}{6})^2$.

We now consider

$$sd(fx,gy) = 2(\tfrac{1}{2}+\tfrac{y}{6})^2 \leq \tfrac{1}{2+(x+\frac{1}{2})^2}(x+\tfrac{1}{2}) + L(y+\tfrac{y}{6})^2.$$

Subcase (b). $y > \frac{1}{2}$.

$M_1(x,y) = M_2(x,y) = (x+y)^2$ and $N_1(x,y) = (y+\frac{y}{6})^2$.

We consider

$$sd(fx,gy) = 2(\tfrac{1}{2}+\tfrac{y}{6})^2 \leq \tfrac{1}{2+(x+y)^2}(x+y)^2 + L(y+\tfrac{y}{6})^2 \leq \beta(M_1(x,y))M_2(x,y) + LN_1(x,y).$$

Therefore from all the above cases the pair (f,g) is an almost Geraghty- Suzuki contraction type (I) maps.

Definition 3.4. Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $f,g : X \to X$ be a selfmaps on X. We say that (f,g) is almost Geraghty-Suzuki contraction type (II) maps, if there exist $L \geq 0$ and $\beta \in \mathfrak{F}$ such that

$$\frac{1}{2s}\min\{d(x,fx),d(y,gy)\} \leq d(x,y) \ \Rightarrow \ sd(fx,gy) \leq \beta(M_3(x,y))M_4(x,y)) + LN_2(x,y) \tag{3.2}$$

where $M_3(x,y) = \max\{d(x,y), \frac{d(y,gy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}\}$.

$M_4(x,y) = \max\{d(x,y), \frac{d(y,gy)[1+d(x,fx)]}{1+d(x,y)}\}$ and $N_2(x,y) = \min\{d(x,fx), d(x,gy)\}$.

Remark 3.5. If $L = 0$ in the inequality (3.2) then we say that the pair (f,g) is a Geraghty-Suzuki contraction type (II) maps.

Example 3.6. Let $X = (0,1)$ and let $d : X \times X \to \mathbb{R}^+$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ (x+y)^2 & \text{if } x \neq y. \end{cases}$$

Then (X,d) is a complete b-metric space with coefficient $s = 2$.

We define $f, g : X \to X$ by

$f(x) = \frac{x(3+x)}{64}$, $g(x) = \frac{x}{8(1+x)}$ and we define $\beta : [0,\infty) \to [0,s)$ by $\beta(t) = \frac{1}{3+t}$

Without loss of generality, we assume that $x \leq y$.

We have $\min\{d(x,fx), d(y,gy)\} = \frac{1}{4}\min\{(x+\frac{x(3+x)}{64})^2, (y+\frac{y}{8(1+y)})^2\} \leq (x+y)^2 = d(x,y)$

$M_3(x,y) = \max\{(x+y)^2, \frac{(y+\frac{y}{8(1+y)})^2[1+(x+\frac{x(3+x)}{64})^2]}{1+(x+y)^2}, \frac{(y+\frac{x(3+x)}{64})^2[1+(x+\frac{x(3+x)}{64})^2]}{4(1+(x+y)^2)}\}$

$M_4(x,y) = \max\{(x+y)^2, \frac{(y+\frac{y}{8(1+y)})^2[1+(x+\frac{x(3+x)}{64})^2]}{1+(x+y)^2}\}$ and

$N_2(x,y) = \min\{(x+\frac{x(3+x)}{64})^2, (x+\frac{y}{8(1+y)})^2\} = (x+\frac{y}{8(1+y)})^2$.

We consider

$$\begin{aligned} sd(fx,gy) &= 2(\frac{x(3+x)}{64}+\frac{y}{8(1+y)})^2 \leq \frac{1}{32}(\frac{x(3+x)}{8}+\frac{y}{1+y})^2 \leq \frac{1}{32}(x+y)^2 \\ &\leq \frac{1}{3+(x+y)^2}(x+y)^2 + L(x+\frac{y}{8(1+y)})^2 \leq \beta(M_3(x,y))M_4(x,y) + LN_2(x,y). \end{aligned}$$

Therefore the pair of (f,g) is an almost Geraghty-Suzuki contraction type (II) maps.

4. Main Results

Proposition 4.1. Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $f, g : X \to X$ be two selfmaps. Assume that the pair (f,g) is an almost Geraghty-Suzuki contraction type (I) maps. Then u is a fixed point of f if and only if u is a fixed point of g. Moreover, in this case u is unique.

Proof. Let u be a fixed point of f. i.e., $fu = u$.

Suppose that $gu \neq u$.

We have

$\frac{1}{2s}\min\{d(u,fu),d(u,gu)\} = \frac{1}{2s}\min\{d(u,u),d(u,gu)\} = 0 = d(u,u)$.

From the inequality (3.1), we get

$$sd(fu,gu) \leq \beta(M_1(u,u))M_2(u,u) + LN_1(u,u) \tag{4.1}$$

where

$M_1(u,u) = \max\{d(u,u),d(u,fu),d(u,gu),\frac{d(u,gu)+d(u,fu)}{2s}\} = d(u,gu)$,

$M_2(u,u) = \max\{d(u,u),d(u,fu),d(u,gu)\} = d(u,gu)$ and

$N_1 = \min\{d(u,fu),d(u,gu),d(u,fu)\} = 0$.

Therefore from the inequality (4.1), we get

$sd(u,gu) \leq \beta(d(u,gu))d(u,gu) < \frac{d(u,gu)}{s}$,

which is a contradiction.

Hence $gu = u$, so that u is a common fixed point of f and g.

Similarly, it is easy to see that if u is a fixed point of g then u is a fixed point of f also.

Suppose u and v are two common fixed points of f and g with $u \neq v$.

Since $\frac{1}{2s}\min\{d(u,fu),d(v,gv)\} \leq d(u,v)$ so that from the inequality (3.1), we get

$$sd(fu,gv) \leq \beta(M_1(u,v))M_2(u,v) + LN_1(u,v) \tag{4.2}$$

where $M_1(u,v) = \max\{d(u,v),d(u,fu),d(v,gv),\frac{d(u,gv)+d(v,fu)}{2s}\} = d(u,v)$,

$M_2(u,v) = \max\{d(u,v),d(u,fu),d(v,gv)\} = d(u,v)$ and

$N_1(u,v) = \max\{d(u,fu),d(v,gv),d(v,fu)\} = 0$.

From the inequality (4.2), we have

$sd(u,v) \leq \beta(d(u,v))d(u,v) < \frac{d(u,v)}{s}$,

it is a contradiction.

Therefore $u = v$.

Hence, f and g have a unique common fixed point in X. □

Proposition 4.2. Let (X,d) be a b-metric space with coefficient $s \geq 1$ and f,g be two self maps on X. Assume that the pair (f,g) is an almost Geraghty-Suzuki contraction type (II) maps. Then u is a fixed point of f if and only if u is a fixed point of g. Moreover, it is unique.

Proof. Let u be a fixed point of f. i.e., $fu = u$. Suppose that $gu \neq u$.

We have $\frac{1}{2s}\min\{d(u,fu),d(u,gu)\} = 0 = d(u,u)$ and hence from the inequality (3.2),we get

$$sd(fu,gu) \leq \beta(M_3(u,u))M_4(u,u) + LN_2(u,u) \tag{4.3}$$

where $M_3(u,u) = \max\{d(u,u), \frac{d(u,gu)[1+d(u,fu)]}{1+d(u,u)}, \frac{d(u,fu)[1+d(u,fu)]}{s^2(1+d(u,u))}\} = d(u,gu)$,

$M_4(u,u) = \max\{d(u,u), \frac{d(u,gu)[1+d(u,fu)]}{s^2(1+d(u,u))}\} = 0$ and $N_2(u,u) = \min\{d(u,fu),d(u,gu)\} = 0$

From the inequality (4.3), we have

$sd(u,gu) \leq \beta(d(u,gu)).0 + L.0 = 0$,

which is a contradiction. Therefore $gu = u$. Hence u is a common fixed point of f and g.

Similarly, it is easy to see that if u is a fixed point of g then u is a fixed point of f also.

Suppose u and v are two common fixed points of f and g with $u \neq v$.

Since $\frac{1}{2s}\min\{d(u,fu),d(v,gv)\} \leq d(u,v)$ so that from the inequality (3.2), we have

$$sd(fu,gv) \leq \beta(M_3(u,v))M_4(u,u) + LN_2(u,v) \tag{4.4}$$

where $M_3(u,v) = \max\{d(u,v), \frac{d(v,gv)[1+d(u,fu)]}{1+d(u,v)}, \frac{d(v,fu)[1+d(u,fv)]}{s^2(1+d(u,v))}\} = d(u,v)$,

$M_4(u,v) = \max\{d(u,v), \frac{d(v,gv)[1+d(u,fu)]}{1+d(u,v)}\} = d(u,v)$ and $N_2(u,v) = \min\{d(u,fu),d(u,gv)\} = 0$.

From the inequality (4.4), we have

$sd(u,v) \leq \beta(d(u,v))d(u,v) < \frac{d(u,v)}{s}$,

which is contradiction.

Therefore $u = v$.

Hence f and g have a unique common fixed point in X. □

Theorem 4.3. Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and let the pair (f,g) be an almost Geraghty-Suzuki contraction type (I) maps. If either f (or) g is b-continuous then f and g have a unique common fixed point in X.

Proof. We take $x_0 \in X$ be arbitrary.

Since $f(X) \subseteq X$ and $g(X) \subseteq X$, there exist $x_1, x_2 \in X$ such that $fx_0 = x_1$ and $gx_1 = x_2$.

Similarly there exist $x_3, x_4 \in X$ such that $fx_2 = x_3$ and $gx_3 = x_4$.

In general, we construct a sequence $\{x_n\}$ in X by $fx_{2n} = x_{2n+1}, gx_{2n+1} = x_{2n+2}$ for $n = 0,1,2,\ldots$.

Suppose $x_{2n} = x_{2n+1}$ for some n, then $x_{2n} = fx_{2n}$ so that x_{2n} is a fixed point of f.

Hence by Proposition 4.1, we have x_{2n} is a fixed point of g.

Therefore x_{2n} is a common fixed point of f and g.

Hence without loss of generality, we assume that $x_n \neq x_{n+1}$ for all n.

Suppose n is even. Then $n = 2m, m \in \mathbb{N}$. Since

$\frac{1}{2s}\min\{d(x_n, fx_n), d(x_{n+1}, gx_{n+1})\} = \frac{1}{2s}\min\{d(x_{2m}, fx_{2m}), d(x_{2m+1}, gx_{2m+1})\} \leq d(x_{2m}, x_{2m+1})$,

it follows from (3.1) that

$$sd(fx_{2m}, gx_{2m+1}) \leq \beta((M_1(x_{2m}, x_{2m+1}))M_2(x_{2m}, x_{2m+1})) + LN_1(x_{2m, x_{2m+1}}) \tag{4.5}$$

where

$$\begin{aligned} M_1(x_{2m}, x_{2m+1}) &= \max\{d(x_{2m}, x_{2m+1}), d(x_{2m}, fx_{2m}), d(x_{2m+1}, gx_{2m+1}), \tfrac{d(x_{2m}, gx_{2m+1}) + d(x_{2m+1}, fx_{2m})}{2s} \\ &= \max\{d(x_{2m}, x_{2m+1}), d(x_{2m}, x_{2m+1}), d(x_{2m+1}, x_{2m+2}), \tfrac{d(x_{2m}, x_{2m+2}) + d(x_{2m+1}, x_{2m+1})}{2s}\} \\ &= \max\{d(x_{2m}, x_{2m+1}), d(x_{2m+1}, x_{2m+2})\}, \end{aligned}$$

$M_2(x_{2m}, x_{2m+1}) = \max\{d(x_{2m}, x_{2m+1}), d(x_{2m}, x_{2m+1}, d(x_{2m+1}, x_{2m+2})\}$ and

$N_1(x_{2m}, x_{2m+1}) = \min\{d(x_{2m}, fx_{2m}), d(x_{2m+1}, gx_{2m+1}), d(x_{2m+1}, fx_{2m})\} = 0$

If $d(x_{2m}, x_{2m+1}) < d(x_{2m+1}, x_{2m+2})$ then

$M_1(x_{2m}, x_{2m+1}) = M_2(x_{2m}, x_{2m+1}) = d(x_{2m+1}, x_{2m+2})$.

Therefore from the inequality (4.5),

$sd(x_{2m+1}, x_{2m+2}) \leq \beta(d(x_{2m+1}, x_{2m+2}))d(x_{2m+1}, x_{2m+2}) < \frac{1}{s}d(x_{2m+1}, x_{2m+2})$

which is contradiction.

Therefore $M_1(x_{2m}, x_{2m+1}) = M_2(x_{2m}, x_{2m+1}) = d(x_{2m}, x_{2m+1})$.

Hence from the inequality (4.5), we have

$$sd(x_{2m+1}, x_{2m+2}) \leq \beta(d(x_{2m}, x_{2m+1}))d(x_{2m}, x_{2m+1}) < \frac{1}{s}d(x_{2m}, x_{2m+1}) \tag{4.6}$$

Therefor $d(x_{2m+1}, x_{2m+2}) \leq d(x_{2m}, x_{2m+1})$

Similarly, we obtain $d(x_{2m+2}, x_{2m+3}) \leq d(x_{2m+1}, x_{2m+2})$

Hence $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$

Thus $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of non negative reals and bounded below by 0.

Hence there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$.

Now on taking limit superior as $n \to \infty$ in (4.6),we get

$sr \leq \limsup_{n\to\infty} \beta(d(x_{2m}, x_{m+1}))r$ implies that $s \leq \limsup_{n\to\infty} \beta(d(x_{2m}, x_{m+1})) \leq \frac{1}{s}$

which implies that $\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta(d(x_n, x_{n+1})) \leq \frac{1}{s}$.
Therefore $\limsup_{n\to\infty} \beta(d(x_n, x_{n+1})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{4.7}$$

we know prove that $\{x_n\}$ is a b-Cauchy sequence in X.

It is sufficient to show that $\{x_{2n}\}$ is a b-Cauchy.

Suppose that $\{x_{2n}\}$ is not a b-Cauchy sequence.

Then by Lemma 2.9 there exist an $\varepsilon > 0$ for which we can find subsequences $\{x_{2m_k}\}$ and $\{x_{2n_k}\}$ of $\{x_{2n}\}$ with $n_k > m_k \geq k$ such that

$$d(x_{2m_k}, x_{2n_k}) \geq \varepsilon \;\text{ and } d(x_{2m_k}, x_{2n_{k-2}}) < \varepsilon \tag{4.8}$$

satisfying (i)-(iv) of Lemma 2.9.

Suppose that there exists a $k_1 \in \mathbb{N}$ with $k \geq k_1$ such that

$$\frac{1}{2s}\min\{d(x_{2m_k}, fx_{2m_k}), d(x_{2n_k-1}, gx_{2n_k-1})\} > d(x_{2m_k}, x_{2n_k-1}). \tag{4.9}$$

On letting as $k \to \infty$ in (4.9), we get that $\varepsilon \leq 0$,

a contradiction.

Therefore $\frac{1}{2s}\min\{d(x_{2m_k}, fx_{2m_k}), d(x_{2n_k-1}, gx_{2n_k-1})\} \leq d(x_{2m_k}, x_{2n_k-1})$.
and from (3.1), we have

$$sd(fx_{2m_k}, gx_{2n_{k-1}}) \leq \beta(M_1(x_{2m_k}, x_{2n_k-1}))M_2(x_{2m_k}, x_{2n_k-1}) + LN_1(x_{2m_k}, x_{2n_k-1}) \tag{4.10}$$

where
$$\begin{aligned}
M_1(x_{2m_k}, x_{2n_{k-1}}) &= \max\{d(x_{2m_k}, x_{2n_{k-1}}), d(x_{2m_k}, fx_{2m_k}), d(x_{2n_{k-1}}, gx_{2n_{k-1}}),\\
&\qquad \tfrac{1}{2s}[d(x_{2m_k}, gx_{2n_{k-1}}) + d(x_{2n_{k-1}}, fx_{2m_k})]\}\\
&= \max\{d(x_{2m_k}, x_{2n_{k-1}}), d(x_{2m_k}, x_{2m_{k+1}}), d(x_{2n_{k-1}}, x_{2n_k}),\\
&\qquad \tfrac{1}{2s}[d(x_{2m_k}, gx_{2n_k}) + d(x_{2n_{k-1}}, x_{2m_{k+1}})]\}\\
&\leq \max\{d(x_{2m_k}, x_{2n_{k-1}}), d(x_{2m_k}, x_{2m_{k+1}}), d(x_{2n_{k-1}}, x_{2n_k}),\\
&\qquad \tfrac{1}{2s}s[d(x_{2m_k}, x_{2n_{k-1}}) + d(x_{2n_{k-1}}, x_{2n_k}) + d(x_{2n_{k-1}}, x_{2m_k}) + d(x_{2m_k}, x_{2m_{k+1}})]\},
\end{aligned}$$

$$\begin{aligned}
M_2(x_{2m_k}, x_{2n_{k-1}}) &= \max\{d(x_{2m_k}, x_{2n_{k-1}}), d(x_{2m_k}, fx_{2m_k}), d(x_{2n_{k-1}}, gx_{2n_{k-1}})\}\\
&= \max\{d(x_{2m_k}, x_{2n_{k-1}}), d(x_{2m_k}, fx_{2m_{k+1}}), d(x_{2n_{k-1}}, x_{2n_k})\}
\end{aligned}$$

and $N_1(x_{2m_k}, x_{2n_{k-1}}) = \min\{d(x_{2n_{k-1}}, gx_{2n_{k-1}}), d(x_{2n_{k-1}}, fx_{2m_k})\}$.

On taking limit superior as $k \to \infty$, $M_1(x_{2m_k}, x_{2n_{k-1}}), M_2(x_{2m_k}, x_{2n_{k-1}})$ and $N_1(x_{2m_k}, x_{2n_{k-1}})$ and using (4.7), we get

$\limsup\limits_{k\to\infty} M_1(x_{2m_k}, x_{2n_{k-1}}) = \limsup\limits_{k\to\infty} d(x_{2m_k}, x_{2n_{k-1}}) = \limsup\limits_{k\to\infty} M_2(x_{2m_k}, x_{2n_{k-1}})$ and $\limsup\limits_{k\to\infty} N_1(x_{2m_k}, x_{2n_{k-1}}) = 0$.

We consider $d(x_{2m_k}, x_{2n_{k-1}}) \leq s[d(x_{2m_k}, x_{2n_{k-2}}) + d(x_{2n_{k-2}}, x_{2n_{k-1}})]$.

On taking limit superior as $k \to \infty$, we get

$\limsup\limits_{k\to\infty} d(x_{2m_k}, x_{2n_{k-1}}) \leq s\varepsilon$.

On letting limit superior as $k \to \infty$ in (4.10), we get

$$\begin{aligned} s(\tfrac{\varepsilon}{s}) &\leq \limsup_{k\to\infty} \beta(M_1(x_{2m_k}, x_{2n_{k-1}})) \limsup_{k\to\infty} M_2(x_{2m_k}, x_{2n_{k-1}}) + L \limsup_{k\to\infty} N_1(x_{2m_k}, x_{2n_{k-1}}) \\ &= \limsup_{k\to\infty} \beta(M_1(x_{2m_k}, x_{2n_{k-1}})) \limsup_{k\to\infty} d(x_{2m_k}, x_{2n_{k-1}}) \leq \limsup_{k\to\infty} \beta(M_1(x_{2m_k}, x_{2n_{k-1}}))\varepsilon s \end{aligned}$$

Therefore $\frac{1}{s} \leq \limsup\limits_{k\to\infty} \beta(M_1(x_{2m_k}, x_{2n_{k-1}})) \leq \frac{1}{s}$, which implies $\limsup\limits_{k\to\infty} \beta(M_1(x_{2m_k}, x_{2n_{k-1}})) = \frac{1}{s}$.

Since $\beta \in \mathfrak{F}$, we have $\lim\limits_{k\to\infty} M_1(x_{2m_k}, x_{2n_{k-1}}) = 0$.

$$\text{i.e., } \lim_{k\to\infty} d(x_{2m_k}, x_{2n_{k-1}}) = 0. \tag{4.11}$$

From the inequality (4.8) and using b-triangular property, we have

$0 < \varepsilon \leq d(x_{2m_k}, x_{2n_k}) \leq s[d(x_{2m_k}, x_{2n_{k-1}}) + d(x_{2n_{k-1}}, x_{2n_k})]$.

Taking limit as $k \to \infty$, using (4.6) and (4.11), we get $0 < \varepsilon \leq 0$

which is contradiction.

Therefore $\{x_n\}$ is a b-cauchy sequence in$X$. Since X is b-complete, there exists $x \in X$ such that $\lim\limits_{n\to\infty} x_n = x$.

Suppose f is b-continuous, we have

$fx = \lim\limits_{n\to\infty} fx_{2n} = \lim\limits_{n\to\infty} x_{2n+1} = x$. Therefore x is a fixed point of f.

By proposition 4.1, we get that x is a unique common fixed point of f and g.

Similarly, we can prove that x is a unique common fixed point of f and g whenever g is b-continuous. □

Theorem 4.4. Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and let the pair (f,g) be an almost Geraghty-Suzuki contraction type (II) maps. If either f or g is b-continuous then f and g have a unique common fixed point in X.

Proof. We take $x_0 \in X$ be arbitrary.

Since $f(X) \subseteq X$ and $g(X) \subseteq X$, there exist $x_1, x_2 \in X$ such that $fx_0 = x_1$ and $gx_1 = x_2$.

Similarly there exist $x_3, x_4 \in X$ such that $fx_2 = x_3$ and $gx_3 = x_4$.

In general, we construct a sequence $\{x_n\}$ in X by $fx_{2n} = x_{2n+1}, gx_{2n+1} = x_{2n+2}$ for $n = 0, 1, 2, \ldots$.

Suppose $x_{2n} = x_{2n+1}$ for some n, then $x_{2n} = fx_{2n}$ so that x_{2n} is a fixed point of f.

Hence by Proposition 4.2, we have x_{2n} is a fixed point of g.

Therefore x_{2n} is a common fixed point of f and g.

Hence without loss of generality, we assume that $x_n \neq x_{n+1}$ for all n.

Suppose n is even. Then $n = 2m, m \in \mathbb{N}$. Since

$\frac{1}{2s}\min\{d(x_{2n}, fx_{2n}), d(x_{2n+1}, gx_{2n+1})\} \leq d(x_{2n}, x_{2n+1})$, it follows from (3.2) that

$$sd(fx_{2n}, gx_{2n+1}) \leq \beta((M_3(x_{2n}, x_{2n+1}))M_4(x_{2n}, x_{2n+1})) + LN_2(x_{2n}, x_{2n+1}) \tag{4.12}$$

where

$$\begin{aligned} M_3(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), \tfrac{d(x_{2n+1}, gx_{2n+1})[1+d(x_{2n}, fx_{2n})]}{1+d(x_{2n}, x_{2n+1})}, \tfrac{d(x_{2n+1}, fx_{2n})[1+d(x_{2n}, fx_{2n})]}{s^2(1+d(x_{2n}, x_{2n+1}))}\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}, \\ M_4(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}), \tfrac{d(x_{2n+1}, gx_{2n+1})[1+d(x_{2n}, fx_{2n})]}{1+d(x_{2n}, x_{2n+1})}\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \text{ and } N_2(x_{2n}, x_{2n+1}) = 0. \end{aligned}$$

If $d(x_{2n}, x_{2n+1}) < d(x_{2n+1}, x_{2n+2})$ then

$M_3(x_{2n}, x_{2n+1}) = M_4(x_{2n}, x_{2n+1}) = d(x_{2n+1}, x_{2n+2})$.

From the inequality (4.12), we get

$sd(x_{2n+1}, x_{2n+2}) \leq \beta(d(x_{2n+1}, x_{2n+2}))d(x_{2n+1}, x_{2n+2}) < \frac{1}{s}d(x_{2n+1}, x_{2n+2})$

which is a contradiction.

Therefore $M_3(x_{2n}, x_{2n+1}) = M_4(x_{2n}, x_{2n+1}) = d(x_{2n}, x_{2n+1})$.

Again from inequality (4.12), we have

$$sd(x_{2n+1}, x_{2n+2}) \leq \beta(d(x_{2n}, x_{2n+1}))d(x_{2n}, x_{2n+1}) < \frac{1}{s}d(x_{2n}, x_{2n+1}) \tag{4.13}$$

Therefore $d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1})$.

Similarly, we obtain $d(x_{2n+2}, x_{2n+3}) \leq d(x_{2n+1}, x_{2n+2})$.

Hence $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$.

Thus $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of non negative reals and bounded below by 0.

Hence there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = r$

Now on letting limit superior as $n \to \infty$ in (4.13), we get

$sr \leq \limsup\limits_{n\to\infty} \beta(d(x_{2n}, x_{2n+1}))r$ implies that $s \leq \limsup\limits_{n\to\infty} \beta(d(x_{2n}, x_{2n+1})) \leq \frac{1}{s}$ implies that $\frac{1}{s} \leq 1 \leq \limsup\limits_{n\to\infty} \beta(d(x_{2n}, x_{2n+1})) \leq \frac{1}{s}$ which implise that $\limsup\limits_{n\to\infty} \beta(d(x_{2n}, x_{2n+1})) = \frac{1}{s}$.

Since $\beta \in \mathfrak{F}$, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{4.14}$$

We now prove that $\{x_n\}$ is a b-Cauchy sequence in X.

It is sufficient to show that $\{x_{2n}\}$ is b-Cauchy.

Suppose that $\{x_{2n}\}$ is not a b-Cauchy sequence.

Then by Lemma 2.9 there exist an $\varepsilon > 0$ for which we can find subsequences $\{x_{2m_k}\}$ and$\{x_{2n_k}\}$ of $\{x_{2n}\}$ with $n_k > m_k \geq k$ such that

$d(x_{2m_k}, x_{2n_k}) \geq \varepsilon$ and $d(x_{2m_k}, x_{2n_{k-2}}) < \varepsilon$ satisfying $(i)-(iv)$ of Lemma 2.9.

Suppose that there exists a $k_1 \in \mathbb{N}$ with $k \geq k_1$ such that

$$\frac{1}{2s}\min\{d(x_{2m_k}, fx_{2m_k}), d(x_{2n_{k-1}}, gx_{2n_{k-1}})\} > d(x_{2m_k}, x_{2n_{k-1}}) \tag{4.15}$$

On letting limit superior as $k \to \infty$ in (4.15), we get that $\varepsilon \leq 0$,

which is contradiction.

Therefore $\frac{1}{2s}\min\{d(x_{2m_k}, fx_{2m_k}), d(x_{2n_{k-1}}, gx_{2n_{k-1}})\} \leq d(x_{2m_k}, x_{2n_{k-1}})$

and from (3.2), we have

$$sd(fx_{2m_k}, gx_{2n_{k-1}}) \leq \beta(M_3(x_{2m_k}, x_{2n_{k-1}}))M_4(x_{2m_k}, x_{2n_{k-1}}) + LN_2(x_{2m_k}, x_{2n_{k-1}}) \tag{4.16}$$

where $M_3(x_{2m_k}, x_{2n_{k-1}}) = \max\{d(x_{2m_k}, x_{2n_{k-1}}), \frac{d(x_{2n_{k-1}}, gx_{2n_{k-1}})[1+d(x_{2m_k}, fx_{2m_k})]}{1+d(x_{2m_k}, x_{2n_{k-1}})},$

$\frac{d(x_{2n_{k-1}}, fx_{2m_k})[1+d(x_{2m_k}, fx_{2m_k})]}{s^2[1+d(x_{2m_k}, x_{2n_{k-1}})]}\},$

$= \max\{d(x_{2m_k}, x_{2n_{k-1}}), \frac{d(x_{2n_{k-1}}, x_{2n_k})[1+d(x_{2m_k}, x_{2m_{k+1}})]}{1+d(x_{2m_k}, x_{2n_{k-1}})},$

$\frac{d(x_{2n_{k-1}}, x_{2m_{k+1}})[1+d(x_{2m_k}, x_{2m_{k+1}})]}{s^2[1+d(x_{2m_k}, x_{2n_{k-1}})]}\},$

$M_4(x_{2m_k}, x_{2n_{k-1}}) = \max\{d(x_{2m_k}, x_{2n_{k-1}}), \frac{d(x_{2n_{k-1}}, gx_{2n_{k-1}})[1+d(x_{2m_k}, fx_{2m_k})]}{1+d(x_{2m_k}, x_{2n_{k-1}})}\},$

and $N_2(x_{2m_k}, x_{2n_{k-1}}) = \min\{d(x_{2m_k}, fx_{2m_k}), d(x_{2m_k}, gx_{2n_{k-1}})\}$.

On letting limit superior as $k \to \infty$ and using (4.14), we get

$\limsup_{k\to\infty} M_3(x_{2m_k}, x_{2n_{k-1}}) = \limsup_{k\to\infty} M_4(x_{2m_k}, x_{2n_{k-1}}) = \limsup_{k\to\infty} d(x_{2m_k}, x_{2n_{k-1}})$.
and $\limsup_{k\to\infty} N_2(x_{2m_k}, x_{2n_{k-1}}) = 0$.
Now, we consider $d(x_{2m_k}, x_{2n_{k-1}}) \le s[d(x_{2m_k}, x_{2n_{k-2}}) + d(x_{2n_{k-2}}, x_{2n_{k-1}})]$.
On taking limit superior as $k \to \infty$, we get $\limsup_{k\to\infty} d(x_{2m_k}, x_{2n_{k-1}}) \le s\varepsilon$.
On letting limit superior as $k \to \infty$ in (4.16), we get

$$\begin{aligned} s(\tfrac{\varepsilon}{s}) &\le \limsup_{k\to\infty} \beta(M_3(x_{2m_k}, x_{2n_{k-1}})) \limsup_{k\to\infty} M_4(x_{2m_k}, x_{2n_{k-1}}) + L \limsup_{k\to\infty} N_2(x_{2m_k}, x_{2n_{k-1}}) \\ &= \limsup_{k\to\infty} \beta(M_3(x_{2m_k}, x_{2n_{k-1}})) \limsup_{k\to\infty} d(x_{2m_k}, x_{2n_{k-1}}) \le \limsup_{k\to\infty} \beta(M_3(x_{2m_k}, x_{2n_{k-1}}))\varepsilon s. \end{aligned}$$

Therefore $\frac{1}{s} \le \limsup_{k\to\infty} \beta(M_3(x_{2m_k}, x_{2n_{k-1}})) \le \frac{1}{s}$, which implies $\limsup_{k\to\infty} \beta(M_3(x_{2m_k}, x_{2n_{k-1}})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, we have $\lim_{k\to\infty} M_3(x_{2m_k}, x_{2n_{k-1}}) = 0$.

$$\text{i.e., } \lim_{k\to\infty} d(x_{2m_k}, x_{2n_{k-1}}) = 0. \tag{4.17}$$

From the inequality (4.14) and using b-triangular property, we have

$0 < \varepsilon \le d(x_{2m_k}, x_{2n_k}) \le s[d(x_{2m_k}, x_{2n_{k-1}}) + d(x_{2n_{k-1}}, x_{2n_k})]$.

Taking limit as $k \to \infty$, using (4.14) and (4.17), we get $\varepsilon \le 0$,

which is contradiction.

Therefore $\{x_n\}$ is a b-cauchy sequence in$X$.

Since X is b-complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

Suppose f is b-continuous, we have

$fx = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} x_{2n+1} = x$.

Therefore x is a fixed point of f.

By proposition 4.2, we get that x is a unique common fixed point of f and g.

Similarly, we can prove that x is a unique common fixed point of f and g whenever g is b-continuous. □

5. Examples and Corollaries

The importance of the class of almost Geraghty-Suzuki contraction type maps is that this class properly includes the class of Geraghty-Suzuki contraction type maps so that the class of almost Geraghty-Suzuki contraction type maps is larger then the class of Geraghty-Suzuki contraction type maps, which illustrated in Example 5.1 and Example 5.2.

The following is an example in support of Theorem 4.3.

Example 5.1. Let $X=[0,1]$ and let $d: X\times X\to\mathbb{R}^+$ defined by

$$d(x,y)=\begin{cases} 0 & \text{if } x=y,\\ \frac{12}{15} & \text{if } x,y\in[0,\frac{2}{3}],\\ \frac{45}{50}+\frac{x+y}{21} & \text{if } x,y\in(\frac{2}{3},1],\\ \frac{61}{125} & \text{if otherwise.}\end{cases}$$

Then clearly(X,d) is a complete b-metric space with $s=\frac{101}{98}$.

Here we observe that when $x=\frac{9}{10}, z=1\in(\frac{2}{3},1]$ and $y\in(0,\frac{2}{3}]$,we have

$d(x,z)=\frac{45}{50}+\frac{x+z}{21}=\frac{104}{105}\nleq\frac{122}{125}=\frac{61}{125}+\frac{61}{125}=d(x,y)+d(y,z)$ so that d is not a metric.

We define $f,g: X\to X$ by

$$f(x)=\begin{cases} \frac{x(3-x)}{4} & \text{if } x\in[0,\frac{2}{3}),\\ \frac{4}{34}-x & \text{if } x\in[\frac{2}{3},1],\end{cases} \quad\text{and}\quad g(x)=\begin{cases} \frac{x+3}{4} & \text{if } x\in[0,\frac{2}{3}),\\ 1-\frac{x}{2} & \text{if } x\in[\frac{2}{3},1].\end{cases}$$

We define $\beta:[0.\infty)\to\ [0,\frac{1}{s})$ by $\beta(t)=\frac{e^{-t}}{3}$. Then $\beta\in\mathfrak{F}$.

We take $L=2$. Without loss of generality, we assume that $x\geq y$.

Case (i). $x,y\in[0,\frac{2}{3})$.

$\frac{1}{2s}\min\{d(x,fx),d(y,gy)\}=\frac{49}{101}\frac{61}{125}\leq\frac{12}{15}=d(x,y)$.

$M_1(x,y)=\max\{\frac{12}{15},\frac{12}{15},\frac{61}{125},\frac{(\frac{61}{125}+\frac{12}{15})49}{101}\}=\frac{12}{15}$

$M_2(x,y)=\max\{\frac{12}{15},\frac{12}{15},\frac{61}{125}\}=\frac{12}{15}$ and $N_1(x,y)=\min\{\frac{12}{15},\frac{61}{125},\frac{12}{15}\}=\frac{61}{125}$.

We consider

$sd(fx,gy)=\frac{101}{98}(\frac{61}{125})\leq(\frac{e^{-\frac{12}{15}}}{3})\frac{12}{15}+2(\frac{61}{125})\leq\beta(M_1(x,y))M_2(x,y)+LN_1(x,y)$.

Case (ii). $x,y\in(\frac{2}{3},1]$.

$\frac{1}{2s}\min\{d(x,fx),d(y,gy)\}=\frac{49}{101}(\frac{61}{125})\leq\frac{45}{50}+\frac{x+y}{21}=d(x,y)$.

$M_1(x,y)=\max\{\frac{45}{50}+\frac{x+y}{21},\frac{61}{125},\frac{61}{125},\frac{(\frac{61}{125}+\frac{61}{125})49}{101}\}=\frac{45}{50}+\frac{x+y}{21}$

$M_2(x,y)=\max\{\frac{45}{50}+\frac{x+y}{21},\frac{61}{125},\frac{61}{125}\}=\frac{45}{50}+\frac{x+y}{21}$ and $N_1(x,y)=\min\{\frac{61}{125},\frac{61}{125},\frac{61}{125}\}=\frac{61}{125}$.

We now consider

$$\begin{aligned} sd(fx,gy)&=\frac{101}{98}(\frac{12}{15})\\ &\leq(\frac{e^{-(\frac{45}{50}+\frac{x+y}{21})}}{3})(\frac{45}{50}+\frac{x+y}{21})+2(\frac{61}{125})\leq\beta(M_1(x,y))M_2(x,y)+LN_1(x,y).\end{aligned}$$

Case (iii). $x\in(\frac{2}{3},1], y\in[0,\frac{2}{3})$

$\frac{1}{2s}\min\{d(x,fx),d(y,gy)\}=\frac{49}{101}(\frac{61}{125})\leq\frac{61}{125}=d(x,y)$.

$M_1(x,y) = \max\{\frac{61}{125}, \frac{61}{125}, \frac{12}{15}, \frac{(\frac{12}{15}+\frac{12}{15})49}{101}\} = \frac{12}{15}$

$M_2(x,y) = \max\{\frac{61}{125}, \frac{61}{125}, \frac{12}{15}\} = \frac{12}{15}$ and $N_1(x,y) = \min\{\frac{61}{125}, \frac{12}{15}, \frac{12}{15}\} = \frac{12}{15}$.

Now we consider

$sd(fx,gy) = \frac{101}{98}(\frac{12}{15}) \leq (\frac{(e^{-\frac{12}{15}}}{3})(\frac{12}{15}) + 2(\frac{61}{125}) \leq \beta(M_1(x,y))M_2(x,y) + LN_1(x,y)$.

Case (iv). $x = \frac{2}{3}, y \in [0, \frac{2}{3})$.

$\frac{1}{2s}\min\{d(x,fx), d(y,gy)\} = 0 \leq \frac{12}{15} = d(x,y)$.

$M_1(x,y) = \max\{\frac{12}{15}, 0, \frac{61}{125}, \frac{(\frac{12}{15}+\frac{12}{15})49}{101}\} = \frac{12}{15}$

$M_2(x,y) = \max\{\frac{12}{15}, 0, \frac{61}{125}\} = \frac{12}{15}$ and $N_1(x,y) = \min\{\frac{61}{125}, \frac{12}{15}\} = \frac{61}{125}$

We now consider

$sd(fx,gy) = \frac{101}{98}(\frac{12}{15}) \leq \frac{(e^{-\frac{12}{15})}}{3}(\frac{12}{15}) + 2(\frac{61}{125}) \leq \beta(M_1(x,y))M_2(x,y) + LN_1(x,y)$.

From all the above cases we conclude that the point (f,g) is an almost Geraghty-Suzuki contraction type (I) maps.

Therefore f and g satisfy all the hypotheses of Theorem 4.3 and $\frac{2}{3}$ is unique common fixed point.

Here we observe that if $L = 0$ then the inequality (3.1) fails to hold.

For, we choose, $x = 0$ and $y = \frac{1}{2}$ we have

$d(fx,gy) = \frac{61}{125}$, $M_1(x,y) = \max\{\frac{12}{15}, 0, \frac{61}{125}, \frac{[\frac{61}{125}+\frac{12}{15}]49}{101}\} = \frac{12}{15}$ and

$M_2(x,y) = \max\{\frac{12}{15}, 0, \frac{61}{125}\} = \frac{12}{15}$

Hence we note that

$sd(fx,gy) = \frac{101}{98}(\frac{61}{125}) \not\leq \beta(\frac{12}{15})\frac{12}{15} = \beta(M_1(x,y))M_2(x,y)$ for any $\beta \in \mathfrak{F}$.

The following is an example in support of Theorem 4.4.

Example 5.2. Let $X = [0,\infty)$ and let $d : X \times X \to \mathbb{R}^+$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 4 & \text{if } x,y \in (0,1), \\ 5 + \frac{1}{x+y} & \text{if } x,y \in [1,\infty) \\ \frac{68}{25} & \text{if } otherwise. \end{cases}$$

Then clearly(X,d) is a complete b-metric space with $s = \frac{243}{240}$.

We define $f,g : X \to X$ by

$$f(x)=\begin{cases}\frac{x^3}{3}+2 & \text{if } x\in[0,1),\\ 3x^3-2 & \text{if } x\in[1,\infty),\end{cases} \quad \text{and } g(x)=\begin{cases}x^2 & \text{if } x\in[0,1),\\ \frac{1}{x^2} & \text{if } x\in[1,\infty).\end{cases}$$

Here we observe that clearly g is b-continuous..

We define $\beta:\mathbb{R}^+\to[0,\frac{1}{s})$ by $\beta(t)=\frac{e^{-t}}{3}$. Then $\beta\in\mathfrak{F}$

Case (i). $x,y\in[0,1)$

$\frac{1}{2s}\min\{d(x,fx),d(y,gy)\}=\frac{120}{243}\min\{\frac{68}{25},4\}=(\frac{120}{243})(\frac{68}{25})\leq 4=d(x,y)$

Here $M_3(x,y)=\max\{4,\frac{4(1+\frac{68}{25})}{1+4},\frac{\frac{68}{25}(1+\frac{68}{25})}{1+4}\}=4, M_4(x,y)=\max\{4,\frac{4(1+\frac{68}{25})}{1+4}\}=4$

and $N_2(x,y)=\min\{\frac{68}{25},4\}=\frac{68}{25}$

We now consider

$sd(fx,gy)=(\frac{243}{240})(\frac{68}{25})\leq\frac{e^{-4}}{3}(4)+3.(\frac{68}{25})\leq\beta(M_3(x,y))M_4(x,y)+LN_2(x,y).$

Case (ii). $x,y\in(1,\infty)$.

$\frac{1}{2s}\min\{d(x,fx),d(y,gy)\}=\frac{120}{243}\min\{5+\frac{1}{x+y},\frac{68}{25}\}=(\frac{120}{243})(\frac{68}{25})\leq 5+\frac{1}{x+y}=d(x,y).$

Here $M_3(x,y)=\max\{5+\frac{1}{x+y},\frac{\frac{68}{25}[1+5+\frac{1}{x+y}]}{1+5+\frac{1}{x+y}},\frac{5+\frac{1}{x+y}[1+5+\frac{1}{x+y}]}{(\frac{243}{240})^2(1+5+\frac{1}{x+y})}\}=5+\frac{1}{x+y},$

$M_4(x,y)=\max\{5+\frac{1}{x+y},\frac{\frac{68}{25}[1+5+\frac{1}{x+y}]}{1+5+\frac{1}{x+y}}\}=5+\frac{1}{x+y}$ and $N_2(x,y)=\min\{\frac{68}{25},5+\frac{1}{x+y}\}=\frac{68}{25}.$

We now consider

$sd(fx,gy)=\frac{243}{240}(\frac{68}{25})\leq\frac{e^{-(5+\frac{1}{x+y})}}{3}(5+\frac{1}{x+y})+3(\frac{68}{25})\leq\beta(M_3(x,y))M_4(x,y)+LN_2(x,y).$

Case (iii). $x\in[0,1),y\in(1,\infty)$

$\frac{1}{2s}\min\{d(x,fx),d(y,gy)\}=\frac{120}{243}\min\{\frac{68}{25},\frac{68}{25}\}=(\frac{120}{243})(\frac{68}{25})\leq\frac{68}{25}=d(x,y).$

$M_3(x,y)=\max\{\frac{68}{25},\frac{\frac{68}{25}[1+\frac{68}{25}]}{1+\frac{68}{25}},\frac{5+\frac{1}{x+y}[1+\frac{68}{25}]}{(\frac{243}{240})^2(1+\frac{68}{25})}\}=(5+\frac{1}{x+y})(\frac{240}{243})^2,$

$M_4(x,y)=\max\{\frac{68}{25},\frac{\frac{68}{25}[1+\frac{68}{25}]}{1+\frac{68}{25}}\}=\frac{68}{25}$ and $N_2(x,y)=\min\{5+\frac{1}{x+y},4\}=4.$

Now we consider

$sd(fx,gy)=(\frac{243}{240})(\frac{68}{25})\leq\frac{e^{-(\frac{25+\frac{5}{x+y}}{1+\frac{68}{25}})}}{3}(\frac{68}{25})+3\times 4\leq\beta(M_3(x,y))M_4(x,y)+LN_2(x,y).$

Case (iv). $x\in(1,\infty),y\in[0,1)$.

$\frac{1}{2s}\min\{d(x,fx),d(y,gy)\}=\frac{120}{243}\min\{5+\frac{1}{x+y},4\}=\frac{480}{243}\leq\frac{68}{25}=d(x,y).$

Here $M_3(x,y)=\max\{\frac{68}{25},\frac{4(1+5+\frac{1}{x+y})}{1+\frac{68}{25}},\frac{\frac{68}{25}[1+5+\frac{1}{x+y}]}{(\frac{243}{240})^2[1+\frac{68}{25}]}\}=\frac{100(6+\frac{1}{x+y})}{93},$

$M_4(x,y)=\max\{\frac{68}{25},\frac{4(1+5+\frac{1}{x+y})}{1+\frac{68}{25}}\}=\frac{100(6+\frac{1}{x+y})}{93}$ and $N_2(x,y)=\min\{\frac{68}{25},\frac{68}{25}\}=\frac{68}{25}.$

We now consider

$sd(fx,gy)=(\frac{243}{240})(\frac{68}{25})\leq\frac{e^{-(\frac{100(6+\frac{1}{x+y})}{93})}}{3}(\frac{100(6+\frac{1}{x+y})}{93})+3\times(\frac{68}{25})\leq\beta(M_3(x,y))M_4(x,y)+$

$LN_2(x,y)$.

Case (v). $x=1, y=[0,1)$

$\frac{1}{2s}\min\{d(x,fx),d(y,gy)\}=0\leq d(x,y)$

$M_3(x,y)=\max\{\frac{68}{25},\frac{4(1+0)}{1+\frac{68}{25}},\frac{\frac{68}{25}(1+0)}{(\frac{243}{240})^2(1+\frac{68}{25})}\}=\frac{100}{93}$,

$M_4(x,y)=\max\{\frac{68}{25},\frac{4(1+0)}{1+\frac{68}{25}}\}=\frac{100}{93}$ and $N_2(x,y)=\min\{\frac{68}{25},\frac{68}{25}\}=\frac{68}{25}$.

Now we consider

$sd(fx,gy)=\frac{243}{240}(\frac{68}{25})\leq\frac{e^{-\frac{100}{93}}}{3}(\frac{100}{93})+3\times\frac{68}{25}\leq\beta(M_3(x,y))+LN_2(x,y)$.

Case (vi). $x\in[0,1), y=1$.

$\frac{1}{2s}\min\{d(x,fx),d(y,gy)\}=0\leq d(x,y)$.

$M_3(x,y)=\max\{\frac{68}{25},0,\frac{(5+\frac{1}{x+y})(1+\frac{68}{25})}{(\frac{243}{240})^2(1+\frac{68}{25})}\}=\frac{(5+\frac{1}{x+y})(240)^2}{(240)^2}, M_4(x,y)=\max\{\frac{68}{25},0\}=\frac{68}{25}$ and

$N_2(x,y)=\min\{5+\frac{1}{x+y},\frac{68}{25}\}=\frac{68}{25}$.

We consider

$sd(fx,gy)=\frac{243}{240}(5+\frac{1}{x+y})\leq\frac{e^{-5+\frac{1}{x+y}\frac{(240)^2}{(243)^2}}}{3}(\frac{68}{25})+3\times\frac{68}{25}\leq\beta(M_3(x,y))M_4(x,y)+LN_2(x,y)$.

From all the above cases we conclude that the pair (f,g) is an almost Geraghty-Suzuki contraction type (II) maps. Therefore f and g satisfy all the hypotheses of Theorem 1 and $\frac{2}{3}$ is unique common fixed point.

Hence we observe that if $L=0$ then the inequality (2.3.1) fails to hold.

For, we choose $x=0$ and $y=2$ we have

$d(fx,gy)=\frac{68}{25}, M_3(x,y)=\frac{79200000}{10983114}, M_4(x,y)=\frac{68}{25}$

Here we note that

$sd(fx,gy)=\frac{4131}{1500}\nleq\beta(\frac{79200000}{10983114})\frac{68}{25}=\beta(M_3(x,y))M_4(x,y)$ for any $\beta\in\mathfrak{F}$.

Corollary 5.3. Let (X,d) be a complete b-metric space with coefficient $s\geq 1$ and let $f:X\to X$ be a selfmap satisfies the following condition:

$$\frac{1}{2s}d(x,fx)<d(x,y) \text{ implies that } sd(fx,fy)\leq\beta(M_1(x,y))M_2(x,y)+LN_1(x,y),$$

where $M_1(x,y)=\max\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(fx,y)}{2s}\}$,

$M_1(x,y)=\max\{d(x,y),d(x,fx),d(y,fy)\}$ and $N_1(x,y)=\min\{d(y,fx),d(y,fy)\}$.

Then f has a unique fixed point in X.

Proof. Follows by choosing $g=f$ in Theorem 4.3. □

Corollary 5.4. Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and let $f : X \to X$ be a selfmap satisfies the following condition:

$$\frac{1}{2s}d(x,fx) < d(x,y) \text{ implies that } sd(fx,fy) \leq \beta(M_3(x,y))M_4(x,y) + LN_2(x,y),$$

where $M_3(x,y) = \max\{d(x,y), \frac{d(y,fy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(x,fy)[1+d(x,fx)]}{s^2(1+d(x,y))}\}$,
$M_4(x,y) = \max\{d(x,y), \frac{d(y,fy)[1+d(x,fx)]}{1+d(x,y)}\}$ and $N_2(x,y) = \min\{d(x,fx), d(x,fy)\}$.
Then f has a unique fixed point in X.

Proof. Follows by choosing $g = f$ in Theorem 4.4. □

The following corollaries follows by taking $L = 0$ in Theorem 4.3 and Theorem 4.4.

Corollary 5.5. Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and let (f,g) be an Geraghty-Suzuki contraction type (I) maps. If either f (or) g is b-continuous then f and g have a unique common fixed point in X.

Corollary 5.6. Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and let (f,g) be an Geraghty-Suzuki contraction type (II) pair of maps. If either f (or) g is b-continuous then f and g have a unique common fixed point in X.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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