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PROPERTIES OF CONFORMABLE FRACTIONAL CHI-SQUARE PROBABILITY DISTRIBUTION

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Abstract. The paper introduces conformable fractional analogs of some basic concepts related to probability distributions of random variables, namely density, cumulative distribution, survival and hazard functions. Moreover, it introduces conformable fractional analogs to expected values, r^{th} moments, r^{th} central moments, mean, variance, skewness and kurtosis. In addition, it introduces conformable fractional analogs to some entropy measures, namely, Shannon, Renyi and Tsallis entropy measures together with Awad sup- entropy counter parts of these three measures. All these concepts had been applied to the conformable fractional chi- square probability distribution.

Keywords: conformable derivative; probability distribution; entropy.

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1. Introduction

Fractional derivatives proved to be very useful in many branches of sciences. Conformable fractional derivative which is a very recent definition introduced in 2014 by khalil et. al, [9] as follows:

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Given a function $f : [0,\infty) \longrightarrow \mathbb{R}$. Then for all $t > 0$, $\alpha \in (0,1)$, let

$$D_\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

$D_\alpha f$ is called the conformable fractional derivative of f of order α.

Let $f^{(\alpha)}(t)$ stand for $D_\alpha(f)(t)$.

If f is $\alpha-$ differentiable in some $(0,b)$, $b > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then let

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t).$$

The conformable derivative satisfies all the classical properties of the usaual first derivative. Further, according to this derivative, the following statements are true, see [9].

1. $D_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$,
2. $D_\alpha(\sin \frac{1}{\alpha}t^\alpha) = \cos \frac{1}{\alpha}t^\alpha$,
3. $D_\alpha(\cos \frac{1}{\alpha}t^\alpha) = -\sin \frac{1}{\alpha}t^\alpha$,
4. $D_\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}$.

We refer [1] and [9] for more on conformable fractional derivative.

2. Main Concepts

In this section we introduce basic dual definition of conformable fraction concepts of probability theory. Then we apply then to the theory of fractional distributions introduced in [10].

Definition 2.1. Let X be a random variable taking values in the interval $[L,U] \subset R^+$. Assume $0 < \alpha < 1$. A function $f : [L,U] \to R^+$ is called conformable fractional probability density function of X, $(CFPDF)$, if $f(x) \succeq 0$ for all $x \in [L,U]$, and its conformable integral $\int_L^U f(x)$ $d^\alpha x = 1$,[2].

Note that the domain of f may be open or semi-open interval, moreover U may be equal to infinity.

Definition 2.2. Let X be a random variable with domain $[L,U] \subset R^+$. Assume $0 < \alpha < 1$. A function F: $[L,U] \to R^+$ is called conformable fractional cumulative distribution function of X, $(CFCDF)$, if

1. $F(x)$ is a non-decreasing function in $[L,U]$.
2. $\lim_{x\to L^+} F(x) = 0$,

3. $\lim_{x \to U^-} F(x) = 1$,

4. $F(x)$ is right continuous.

Now we introduce some statistical concepts associated with the conformable derivative

Definition 2.3: A random variable X is called continuous if its $CFCDF$, denoted by $F(x)$ is continuous in X.

Definition 2.4: If X is a continuous random variable with domain $[L,U]$ and $CFPDF$ function $f(x)$, then it $CFCDF$ function $F(x) = \int_L^x f(y) d^\alpha y$.

Definition 2.5: Conformable fractional expectation E_α of a function $g(x)$ of a continuous random variable X whose CFPDF $f(x)$ is defined as

$E_\alpha(g(X)) = \int_L^U g(x)\, f(x)\, d^\alpha x.$

Definition 2.6: The Mode of the conformable fraction distribution is the point in its domain that maximizes the CFPDF.

It is interesting to note that all these definitions reduce to their traditional counterparts as $\lambda \to 1$.

now, we introduice three conformable fractional entropy measures:

Definition 2.7: Conformable Fractional Shannon entropy of a random variable X whose CFPDF f(x) is defined as

$\alpha H = -E_\alpha(Log(f(X))).$

Definition 2.8: Conformable Fractional Tsallis entropy of a random variable X whose CF-PDF f(x) is defined as

$\alpha T = \frac{1}{1-\lambda}(E_\alpha((f(X))^{\lambda-1}) - 1).$

Definition 2.9: Conformable Fractional Renyi entropy of a random variable X whose CFPDF f(x) is defined as

$\alpha R = \frac{1}{1-\lambda} Log(E_\alpha((f(X))^{\lambda-1})).$

It is interesting to note that all these entropy as $\lambda \to 1$ these measures reduces to the traditional forms.

It is well known that each of the traditional Renyi entropy R and the traditional Tsallis entropy T approaches Shannon entropy H as $\lambda \to 1$. Because in the traditional case the Shannon, the Renyi and the Tsallis entropy measures may be negative for some continuous distributions, Awad at. al. [2],[4] modified the definitions of these entropies to what is called sup- entropy measures assuming that the $sup_x(f(x))$ is finite. In the same token.Definition (2.10) provides the corresponding modified conformable fractional entropy measures. Moreover, it is well known that each of the traditional Renyi entropy R and the traditional Tsallis entropy T approaches the traditional Shannon entropy H as $\lambda \to 1$. For more details on information see e.g. [2] and [10].

So the question that arises here is that: Does this fact hold for the conformable fraction counterparts? In this paper, we will investigate this question.

Definition 2.10: Consider a random variable X whose $CFPDF$ is $f(x)$ and assume that its supremum over its domain $[L,U]$ is finite and is denoted by s. Assume $\lambda > 0$ and $\lambda \neq 1$.

The Conformable Fractional Awad-Shannon sup entopy:

$\alpha AH = -E_\alpha(Log(\frac{f(X)}{s}))$.

The Conformable Fractional Awad-Tsallis sup entopy:

$\alpha AT = \frac{1}{1-\lambda}(E_\alpha((\frac{f(X)}{s})^{\lambda-1}) - 1)$.

The Conformable Fractional Awad-Renyi sup entopy:

$\alpha AR = \frac{1}{1-\lambda}Log(E_\alpha((\frac{f(X)}{s})^{\lambda-1}))$.

Concerning αAT and αAR, we will investigate their relations to αAH as $\lambda \to 1$.Moreover we will investigate the values of αAH , αAT and αAR that are obtained from Conformable Fractional Chi-Square Probability Distribution together with their limits when $\alpha \to 1$.

we need in the sequal the following lemma.

Lemma 2.1

$\alpha AH = \alpha H + \log(s)$

$\alpha AT = \frac{1}{1-\lambda}(\frac{1+(1-\lambda)\alpha T}{s^{\lambda-1}} - 1)$

$\alpha AR = \alpha R + \log(s)$

Proof: It is obvious from Definition 2.10.

3. APPLICATIONS TO CONFORMABLE $\alpha-$CHI-SQUARE DISTRIBUTION

Abu Hammad et al.[10], solved the conformable differential equation

$$2xD^\alpha y + (-k + x^\alpha + 2)y = 0, \text{ where } x > 0; \quad 0 < \alpha < 1$$

They obtained that

$$y = Ax^{(\frac{k}{2})-1}\exp(\frac{-x^{\alpha}}{2\alpha}) \text{ where } \quad A > 0$$

Set $f_{\alpha}(x) = Ax^{(\frac{k}{2})-1}\exp(\frac{-x^{\alpha}}{2\alpha})$; for $f_{\alpha}(x)$ to be a conformable fractional probability distribution function ($CFPDF$) with support $(0,\infty)$, we need $\int_0^{\infty} f_{\alpha}(x)\, d^{\alpha}x = 1$.

To evaluate this integral use the substitution $z = \frac{x^{\alpha}}{2\alpha}$ to get

$$2^{\frac{-2+k+2\alpha}{2\alpha}}A\alpha^{\frac{-2+k}{2\alpha}}\Gamma(\frac{-2+k+2\alpha}{2\alpha}) = 1, \text{provided that } k > 2(1-\alpha).$$

Hence the normalizing factor is $A = \frac{(2\alpha)^{\frac{1}{\alpha}-\frac{k}{2\alpha}}}{2\Gamma(1+\frac{k-2}{2\alpha})}$ when $k > 2(1-\alpha)$.

So, if $x > 0,\ 0 < \alpha < 1$, and $k > 2(1-\alpha)$,

$$f_{\alpha}(x) = \frac{\alpha(2\alpha)^{\frac{1}{\alpha}-\frac{k}{2\alpha}-1}x^{\frac{k}{2}-1}\exp(\frac{-x^{\alpha}}{2\alpha})}{\Gamma(1+\frac{k-2}{2\alpha})} \tag{1}$$

is a (CFPDF) of a random variable X with support $(0,\infty)$.

For the rest of this paper we use the notation $X : \alpha G(\frac{k}{2}, 2\alpha)$ to denoted a continuous random variable with $CFPDF$ given by (1).

It is interesting to note that:

$$\lim_{\alpha\to 1}(\, f_{\alpha}(x)) = \frac{(2)^{-\frac{k}{2}}\, x^{\frac{k}{2}-1}\exp(\frac{-x}{2})}{\Gamma(\frac{k}{2})} \tag{2}$$

provided that $k > 0$ which is a probability distribution function of gamma distribution with parameters $(\frac{k}{2}, 2)$ because k could be any positive real number.

In particular if k is a natural number then the the distribution is called Chi- square with k degrees of freedom. It can also be called $CFPDF$ of a conformable fractional two parameter gamma distribution.

In this section we obtain the basic conformable fractional probabilistic properties of this distribution.

3.1 The mode of the conformable fractional distribution .

The $CFPDF$ of X is given by (1). Because f_{α} is differentiable in the classical sense we can evaluate its critical points and classify them in the same procedure that is used in classical calculus.

The logarithm to base e of $f_\alpha(x)$ is

$$\begin{aligned} log(f_\alpha(x)) &= \frac{-1}{2\alpha}(x^\alpha + klog(2) - log(4) + 2\alpha log(k-2) - (k-2)\alpha log(x) - 2log(\alpha) \\ &\quad + klog(\alpha) - 2\alpha log(\alpha) + 2\alpha log(\Gamma(\frac{k-2}{2\alpha}))) \qquad (3) \end{aligned}$$

The conformable fractional derivative of $log(f_\alpha(x))$ is

$$D^\alpha(log(f_\alpha(x))) = -\frac{1}{2}\,(2-k+x^\alpha)\,x^{-\alpha} \qquad (4)$$

Set this derivative equals zero to get the critical point $x_0 = (k-2)^{\frac{1}{\alpha}}$ which is an interior point of the domain of f_α when $k > 2$.

Note that $k = 0$ is not allowed but k may be very close to zero when α is very close to 1 because we imposed in the definition of $f_\alpha(x)$ the condition: $k > 2(1-\alpha)$. If $k < 2$, $D^\alpha(f_\alpha(x)) < 0$ for all x. Hence both $log(f_\alpha(x))$ and $f_\alpha(x)$ are decreasing in x and the supermum of $f_\alpha(x)$ does not exist as a finite real number.

The classical second derivative test ensures that $x_0 = (k-2)^{\frac{1}{\alpha}}$ is a point at which $f_\alpha(x)$ has its maximum value. This discussion allows us to state the following lemma.

Lemma 3.1

Let $X : \alpha G(\frac{k}{2}, 2\alpha)$ and denote by s the $\sup_x(f_\alpha(x))$ in the domain of $f_\alpha(x)$. Assume that $k > 2$. Then

$$s = f_\alpha(x_0) = \frac{\alpha(\frac{2\alpha e}{k-2})^{\frac{1}{\alpha}-\frac{k}{2\alpha}}}{(k-2)\Gamma(\frac{k}{2\alpha}-\frac{1}{\alpha})} \qquad (5)$$

3.2 The conformable fractional cumulative distribution function (CFCDF) αCDF.

Let $X : \alpha G(\frac{k}{2}, 2\alpha)$. Because $f_\alpha(y)$ is differentiable in the classical sense, the $CFCDF$ of X is

$$F_\alpha(x) = \int_0^x f_\alpha(y)d^\alpha y = \int_0^x \frac{f_\alpha(y)}{y^{1-\alpha}}\,dy$$

To evaluate this integral use the transformation $z = \frac{y^\alpha}{2\alpha}$ to get

$$F_\alpha(x) = \int_0^{\frac{x^\alpha}{2\alpha}} \frac{z^{\frac{k-2}{2\alpha}} e^{-z}}{\Gamma(1+\frac{k-2}{2\alpha})}dz = 1 - \frac{\Gamma(1+\frac{k-2}{2\alpha}, \frac{x^\alpha}{2\alpha})}{\Gamma(1+\frac{k-2}{2\alpha})} \qquad (6)$$

It is easy to see that $F_\alpha(x)$ satisfies Definition 2.2.

3.3 The conformable fractional survival function ($CFSF$) S_α.

The conformable fractional survival function ($CFSF$) of X is defined as $S_\alpha(x) = 1 - F_\alpha(x)$.

Let $X : \alpha G(\frac{k}{2}, 2\alpha)$. Hence

$$S_\alpha(X) = \frac{\Gamma(1+\frac{k-2}{2\alpha}, \frac{x^\alpha}{2\alpha})}{\Gamma(1+\frac{k-2}{2\alpha})} \text{ provided that } k > 2(1-\alpha). \tag{7}$$

3.4 The conformable fractional hazard function ($CFHF$) h_α.

The conformable fractional hazard function ($CFHF$) of X is defined as $h_\alpha(x) = \frac{f_\alpha(x)}{S_\alpha(x)}$

Let $X : \alpha G(\frac{k}{2}, 2\alpha)$. Hence

$$h_\alpha(x) = \frac{(2\alpha)^{\frac{1}{\alpha}-\frac{k}{2\alpha}} x^{\frac{k}{2}-1} \exp(\frac{-x^\alpha}{2\alpha})}{2\Gamma(1+\frac{k-2}{2\alpha}, \frac{x^\alpha}{2\alpha})} \text{ provided that } k > 2(1-\alpha). \tag{8}$$

It can be seen that for $k > 2$, $h_\alpha(x) \to 0$ as $x \to 0$ and $h_\alpha(x) \to \frac{1}{2}$, as $x \to \infty$ and $h_\alpha(x)$ takes values between 0 and $\frac{1}{2}$.

3.5 Conformable fractional expectation E_α :

3.5.1 r^{th} $\alpha-$ Moment $(E_\alpha(X^r))$.

Using Definition 5.1 the conformable fractional r^{th} moment of $X : \alpha G(\frac{k}{2}, 2\alpha)$ is defined as $\alpha\mu(r) = E_\alpha(X^r)$. Because $f_\alpha(y)$ is differentiable in the classical sense, then using the transformation $z = \frac{y^\alpha}{2\alpha}$ we get

$$\alpha\mu(r) = E_\alpha(X^r) = \int_0^\infty x^r \; f_\alpha(x) d^\alpha x = \int_0^\infty \frac{2^{-\frac{-2+k+2\alpha}{2\alpha}} x^{\frac{k}{2}+r+\alpha-2} \exp(\frac{-x^\alpha}{2\alpha}) \; \alpha^{-\frac{-2+k}{2\alpha}}}{\Gamma(1+\frac{k-2}{2\alpha})} \, dx$$

$$= \frac{2^{\frac{r}{\alpha}} \; \alpha^{\frac{r}{\alpha}} \Gamma(\frac{k+2(r+\alpha-1)}{2\,\alpha})}{\Gamma(1+\frac{k-2}{2\alpha})} \tag{9}$$

The first four conformable fractional moment of $X : \alpha G(\frac{k}{2}, 2\alpha)$ are given bellow

$$\alpha\mu(1) = \frac{2^{\frac{1}{\alpha}} \; \alpha^{\frac{1}{\alpha}} \; k \, \Gamma(\frac{k}{2\,\alpha})}{(k-2)\Gamma(\frac{k-2}{2\alpha})} \tag{10}$$

$$\alpha\mu(2) = \frac{2^{\frac{2}{\alpha}} \; \alpha^{\frac{2}{\alpha}} (k+2)\Gamma(\frac{k+2}{2\,\alpha})}{(k-2)\Gamma(\frac{k-2}{2\alpha})} \tag{11}$$

$$\alpha\mu(3) = \frac{2^{\frac{3}{\alpha}} \; \alpha^{\frac{3}{\alpha}} (k+4)\Gamma(\frac{k+4}{2\,\alpha})}{(k-2)\Gamma(\frac{k-2}{2\alpha})} \tag{12}$$

$$\alpha\mu(4) = \frac{2^{\frac{4}{\alpha}}\ \alpha^{\frac{4}{\alpha}}(k+6)\Gamma(\frac{k+6}{2\,\alpha})}{(k-2)\Gamma(\frac{k-2}{2\alpha})} \tag{13}$$

3.5.2 The conformable fractional r^{th} $\alpha-$ central moment $E_\alpha((X-\alpha\mu(1))^r)$:

The conformable fractional r^{th} central moment of X is defined as $\alpha c\mu(x) = E_\alpha((X - \alpha\mu(1))^r)$. It is clear that $\alpha c\mu(1) = 0$.For $r \geqslant 2$, these moments can be computed from r^{th} moments through the binomial expansion of $(X-\alpha\mu(1))^r$.

3.5.3 The conformable fractional standard deviation and variance.

Let $X : \alpha G(\frac{k}{2}, 2\alpha)$.Then the conformable fractional standard deviation $\alpha\sigma$ of X is defined as $\alpha\sigma = \sqrt[2]{\alpha c\mu(2)}$

$$\alpha\sigma = \frac{2^{\frac{1}{\alpha}}\ \alpha^{\frac{1}{\alpha}} \sqrt[2]{-\Gamma(1+\frac{k}{2\alpha})^2 + \Gamma(\frac{k+2+2\,\alpha}{2\,\alpha})\Gamma(\frac{k-2+2\,\alpha}{2\,\alpha})}}{\Gamma(\frac{k-2+2\,\alpha}{2\,\alpha})} \tag{14}$$

The conformable fractional variance $\alpha\sigma^2$ of X is defined as $\alpha\sigma^2 = \alpha c\mu(2)$

$$\alpha\sigma^2 = \frac{4^{\frac{1}{\alpha}}\ \alpha^{\frac{2}{\alpha}}(-\Gamma(1+\frac{k}{2\alpha})^2 + \Gamma(\frac{k+2+2\,\alpha}{2\,\alpha})\Gamma(\frac{k-2+2\,\alpha}{2\,\alpha}))}{\Gamma(\frac{k-2+2\,\alpha}{2\,\alpha})^2} \tag{15}$$

3.5.4 The conformable fractional Skewness αsk of X is defined as $\alpha sk = \frac{\alpha c\mu(3)}{\alpha\sigma^3}$

$$\alpha sk = \frac{(2\Gamma(1+\frac{k}{2\alpha})^3 - 3\Gamma(1+\frac{k-2}{2\alpha})\Gamma(1+\frac{k}{2\alpha})\Gamma(1+\frac{k+2}{2\alpha}) + \Gamma(1+\frac{k-2}{2\alpha})^2\Gamma(1+\frac{k+4}{2\alpha})}{(-\Gamma(1+\frac{k}{2\alpha})^2 + \Gamma(1+\frac{k-2}{2\alpha})\Gamma(1+\frac{k+2}{2\alpha}))^{\frac{3}{2}}} \tag{16}$$

3.5.5 The conformable fractional Kurtosis αku of X is defined as $\alpha ku = \frac{\alpha c\mu(4)}{\alpha\sigma 4}$

$$\begin{aligned}\alpha ku \quad = \quad & \frac{(-3\Gamma(1+\frac{k}{2\alpha})^4 + 6\Gamma(1+\frac{k-2}{2\alpha})\Gamma(1+\frac{k}{2\alpha})^2\Gamma(1+\frac{k+2}{2\alpha}) - 4\Gamma(1+\frac{k-2}{2\alpha})^2\Gamma(1+\frac{k}{2\alpha})\Gamma(1+\frac{k+4}{2\alpha})}{(\Gamma(1+\frac{k}{2\alpha})^2 - \Gamma(1+\frac{k-2}{2\alpha})\Gamma(1+\frac{k+2}{2\alpha}))^2} \\ & + \frac{\Gamma(1+\frac{k-2}{2\alpha})^3\Gamma(1+\frac{k+6}{2\alpha}))}{(\Gamma(1+\frac{k}{2\alpha})^2 - \Gamma(1+\frac{k-2}{2\alpha})\Gamma(1+\frac{k+2}{2\alpha}))^2} \end{aligned} \tag{17}$$

3.5.6 The conformable fractional expected value of $Log(x)$, using Mathematica 12 package we find that

$$\alpha\mu\log = E_\alpha(\log(X)) = \frac{log(2\alpha) + \psi(1+\frac{k-2}{2\alpha})}{\alpha} \tag{18}$$

where $\psi(z) = \frac{\Gamma^{\backslash}(z)}{\Gamma(z)}$ is the psi- function (polylog function).

3.6 Conformable Entropy Measures.

3.6.1 Conformable Fractional Shannon entropy αH of X:

Using definition (2.7), we get

$$\begin{aligned}\alpha H &= \frac{1}{2\alpha}(\alpha\mu(\alpha)+k\log(2)-\log(4)+2\alpha\log(k-2)-(k-2)\alpha\ \alpha\mu\log-2\log(\alpha)\\ &\quad +k\log(\alpha)-2\alpha\log(\alpha)+2\alpha\log(\Gamma(\frac{k-2}{2\alpha}))) \qquad (19)\end{aligned}$$

$\alpha\mu(\alpha)$ is given by formula (9) with $r=\alpha$, and $\alpha\mu\log$ is given by (18). Put their values in (19) and do some simplifications to get

$$\alpha H=\frac{1}{2\alpha}(k-2+2\alpha+2\alpha\log(k-2)-2\alpha\log(\alpha)+2\alpha\log(\Gamma(\frac{k-2}{2\alpha}))-(k-2)\psi(\frac{k-2+2\alpha}{2\alpha})) \qquad (20)$$

$$\lim_{\alpha\to 1}\alpha H=\frac{k}{2}+\log(k-2)+\log(\Gamma(\frac{k-2}{2}))-\frac{1}{2}(k-2)\psi(\frac{k}{2}) \qquad (21)$$

This is equal to the usual Shannon entropy of a usual gamma distribution with parameters $(\frac{k}{2},2)$.

3.6.2 Awad- Shannon sup- entropy of $X:\alpha G(\frac{k}{2},2\alpha)$.

Using (20) and the relation $\alpha AH=\alpha H+\log(s)$, we get

$$\alpha AH=\frac{1}{2\alpha}(2\alpha+\log(4)+(k-2)\log(k-2)+2\log(\alpha)-k\log(2\alpha)-(k-2)\psi(1+\frac{k-2}{2\alpha})) \qquad (22)$$

and

$$\lim_{\alpha\to 1}\alpha AH=1+\frac{1}{2}(k-2)(\log((\frac{k-2}{2}))-\psi(\frac{k}{2})) \qquad (23)$$

This limit is equal to the usual Awad sup- entropy of a usual gamma distribution with parameters $(\frac{k}{2},2)$.

3.6.3 Tsallis conformable fractional entropy

$\alpha T=\frac{1}{1-\lambda}\ (\int_0^\infty (f_\alpha(x))^\lambda d^\alpha x-1)$ **of** $X:\alpha G(\frac{k}{2},2\alpha)$.

Based on the definition (2.8), the computation the conformable fractional Tsallis entropy αT requires the calculation of $(f_\alpha(x))^\lambda$. It is clear that

$$(f_\alpha(x))^\lambda=(2)^{-\frac{(k-2)\lambda}{2\alpha}}\ (\alpha)^{\frac{(2-k+2\alpha)\lambda}{2\alpha}}(k-2)^{-\lambda}(\Gamma(\frac{k-2}{2\alpha}))^{-\lambda}\ x^{\frac{(k-2)\lambda}{2}}\exp(\frac{-x^\alpha\lambda}{2\alpha}) \qquad (24)$$

Using the transformation $z = \frac{x^\alpha \lambda}{2\alpha}$ to get

$$E_\alpha(f_\alpha(x))^\lambda = 2\alpha^\lambda \lambda^{-1-\frac{(k-2)\lambda}{2\alpha}}((k-2)\Gamma(\frac{k-2}{2\alpha}))^{-\lambda}\Gamma(\frac{(k-2)\lambda}{2\alpha}+1) \quad (25)$$

Hence from definition (2.8) we get

$$\alpha T = \frac{2(k-2)^{-\lambda}\alpha^\lambda \lambda^{-1+\frac{\frac{\lambda-k\lambda}{2}}{\alpha}}\Gamma(\frac{k-2}{2\alpha})^{-\lambda}\Gamma(\frac{(k-2)\lambda}{2\alpha}+1)-1}{1-\lambda} \quad (26)$$

and

$$\lim_{\lambda\to 1} \alpha T = \frac{1}{2\alpha}(k-2+2\alpha\log(k-2)-2\alpha\log(\alpha)+2\alpha\log(\Gamma(\frac{k-2}{2\alpha}))-(k-2)\psi(\frac{k-2}{2\alpha}))$$

Hence $\lim_{\lambda\to 1} \alpha T = \alpha H$.

So the limit of conformable fractional Tsallis entropy is equal to the conformable fractional Shannon entropy as expected

$$\lim_{\alpha\to 1} \alpha T = \frac{2(k-2)^{-\lambda}\lambda^{-1+\lambda-\frac{\lambda k}{2}}\Gamma(\frac{k-2}{2})^{-\lambda}\Gamma(\frac{(k-2)\lambda}{2}+1)-1}{1-\lambda}$$

This is equal to the usual Tsallis entropy of a usual gamma distribution with parameters $(\frac{k}{2}, 2)$.

3.6.4 Conformable Fractional Awad- Tsallis sup entropy αAT of $X : \alpha G(\frac{k}{2}, 2\alpha)$.

Using the relation $\alpha AT = \frac{1}{1-\lambda}(\frac{1+(1-\lambda)\alpha T}{s^{\lambda-1}} - 1)$ together with formulas (5) and (26) we get

$$\alpha AT = \frac{-1}{1-\lambda}(1-\frac{1}{\Gamma(\frac{k-2+2\alpha}{2\alpha})}(2e)^{\frac{(k-2)(\lambda-1)}{2\alpha}}\lambda^{-\frac{k-2+2\alpha}{2\alpha}}(\frac{(k-2)\lambda}{\alpha})^{-\frac{(k-2)(\lambda-1)}{2\alpha}}\Gamma(\frac{(k-2)\lambda}{2\alpha}+1) \quad (27)$$

Using L'Hopital rule we observe after some simplifications that as $\lambda \to 1$, we get that limit of αAT as $\lambda \to 1$ is equal to αA as expected.

$$\lim_{\alpha\to 1} \alpha AT = \frac{-1}{1-\lambda}(1-\frac{1}{\Gamma(\frac{k}{2})}(2e)^{\frac{(k-2)(\lambda-1)}{2}}\lambda^{-\frac{k}{2}}((k-2)\lambda)^{-\frac{(k-2)(\lambda-1)}{2}}\Gamma(\frac{(k-2)\lambda}{2}+1)$$

This is equal to the usual Awad-Tsallis sup entropy of a usual gamma distribution with parameters $(\frac{k}{2}, 2)$.

3.6.5 Conformable Fractional Renyi entropy αR of $X : \alpha G(\frac{k}{2}, 2\alpha)$.

Using definition (2.9) and formula (25) we get

$$\begin{aligned}\alpha R &= \frac{1}{1-\lambda}(\log(2)+\lambda\log(\alpha)+(-1-\frac{(k-2)\lambda}{2\alpha})\log(\lambda)-\lambda(\log(k-2)\\&+\log(\Gamma(\frac{(k-2)}{2\alpha}))+\log(\Gamma(1-\frac{\lambda}{\alpha}+\frac{k\lambda}{2\alpha}))) \quad (28)\end{aligned}$$

Using L'Hopital rule we observe after some simplifications that as $\lambda \to 1$, the limit of conformable fractional Renyi entropy approaches the conformable fractional Shannon entropy as expected.

3.6.6 Conformable Fractional Awad- Renyi sup entropy αAR of $X : \alpha G(\frac{k}{2}, 2\alpha)$.

Using the relation $\alpha AR = \alpha R + \log(s)$ and formulas (5) and (28) we get

$$\begin{aligned}\alpha AR &= \frac{1}{2\alpha(\lambda-1)}((k-2)(\lambda-1)\log(k-2)-(2-k)(1-\lambda)\log(\alpha)+(k-2)\lambda\log(\lambda)-(2-k)\\&+(2-k)\lambda-(2-k)(1-\lambda)\log(2)+2\alpha(\log(\Gamma(\frac{(k-2)}{2\alpha}))-\log(\Gamma(\frac{(k-2)\lambda}{2\alpha})))) \quad (29)\end{aligned}$$

Using L'Hopital rule we observe after some simplifications that as $\lambda \to 1$, the limit of conformable fractional Awad- Renyi entropy approaches the conformable fractional Awad- Shannon entropy as expected.

$$\lim_{\alpha\to 1}\alpha AR = \frac{1}{2(\lambda-1)}(k-2+(2-k)\lambda-(2-k)(1-\lambda)\log(2)$$
$$+(k-2)(\lambda-1)\log(k-2)+(k-2)\lambda\log(\lambda)+2(\log(\Gamma(\frac{(k-2)}{2}))-\log(\Gamma(\frac{(k-2)\lambda}{2}))))$$

This is equal to the usual Awad-Renyi sup entropy of a usual gamma distribution with parameters $(\frac{k}{2}, 2)$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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