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ADJACENCY MATRIX AND EIGENVALUES OF THE ZERO DIVISOR GRAPH $\Gamma(\mathbb{Z}_n)$

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Abstract. Let R be a commutative ring with non-zero identity and $Z^*(R)$ be the set of non-zero zero-divisors of R. The zero-divisor graph of R, denoted by $\Gamma(R)$, is a simple undirected graph with all non-zero zero-divisors as vertices and two distinct vertices $x,y \in Z^*(R)$ are adjacent if and only if $xy = 0$. In this paper, the eigenvalues of $\Gamma(\mathbb{Z}_n)$ for $n = p^2q^2$, where p and q are distinct primes, are investigated. Also, the girth, diameter, clique number and stability number of this graph are found.

Keywords: eigenvalues; zero-divisor graph; block matrix; adjacency matrix.

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1. Introduction

Beck [4] associated to a commutative ring R its zero-divisor graph $G(R)$ whose vertices are the zero-divisors of R (including 0) and two distinct vertices a and b are adjacent if and only if $ab = 0$. Anderson and Livingston [1] introduced and studied the subgraph $\Gamma(R)$ (of $G(R)$) whose vertices are the non-zero zero-divisors of R and the authors studied the interplay between the ring-theoretic properties of a commutative ring and the graph theoretic properties

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of its zero-divisor graph. Let $Z^*(R) = Z(R) \setminus (0)$, be the set of non-zero zero-divisors of R. The zero-divisor graph of R, denoted by $\Gamma(R)$, is a simple undirected graph with all non-zero zero-divisors as vertices and two distinct vertices $x, y \in Z^*(R)$ are adjacent if and only if $xy = 0$. Thus $\Gamma(R)$ is the null graph if and only if R is an integral domain. Mohammed Reza Ahmadi and Reza Jahani-Nehad [6] studied the energy and Wiener index of the zero divisor graph for the ring of integers modulo n, for $n = p^2$, $n = pq$, where p and q are distinct prime numbers. B. Surendranath Reddy et al.[9] have studied the eigenvalues and wiener index of $\Gamma(\mathbb{Z}_n)$, for $n = p^3$ and $n = p^2q$. In this paper, the adjacency matrix, the characteristic polynomial and the eigenvalues of $\Gamma(\mathbb{Z}_n)$ for $n = p^2q^2$ where p and q are distinct prime numbers, with $p < q$, are investigated. Also, the girth, diameter, stability number and clique number of this graph are traced.

The notations and basic definitions in graph theory are standard and are taken from the books of graph theory such as, e.g.[5] and [7].

Let G be a graph. For vertices x and y of G, let $d(x,y)$ be the length of a shortest path from x to y. Clearly $d(x,x) = 0$ and $d(x,y) = \infty$, if there is no path connecting x and y. The diameter of G is defined as $diam(G) = Sup\{d(x,y)$: x and y are vertices of $G\}$. Clique of a graph is a set of mutually adjacent vertices. The maximum size of a clique of a graph G, called the clique number of G, is denoted by $\omega(G)$. For a graph G, a stable set is a set of vertices, no two of which are adjacent. A stable set in a graph is maximum if the graph contains no larger stable set. The cardinality of a maximum stable set in a graph G is called the stability number, denoted by $\alpha(G)$. The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G. ($gr(G) = \infty$ if G contains no cycles).

The adjacency matrix of G with n vertices is the $n \times n$ matrix $A(G) = (a_{uv})$,where a_{uv} is the number of edges joining vertices u and v, each loop counting as two edges. For a simple graph, $A(G)$ is real and symmetric with entries 0 and 1, where all diagonal entries are zeroes. The eigenvalues of a square matrix A are the roots of its characteristic polynomial $det(A - \lambda I)$. The characteristic polynomial of a graph G is the characteristic polynomial of its adjacency matrix;

denoted by $\Phi(G;\lambda) = det(A - \lambda I)$.

For a natural number n, $\phi(n)$ is the number of positive integers less than n and relatively prime to n. $M_n(F)$ denotes the vector space of all square matrices of size $n \times n$ with entries from a field F. A circulant matrix of size $n \times n$, with entries a and b, $a, b \in \mathbb{R}$, denoted by $C_{(a,b,n)}$ is of the form

$$C_{(a,b,n)} = \begin{bmatrix} a & b & \dots & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{bmatrix}_{n\times n}$$

2. Preliminaries

The complexity of computing the characteristic polynomial of a $n \times n$ block matrix is often reduced to some extent by the application of the following lemmas. The basic concepts of matrix theory and lemma 2.1 and lemma 2.2 are from [8].

2.1. Block diagonal matrices.

A matrix $A \in M_n(F)$, where F is a field of numbers(real or complex); of the form

$$A = \begin{bmatrix} A_{11} & O & \dots & O \\ O & A_{22} & \dots & O \\ \vdots & & \ddots & \vdots \\ O & \dots & & A_{nn} \end{bmatrix}$$

in which $A_{ii} \in M_{n_i}(F), i = 1, 2, ...k$, $\Sigma_{i=1}^{k} n_i = n$, and all blocks above and below the block diagonal are the zero blocks, is called a block diagonal matrix.

Thus $A = A_{11} \oplus A_{22} ... \oplus A_{kk} = \bigoplus_{i=1}^{k} A_{ii}$, is the direct sum of matrices $A_{11}, A_{22}, ...A_{kk}$.

Lemma 2.1. [8] $det(\bigoplus_{i=1}^{k} A_{ii}) = \prod_{i=1}^{k} det(A_{ii})$.

In particular, if $A_{11} \in M_n(F)$ and $A_{22} \in M_m(F)$, then $det \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix} = det(A_{11}) \cdot det(A_{22})$.

Lemma 2.2. [8] If $A_{11} \in M_n(F)$ and $A_{22} \in M_m(F)$ are non singular,
then $\begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix}$

Lemma 2.3. [2, 3] Let M,N,P,Q be matrices and let M be invertible. Let $S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$,
then $detS = detM.\, det(Q - PM^{-1}N)$.
$(Q - PM^{-1}N)$ is called the Schur complement of Q in S, with respect to M.

3. ADJACENCY MATRIX OF $\Gamma(\mathbb{Z}_n)$

The zero-divisor graph of $\mathbb{Z}_n$, the ring of integers modulo n, is a simple and undirected graph. ie. even though $x \cdot x = 0$ for some $x \in Z^*(\mathbb{Z}_n)$, x is not adjacent with itself. Thus the adjacency matrix of $\Gamma(\mathbb{Z}_n)$ is a symmetric matrix with entries 0 and 1, where all diagonal entries are zeroes. In this section, the adjacency matrix of $\Gamma(\mathbb{Z}_n)$ for $n = p^2q^2$, where p and q, are distinct prime numbers with $p < q$, is investigated with an objective of computing its characteristic polynomial. Hence a special interest is taken on rearranging the non-zero zero-divisors of n, so that the vertices of the least degree corresponds to the first row of blocks in the adjacency matrix and that more blocks of zeroes occur in the upper left portion of the adjacency matrix.

Proposition 3.1. The number of nonzero zero divisors of $\mathbb{Z}_n$ is $n - \phi(n) - 1$

Proof: Let m be a positive integer. Then $m \in Z^*(\mathbb{Z}_n)$ if and only if m and n have at least one common prime factor. The co-totient function, $n - \phi(n)$ counts the number of positive integers less than or equal to n which have at least one prime factor in common with n. Hence the number of nonzero zero divisors of n is $n - \phi(n) - 1$.

Theorem 3.1. The adjacency matrix of $\Gamma(\mathbb{Z}_n)$ for $n = p^2q^2$, where p and q are distinct primes, $p < q$, is $A(\Gamma(\mathbb{Z}_n))$ =

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_4 \\ A_5 \\ A_3 \\ A_6 \\ A_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} A_1 & A_2 & A_4 & A_5 & A_3 & A_6 & A_7 \end{array} \\ \begin{bmatrix} O & O & O & O & O & O & J \\ O & O & O & O & O & J & O \\ O & O & O & J & O & O & J \\ O & O & J & O & O & J & O \\ O & O & O & O & J-I & J & J \\ O & J & O & J & J & J-I & J \\ J & O & J & O & J & J & J-I \end{bmatrix} \end{array}$$

where J is a matrix of all ones and I is an identity matrix.
The size of this matrix is $pq(p+q-1)-1$.

Proof: Let $n = p^2q^2$, $p < q$. The divisors of n are $p, q, pq, p^2, q^2, p^2q, pq^2$. By proposition 3.1, the number of non-zero zero-divisors of $\Gamma(\mathbb{Z}_{p^2q^2})$ is $pq(p+q-1)-1$. We partition the non-zero zero-divisors of $\Gamma(\mathbb{Z}_{p^2q^2})$ into seven classes as multiples of the divisors of p^2q^2 as follows.

$A_1 = \{k_1p : k_1 = 1, 2, ... pq^2 - 1$,where $p \nmid k_1$ and $q \nmid k_1\}$.
$A_2 = \{k_2q : k_2 = 1, 2, ... p^2q - 1$,where $p \nmid k_2$ and $q \nmid k_2\}$.
$A_3 = \{k_3pq : k_3 = 1, 2, ... pq - 1$,where $p \nmid k_3$ and $q \nmid k_3\}$.
$A_4 = \{k_4p^2 : k_4 = 1, 2, ... q^2 - 1$,where $q \nmid k_4\}$.
$A_5 = \{k_5q^2 : k_5 = 1, 2, ... p^2 - 1$,where $p \nmid k_5\}$.
$A_6 = \{k_6p^2q : k_6 = 1, 2, ... q - 1\}$.
$A_7 = \{k_7pq^2 : k_7 = 1, 2, ... p - 1\}$.

Using elementary number theory, it can be easily seen that the cardinality of A_1 is
$|A_1| = q(p-1)(q-1)$. Similarly,
$|A_2| = p(p-1)(q-1)$, $|A_3| = (p-1)(q-1)$, $|A_4| = q(q-1)$, $|A_5| = p(p-1)$, $|A_6| = (q-1)$, $|A_7| = (p-1)$.

We also observe that,

(1) $xy \neq 0, \forall x \in A_1$ and $\forall y \in A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$ and $xy = 0, \forall x \in A_1$ and $\forall y \in A_7$.

(2) $xy \neq 0, \forall x \in A_2$ and $\forall y \in A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_7$ and $xy = 0, \forall x \in A_2$ and $\forall y \in A_6$.

(3) $xy \neq 0, \forall x \in A_3$ and $\forall y \in A_1 \cup A_2 \cup A_4 \cup A_5$ and $xy = 0, \forall x \in A_3$ and $\forall y \in A_3 \cup A_6 \cup A_7$.

(4) $xy \neq 0, \forall x \in A_4$ and $\forall y \in A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_6$ and $xy = 0, \forall x \in A_4$ and $\forall y \in A_5 \cup A_7$.

(5) $xy \neq 0, \forall x \in A_5$ and $\forall y \in A_1 \cup A_2 \cup A_3 \cup A_5 \cup A_7$ and $xy = 0, \forall x \in A_5$ and $\forall y \in A_4 \cup A_6$.

(6) $xy \neq 0, \forall x \in A_6$ and $\forall y \in A_1 \cup A_4$ and $xy = 0, \forall x \in A_6$ and $\forall y \in A_2 \cup A_3 \cup A_5 \cup A_6 \cup A_7$.

(7) $xy \neq 0, \forall x \in A_7$ and $\forall y \in A_2 \cup A_5$ and $xy = 0, \forall x \in A_7$ and $\forall y \in A_1 \cup A_3 \cup A_4 \cup A_6 \cup A_7$.

Also for any $x, y \in A_i, i = 1,2,4,5, \quad xy \neq 0$.

These observations describe the adjacency of vertices in $\Gamma(\mathbb{Z}_{p^2q^2})$. For example, no vertex in A_1 is adjacent to vertices in A_1, A_2, A_3, A_4, A_5, or A_6 and correspondingly, we get blocks of zeros in the adjacency matrix of $\Gamma(\mathbb{Z}_{p^2q^2})$. Also, all vertices of A_1 are adjacent to every vertex of A_7 and correspondingly, we get a block of all ones and so on. We recall that the zero divisor graph of a commutative ring is a simple, undirected graph. Since $xy = 0, \forall x, y \in A_i$, for $i = 3,6,7$; $A_3 \cup A_6 \cup A_7$ induces a complete subgraph , but self adjacency of vertices are omitted. The non-zero zero divisors of n are rearranged, such that the elements of A_1 appear first and then A_2, A_4, A_5, A_3, A_6, and A_7. Thus the adjacency matrix of $\Gamma(\mathbb{Z}_{p^2q^2})$ is a 7×7 block matrix consisting of 49 blocks of zeros and ones in the following form,

$$A(\Gamma(\mathbb{Z}_n)) = \begin{bmatrix} O & O & O & O & O & O & J \\ O & O & O & O & O & J & O \\ O & O & O & J & O & O & J \\ O & O & J & O & O & J & O \\ O & O & O & O & J-I & J & J \\ O & J & O & J & J & J-I & J \\ J & O & J & O & J & J & J-I \end{bmatrix} \tag{1}$$

Thus the size of this matrix is $\Sigma_{i=1}^{7} |A_i| = pq(p+q-1) - 1$.

Theorem 3.2. Let $G = \Gamma(\mathbb{Z}_n)$, for $n = p^2q^2$, where p and q are distinct primes, $p < q$. The clique number $\omega(G) = pq - 1$.

Proof: The principal sub matrix $\begin{bmatrix} J-I & J & J \\ J & J-I & J \\ J & J & J-I \end{bmatrix}$ of $A(\Gamma(\mathbb{Z}_n))$ corresponds to a complete subgraph of maximum order , induced by the vertices in $A_3 \cup A_6 \cup A_7$. Hence the clique number of G is $\omega(G) = |A_3| + |A_6| + |A_7| = pq-1$.

Theorem 3.3. Let $G = \Gamma(\mathbb{Z}_n)$, for $n = p^2q^2$, where p and q are distinct primes, $p < q$. The stability number $\alpha(G) = p(q-1)(p+q-1)$.

Proof: Since, for any $x, y \in A_1 \cup A_2 \cup A_4$, $xy \neq 0$; no two vertices of $A_1 \cup A_2 \cup A_4$, are adjacent. Thus $A_1 \cup A_2 \cup A_4$ is a maximum independent set. Hence the stability number of G, $\alpha(G) = |A_1| + |A_2| + |A_4| = p(q-1)(p+q-1)$.

Theorem 3.4. Let $G = \Gamma(\mathbb{Z}_n)$, for $n = p^2q^2$, where p and q are distinct primes, $p < q$. The girth, $gr(G) = 3$.

Proof:Since G has a clique of cardinality $pq-1 > 3$, it contains a cycle of length 3. Hence gr(G)=3. .

It was shown in [1] that, for a commutative ring R, $diam\Gamma(R) \leq 3$. In the next theorem, it is seen that $\Gamma(\mathbb{Z}_{p^2q^2})$ attains this upper bound.

Theorem 3.5. Let $G = \Gamma(\mathbb{Z}_n)$, for $n = p^2q^2$, where p and q are distinct primes, $p < q$. The diameter, $diam(G) = 3$.

Proof: For $x \in A_1, y \in A_2$, any shortest (x,y)-path contains an intermediate vertex from A_7 and a vertex from A_6. Thus $diamG = 3$.

4. EIGENVALUES OF $\Gamma(\mathbb{Z}_{p^2q^2})$

Among the zero-divisor graphs belonging to this class, $\Gamma(\mathbb{Z}_{36})$ is of minimum order, ie for $p = 2, q = 3$ and the order of $\Gamma(\mathbb{Z}_{36})$ is 23. As the order increases, the difficulty level of extracting eigenvalues of the graph increases. In this section, two eigenvalues of $\Gamma(\mathbb{Z}_{p^2q^2})$ are found with multiplicities. Also, the polynomial of degree seven which gives the remaining eigenvalues is explored through a long computation. To make the diction precise, many steps are cut short in the proof of the next theorem.

The circulant matrix of the form $C_{(a,b,n)}$ plays a vital role in the main theorem of this paper. The following lemmas are used to find the the determinant and inverse of $C_{(a,b,n)}$. We use the properties of determinant of a square matrix , to prove the following propositions.

Proposition 4.1. Let $C_{(a,b,n)}$= $\begin{bmatrix} a & b & \dots & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{bmatrix}_{n\times n}$ be a circulant matrix of size $n\times n$; with entries a and b.

Then $detC_{(a,b,n)}$; denoted by δ, is given by $\delta=(a+(n-1)b)\,(a-b)^{n-1}$.

Proposition 4.2. If $C_{(a,b,n)}$ is nonsingular, then its inverse is given by

$$C^{-1}_{(a,b,n)}=\frac{1}{\delta}\begin{bmatrix} \delta_{n-1} & \Delta_{n-1} & \dots & \Delta_{n-1} \\ \Delta_{n-1} & \delta_{n-1} & \dots & \Delta_{n-1} \\ \vdots & & \ddots & \vdots \\ \Delta_{n-1} & \dots & & \delta_{n-1} \end{bmatrix} = \frac{1}{\delta}C(\delta_{n-1},\Delta_{n-1},n),$$

where $\delta_{n-1}=(a+(n-2)b)\,(a-b)^{n-2}$ and $\Delta_{n-1}=-b\cdot(a-b)^{n-2}$.

Theorem 4.1. Let $G=\Gamma(\mathbb{Z}_{p^2q^2})$ and let λ be an eigenvalue of G. Then $\lambda=0$ and $\lambda=-1$ are eigenvalues of G with multiplicities $pq(p+q-2)-4$ and $pq-4$ respectively. If $\lambda\neq 0$, $\lambda\neq -1$, then λ satisfies $\phi(\lambda)=\lambda^7-b_6\lambda^6+b_5\lambda^5+b_4\lambda^4+b_3\lambda^3-b_2\lambda^2+b_1\lambda+b_0=0$, where

$b_6=pq-4, b_5=\{p^2(q-1)-q(2p+1)+6\}-(p-1)(q-1)(3pq+p+1)$,

$b_4=\{pq(p-1)(q-1)(3pq-p-q-8)\}-pq+2$,

$b_3=pq(p-1)(q-1)\left\{pq(3pq-4p-4q+7)+p^2+q^2-9\right\}$,

$b_2=\{pq(p-1)(q-1)\}^2(3pq-3p-3q-2)+pq(p-1)(q-1)\left\{pq(2p+2q+1)-2(p+q)^2+4\right\}$,

$b_1=pq(p-1)^3(q-1)^4(pq+p+q-p^2)+\{pq(p-1)(q-1)\}^2\left\{1-p(p-1)(q-1)^2-p(q-1)\right\}+\left\{pq(p-1)^2(q-1)^2\right\}\{pq-(p+q)\}\{p(p-2)(q-1)-q(q-2)\}$,

$b_0=p^2q^2(p-1)^4(q-1)^4(pq-(p+q))$.

Proof: Let the adjacency matrix of $G=\Gamma(\mathbb{Z}_{p^2q^2})$; given by equation (1) be denoted by M. Then the eigenvalues of G are given by $|M-\lambda I|=0$.

Thus

(2)

$$M-\lambda I=\left[\begin{array}{ccc|cccc} C_{(-\lambda,0)} & O & O & O & O & O & J \\ O & C_{(-\lambda,0)} & O & O & O & J & O \\ O & O & C_{(-\lambda,0)} & J & O & O & J \\ \hline O & O & J & C_{(-\lambda,0)} & O & J & O \\ O & O & O & O & C_{(-\lambda,1)} & J & J \\ O & J & O & J & J & C_{(-\lambda,1)} & J \\ J & O & J & O & J & J & C_{(-\lambda,1)} \end{array}\right]=\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

By Lemma 2.3, if $\lambda \neq 0$,

$$det(M-\lambda I)=detA\cdot det(D-CA^{-1}B) \tag{3}$$

Since $A=\begin{bmatrix} C_{(-\lambda,0)} & O & O \\ O & C_{(-\lambda,0)} & O \\ O & O & C_{(-\lambda,0)} \end{bmatrix}$ is a scalar matrix of size $p(q-1)(p+q-1)$,

$$detA=(-\lambda)^{p(q-1)(p+q-1)} \tag{4}$$

(Note that size of A is sum of cardinalities of A_1, A_2 and A_4 which is equal to $p(q-1)(p+q-1)$.

Also $A^{-1}=\dfrac{-1}{\lambda}I$, where I is the identity matrix of size $p(q-1)(p+q-1)$.

Let $a=|A_4|=q(q-1)$, $\quad b=|A_2|=p(p-1)(q-1)$, $\quad c=|A_1|=q(p-1)(q-1)$.

Then $CA^{-1}B = \dfrac{-1}{\lambda}CB = \dfrac{-1}{\lambda}\begin{bmatrix} a & \cdots & a & 0 & \cdots & 0 & 0 & \cdots & 0 & a & \cdots & a \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a & \cdots & a & 0 & \cdots & 0 & 0 & \cdots & 0 & a & \cdots & a \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & b & \cdots & b & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & b & \cdots & b & 0 & \cdots & 0 \\ a & \cdots & a & 0 & \cdots & 0 & 0 & \cdots & 0 & a+c & \cdots & a+c \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a & \cdots & a & 0 & \cdots & 0 & 0 & \cdots & 0 & a+c & \cdots & a+c \end{bmatrix}$

Hence $D - CA^{-1}B$=

$$\begin{bmatrix} \frac{a}{\lambda}-\lambda & \frac{a}{\lambda} & \cdots & \frac{a}{\lambda} & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & \frac{a}{\lambda} & \cdots & \cdots & \frac{a}{\lambda} \\ \frac{a}{\lambda} & \frac{a}{\lambda}-\lambda & \cdots & \frac{a}{\lambda} & \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\ \frac{a}{\lambda} & \cdots & \cdots & \frac{a}{\lambda}-\lambda & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & \frac{a}{\lambda} & \cdots & \cdots & \frac{a}{\lambda} \\ 0 & \cdots & \cdots & 0 & -\lambda & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 \\ \vdots & & & \vdots & 1 & -\lambda & \cdots & 1 & \vdots & & & \vdots & \vdots & & & \vdots \\ \vdots & & & \vdots & \vdots & & \ddots & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & -\lambda & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & \frac{b}{\lambda}-\lambda & \frac{b}{\lambda}+1 & \cdots & \frac{b}{\lambda}+1 & 1 & \cdots & \cdots & 1 \\ \vdots & & & \vdots & \vdots & & & \vdots & \frac{b}{\lambda}+1 & \frac{b}{\lambda}-\lambda & & \vdots & \vdots & & & \vdots \\ \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & \ddots & \vdots & \vdots & & & \vdots \\ 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & \frac{b}{\lambda}+1 & \cdots & \cdots & \frac{b}{\lambda}-\lambda & 1 & \cdots & \cdots & 1 \\ \frac{a}{\lambda} & \cdots & \cdots & \frac{a}{\lambda} & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & \frac{a+c}{\lambda}-\lambda & \frac{a+c}{\lambda}+1 & \cdots & \frac{a+c}{\lambda}+1 \\ \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots & \frac{a+c}{\lambda}+1 & \frac{a+c}{\lambda}-\lambda & & \vdots \\ \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & \ddots & \vdots \\ \frac{a}{\lambda} & \cdots & \cdots & \frac{a}{\lambda} & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & \frac{a+c}{\lambda}+1 & \cdots & \cdots & \frac{a+c}{\lambda}-\lambda \end{bmatrix}$$

Thus,

$$D-CA^{-1}B=\left[\begin{array}{cc|cc} {X_1}_{p(p-1)} & O_{p(p-1)\times(p-1)(q-1)} & Q_{p(p-1)\times(q-1)} & Y_{p(p-1)\times(p-1)} \\ O_{(p-1)(q-1)\times p(p-1)} & {X_2}_{(p-1)(q-1)} & R_{(p-1)(q-1)\times(q-1)} & S_{(p-1)(q-1)\times(p-1)} \\ \hline Q^T_{(q-1)\times p(p-1)} & R^T_{(q-1)\times(p-1)(q-1)} & X_{3(q-1)} & U_{(q-1)\times(p-1)} \\ Y^T_{(p-1)\times p(p-1)} & S^T_{(p-1)\times(p-1)(q-1)} & U^T_{(p-1)\times(q-1)} & {X_4}_{(p-1)} \end{array}\right]$$

where $X_1=C_{\left(\frac{a}{\lambda}-\lambda,\ \frac{a}{\lambda},\ p(p-1)\right)}$, $X_2=C_{(-\lambda,\ 1,\ (p-1)(q-1))}$, $X_3=C_{\left(\frac{b}{\lambda}-\lambda,\ \frac{b}{\lambda}+1,\ q-1\right)}$, $X_4=C_{\left(\frac{a+c}{\lambda}-\lambda,\ \frac{a+c}{\lambda}+1,\ p-1\right)}$, $Y=\frac{a}{\lambda}J_{p(p-1)\times(p-1)}$, where J is a matrix of all ones and Q, R, S, U are matrices of all ones.

Let n , m be the size of X_1 and X_2 respectively. Hence $n=p(p-1), m=(p-1)(q-1)$. While doing tedious computations hereafter, the following substitutions are made.

$f(\lambda)=na-\lambda^2$,

$g(\lambda)=m-1-\lambda$, and

$h(\lambda)=f(\lambda)g(\lambda)\left\{p(p-1)(q-1)^2+\lambda(q-2)-\lambda^2\right\}-(q-1)\{n\lambda^2g(\lambda)+m\lambda f(\lambda)\}$.

Using Proposition 4.1 and 4.2, it can be seen that ,

$$det X_1=(-1)^{n-1}(\lambda)^{n-2}f(\lambda) \tag{5}$$

$$det X_2=(-1)^{m-1}(\lambda+1)^{m-1}g(\lambda) \tag{6}$$

$X_1^{-1}=\frac{-1}{f(\lambda)}\cdot C_{\left(\frac{(n-1)a}{\lambda}-\lambda,\ \frac{-a}{\lambda},\ n\right)}$, $X_2^{-1}=\frac{-1}{(\lambda+1)g(\lambda)}\cdot C_{(m-\lambda-2,\ -1,\ m)}$.

Using Lemma 2.1, Lemma 2.2 and Lemma 2.3, we see that

(7)

$$det\left(D-CA^{-1}B\right)=det X_1\cdot\ det X_2\cdot\ det\left(\begin{bmatrix} X_3 & U \\ U^T & X_4\end{bmatrix}-\begin{bmatrix} Q^T & R^T \\ Y^T & S^T\end{bmatrix}\begin{bmatrix} X_1^{-1} & O \\ O & X_2^{-1}\end{bmatrix}\begin{bmatrix} Q & Y \\ R & S\end{bmatrix}\right)$$

Applying Lemma 2.3 again, $det\left(\begin{bmatrix} X_3 & U \\ U^T & X_4\end{bmatrix}-\begin{bmatrix} Q^T & R^T \\ Y^T & S^T\end{bmatrix}\begin{bmatrix} X_1^{-1} & O \\ O & X_2^{-1}\end{bmatrix}\begin{bmatrix} Q & Y \\ R & S\end{bmatrix}\right)$ gets surprisingly simplified to be the determinant of the circulant matrix $C_{(v-z,\ w-z,\ p-1)}$, where

$z=\{f(\lambda)g(\lambda)-(nag(\lambda)+mf(\lambda))\}^2\,\dfrac{\lambda(q-1)}{f(\lambda)g(\lambda)h(\lambda)}$ and

$v = \dfrac{f(\lambda)g(\lambda)(a+c-\lambda^2) - m\lambda f(\lambda) - na^2 g(\lambda)}{\lambda f(\lambda)g(\lambda)}$

Also, applying Proposition 4.1,

$$detC_{(v-z,\ w-z,\ p-1)} = (-1)^{p-2}(\lambda+1)^{p-2}\{(v-z)(p-1)+(\lambda+1)(p-2)\} \tag{8}$$

Using eqautions $(4),(5),(6),(7)$ and (8) in (3), the characteristic equation of $\Gamma(Z_n)$ is $(\lambda)^{pq(p+q-2)-4} \cdot (\lambda+1)^{pq-4} \cdot \phi(\lambda) = 0$, where

$$\phi(\lambda) = \lambda^7 - b_6\lambda^6 + b_5\lambda^5 + b_4\lambda^4 + b_3\lambda^3 - b_2\lambda^2 + b_1\lambda + b_0 = 0, \tag{9}$$

where

$b_6 = pq-4, b_5 = \{p^2(q-1) - q(2p+1) + 6\} - (p-1)(q-1)(3pq+p+1),$

$b_4 = \{pq(p-1)(q-1)(3pq-p-q-8)\} - pq + 2,$

$b_3 = pq(p-1)(q-1)\left\{pq(3pq-4p-4q+7) + p^2 + q^2 - 9\right\},$

$b_2 = \{pq(p-1)(q-1)\}^2(3pq-3p-3q-2) + pq(p-1)(q-1)\left\{pq(2p+2q+1) - 2(p+q)^2 + 4\right\},$

$b_1 = pq(p-1)^3(q-1)^4(pq+p+q-p^2) + \{pq(p-1)(q-1)\}^2\left\{1 - p(p-1)(q-1)^2 - p(q-1)\right\} + \left\{pq(p-1)^2(q-1)^2\right\}\{pq-(p+q)\}\{p(p-2)(q-1) - q(q-2)\},$

$b_0 = p^2q^2(p-1)^4(q-1)^4(pq-(p+q))$

Remark 4.1. The algebraic multiplicities of the eigenvalues $\lambda = 0$ and $\lambda = -1$ can be found using the tools of Linear Algebra also. Since $A(G)$ is real and symmetric, the algebraic multiplicity of an eigenvalue is the same as its geometric multiplicity [7]. Performing elementary row transformations, it can be seen that $detM = 0$. Hence $\lambda = 0$ is an eigenvalue of G . The geometric multiplicity of $\lambda = 0$ is the nullity of M. From equation (1), it is clear that the number of zero rows of M is $|A_1| + |A_2| + |A_4| + |A_5| - 4$. Hence the nullity of M is $pq(p+q-2) - 4$.

In a similar way, it can be seen that the algebraic multiplicity of $\lambda = -1$ is the nullity of $M+I$, which is equal to $|A_3| + |A_6| + |A_7| - 3 = pq - 4$.

Since $\phi(\lambda)$ is a polynomial of degree 7, the total number of eigenvalues is $(pq(p+q-2)-4) + (pq-4) + 7$, which is found to be the same as $\sum_{i=1}^{7} |A_i|$.

Remark 4.2. Since M is a symmetric matrix with all diagonal entries zero, the sum of the eigenvalues is equal to the trace of M, which is zero [7]. The non-zero eigenvalues of M are precisely -1 and the zeros of $\phi(\lambda)$. Also, from equation(9), the sum of the roots of $\phi(\lambda)=0$; is $pq-4$. Hence it is convinced that sum of the eigenvalues of M is $(pq-4)(-1)+(pq-4)=0$.

CONCLUSION

This is an attempt to explore the characteristic polynomial and the eigenvalues of the zero-divisor graph of a class of commutative rings $\mathbb{Z}_n$ for $n=p^2q^2$ for any distinct primes p and q, for example, $\mathbb{Z}_{36}$, $\mathbb{Z}_{100}$, $\mathbb{Z}_{225}$, etc.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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