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SOME COMMON FIXED POINT THEOREMS FOR TWO PAIRS OF WEAKLY COMPATIBLE MAPPINGS SATISFYING ϕ-WEAKLY CONTRACTIVE CONDITIONS

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Abstract. In this paper, we introduce the concept of ϕ-weakly contractive condition relative to four mappings A,B,S and T in $b-$metric space. We also prove the existence and uniqueness of common fixed point for two pairs of mappings satisfying ϕ-weakly contractive condition by providing some examples.

Keywords: common fixed point; ϕ-weakly contractive conditions; weakly compatible mappings; $b-$metric space.

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1. Introduction and Preliminaries

Gerald Jungck [1] introduced the concept of compatible mappings by generalizing the concept of commuting mappings. There are various generalizations of compatible mappings and these can be found in the literature ([2]-[4]). Weakly compatible [5] is also one of the weaker form of compatible mappings. Following is the definition of weakly compatible mappings.

Definition 1.1. *([5]) A pair of self mappings f and g in a metric space (X,d) are said to be weakly compatible if $ft = gt$ implies $fgt = gft$ for some $t \in X$.*

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Banach contraction principle is one of the most important result for finding fixed point.Let (X,d) be a metric space and S,T be two self mappings on (X,d). A point $z \in X$ is said to be a common fixed point of S and T if $Sz = Tz = z$.

$b-$metric space or metric type spaces called by some authors was introduced by Bakhtin [6] in 1989 and extended by Czerwik [7] in 1993. Since then, several papers have been published on the fixed point theory in such spaces. The definition of $b-$metric and some properties are given below:

Definition 1.2. [7] *Let X be a non-empty set and $d : X \times X \to [0,\infty)$ be a function satisfying the following conditions :*

(i) $d(x,y) = 0$ *if and only if* $x = y$.

(ii) $d(x,y) = d(y,x)$.

(iii) $d(x,y) \leq s[d(x,z) + d(z,y)], \forall x,y,z \in X$, *where* $s \geq 1$ *is a real number.*

The function d is called a $b-$metric and the space (X,d) is called a $b-$metric space, in short, bMS.

Definition 1.3. [8] *Let (X,d) be a metric space. Then a sequence $(x_n)_{n\in\mathbf{N}}$ in X is said to be*

(i) convergent if and only if there exists $x \in X$ such that $d(x_n,x) \to 0$ as $n \to \infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$.

(ii) Cauchy if and only if $d(x_n,x_m) \to 0$ as $m,n \to \infty$.

(iii) complete if every Cauchy sequence in X converges in X.

Rhoades [9] introduced the concept of ϕ-weakly contractive mappings by generalising the Banach fixed point theorem. In this paper, we introduce the concept of ϕ-weakly contractive condition for two pairs of weakly compatible mappings and proved some unique common fixed point theorems.

Throughout this paper, $\mathbf{N}$ denotes the set of all positive integers, $\mathbf{N}_0 = \{0\} \cup \mathbf{N}, \mathbf{R}^+ = [0,\infty)$ and $\Phi = \big\{\phi : \phi : \mathbf{R}^+ \to \mathbf{R}^+$ is upper semi continuous, and $\lim\limits_{n\to\infty} a_n = 0$ for each sequence $\{a_n\}_{n\in\mathbf{N}} \subset \mathbf{R}^+$ with $a_{n+1} \leq \phi(a_n), \forall n \in \mathbf{N}\big\}$.

Lemma 1.1. [10] *Let $\phi \in \Phi$. Then $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$.*

Zeqing Liu et al. [11] introduced the concept of ψ-weakly contractive conditions relative to four mappings A,B,S and T in a metric space (X,d) as

$$d(Tx,Sy) \leq \psi\big(M_i(x,y)\big),\ \forall x,y \in X, \tag{1}$$

where $i = 1,2,3.$, $\psi \in \Phi$.

$$\begin{aligned} M_1(x,y) &= \max\Big\{d(Ax,By), d(Ax,Tx), d(By,Sy), \frac{1}{2}[d(Ax,Sy)+d(Tx,By)], \frac{d(Ax,Sy)d(Tx,By)}{1+d(Ax,By)}, \\ &\quad \frac{d(Ax,Tx)d(By,Sy)}{1+d(Ax,By)}, \frac{1+d(Ax,Sy)+d(Tx,By)}{1+d(Ax,Tx)+d(By,Sy)}d(Ax,Tx)\Big\}, \forall x,y \in X, \end{aligned} \tag{2}$$

$$\begin{aligned} M_2(x,y) &= \max\Big\{d(Ax,By), d(Ax,Tx), d(By,Sy), \frac{1}{2}[d(Ax,Sy)+d(Tx,By)], \\ &\quad \frac{1+d(Ax,Tx)}{1+d(Ax,By)}d(By,Sy), \frac{1+d(By,Sy)}{1+d(Ax,By)}d(Ax,Tx), \\ &\quad \frac{1+d(Ax,Sy)+d(Tx,By)}{1+d(Ax,Tx)+d(By,Sy)}d(By,Sy)\Big\}, \forall x,y \in X \end{aligned} \tag{3}$$

and

$$M_3(x,y) = max\Big\{d(Ax,By), d(Ax,Tx), d(By,Sy), \frac{1}{2}[d(Ax,Sy)+d(Tx,By)]\Big\}, \forall x,y \in X \tag{4}$$

Now we introduce the following definition of ϕ-weakly contractive condition relative to four mappings A,B,S and T in b-metric space.

Definition 1.4. *Two pairs of self mappings $\{A,B\}$ and $\{S,T\}$ in a $b-$metric space (X,d) are said to be ϕ-weakly contractive mappings if they satisfy*

$$d(Tx,Sy) \leq \phi\big(\Delta_i(x,y)\big),\ \forall x,y \in X, \tag{5}$$

where $i = 1,2,3.$ and $\phi \in \Phi$

$$\begin{aligned} \Delta_1(x,y) &= \max\Big\{d(Ax,By), d(Ax,Tx), d(By,Sy), \frac{1}{2s}[d(Ax,Sy)+d(Tx,By)], \frac{d(Ax,Sy)d(Tx,By)}{1+d(Ax,By)}, \\ &\quad \frac{d(Ax,Tx)d(By,Sy)}{1+d(Ax,By)}, \frac{1+d(Ax,Sy)+d(Tx,By)}{1+s\big(d(Ax,Tx)+d(By,Sy)\big)}d(Ax,Tx)\Big\}, \forall x,y \in X, \end{aligned} \tag{6}$$

$$\begin{aligned}\Delta_2(x,y) &= \max\Big\{d(Ax,By),d(Ax,Tx),d(By,Sy),\frac{1}{2s}[d(Ax,Sy)+d(Tx,By)],\\ &\quad \frac{1+d(Ax,Tx)}{1+d(Ax,By)}d(By,Sy),\frac{1+d(By,Sy)}{1+d(Ax,By)}d(Ax,Tx),\\ &\quad \frac{1+d(Ax,Sy)+d(Tx,By)}{1+s\big(d(Ax,Tx)+d(By,Sy)\big)}d(By,Sy)\Big\},\forall x,y\in X \end{aligned} \tag{7}$$

and

$$\Delta_3(x,y)=max\Big\{d(Ax,By),d(Ax,Tx),d(By,Sy),\frac{1}{2s}[d(Ax,Sy)+d(Tx,By)]\Big\},\forall x,y\in X. \tag{8}$$

2. MAIN RESULTS

Our main results are as follows.

Theorem 2.1. *Let $\{A,B\}$ and $\{S,T\}$ be two pairs of self mappings in a $b-$metric space (X,d) such that*

(i) $\{A,T\}$ and $\{B,S\}$ are weakly compatible;

(ii) $T(X)\subseteq B(X)$ and $S(X)\subseteq A(X)$;

(iii) one of $A(X),B(X),S(X)$ and $T(X)$ is complete;

(iv) $d(Tx,Sy)\le\phi\big(\Delta_1(x,y)\big),\forall x,y\in X$,

where ϕ is in Φ and $s>1$ is a real number. Then, A,B,S and T have a unique common fixed point in X.

Proof. Let $x_0\in X$. It follows from (ii) that there exist two sequences $\{y_n\}_{n\in\mathbf{N}}$ and $\{x_n\}_{n\in\mathbf{N}_0}$ in X such that

$$y_{2n+1}=Bx_{2n+1}=Tx_{2n},\ y_{2n+2}=Ax_{2n+2}=Sx_{2n+1},\forall n\in\mathbf{N}_0 \tag{9}$$

Put $d_n=d(y_n,y_{n+1}),\forall n\in\mathbf{N}$.

Now we prove

$$\lim_{n\to\infty}d_n=0. \tag{10}$$

Using (iv)and (9), we derive

$$d_{2n}=d(Tx_{2n},Sx_{2n-1})\le\phi\big(\Delta_1(x_{2n},x_{2n-1})\big),\forall n\in\mathbf{N} \tag{11}$$

and

$$\begin{aligned}
\Delta_1(x_{2n},x_{2n-1}) &= \max\big\{d(Ax_{2n},Bx_{2n-1}),d(Ax_{2n},Tx_{2n}),d(Bx_{2n-1},Sx_{2n-1}),\\
&\quad \frac{1}{2s}[d(Ax_{2n},Sx_{2n-1})+d(Tx_{2n},Bx_{2n-1})],\\
&\quad \frac{d(Ax_{2n},Sx_{2n-1})d(Tx_{2n},Bx_{2n-1})}{1+d(Ax_{2n},Bx_{2n-1})},\frac{d(Ax_{2n},Tx_{2n})d(Bx_{2n-1},Sx_{2n-1})}{1+d(Ax_{2n},Bx_{2n-1})},\\
&\quad \frac{1+d(Ax_{2n},Sx_{2n-1})+d(Tx_{2n},Bx_{2n-1})}{1+s\big(d(Ax_{2n},Tx_{2n})+d(Bx_{2n-1},Sx_{2n-1})\big)}d(Ax_{2n},Tx_{2n})\big\}\\
&= \max\big\{d(y_{2n},y_{2n-1}),d(y_{2n},y_{2n+1}),d(y_{2n-1},y_{2n}),\frac{1}{2s}[d(y_{2n},y_{2n})+d(y_{2n+1},y_{2n-1})],
\end{aligned}$$

$$\begin{aligned}
&\quad \frac{d(y_{2n},y_{2n})d(y_{2n+1},y_{2n-1})}{1+d(y_{2n},y_{2n-1})},\frac{d(y_{2n},y_{2n+1})d(y_{2n-1},y_{2n})}{1+d(y_{2n},y_{2n-1})},\\
&\quad \frac{1+d(y_{2n},y_{2n})+d(y_{2n+1},y_{2n-1})}{1+s\big(d(y_{2n},y_{2n+1})+d(y_{2n-1},y_{2n})\big)}d(y_{2n},y_{2n+1})\big\}\\
&= \max\big\{d_{2n-1},d_{2n},d_{2n-1},\frac{1}{2s}d(y_{2n+1},y_{2n-1}),0,\frac{d_{2n}d_{2n-1}}{1+d_{2n-1}},\frac{1+d(y_{2n+1},y_{2n-1})}{1+s(d_{2n}+d_{2n-1})}d_{2n}\big\}\\
&= \max\big\{d_{2n-1},d_{2n}\big\},\forall n\in\mathbf{N}. \qquad (12)
\end{aligned}$$

Suppose that $d_{2n_0-1} < d_{2n_0}$ for some $n_0 \in \mathbf{N}$. It follows from (11), (12), $\phi \in \Phi$, and Lemma 1.1 that

$$d_{2n_0} \le \phi\big(\Delta_1(x_{2n_0},x_{2n_0-1})\big) = \phi\big(max\{d_{2n_0-1},d_{2n_0}\}\big) = \phi(d_{2n_0}) < d_{2n_0},$$

which is a contradiction. Hence

$$d_{2n} \le d_{2n-1} = \Delta_1(x_{2n},x_{2n-1}),\forall n\in\mathbf{N}. \qquad (13)$$

Similarly we infer

$$d_{2n+1} \le d_{2n} = \Delta_1(x_{2n},x_{2n+1}),\forall n\in\mathbf{N},$$

which together with (13) ensures

$$d_{n+1} \le d_n,\forall n\in\mathbf{N},$$

which means that the sequence $\{d_n\}_{n\in\mathbf{N}}$ is non-increasing and bounded. Consequently there exists $r \ge 0$ with $\lim_{n\to\infty} d_n = r$. Suppose that $r > 0$. It follows from (11), (13), $\phi \in \Phi$, and Lemma

1.1 that

$$r = \lim_{n\to\infty} \sup d_{2n} \leq \lim_{n\to\infty} \sup \phi\left(\Delta_1(x_{2n}, x_{2n-1})\right) = \lim_{n\to\infty} \sup \phi(d_{2n-1}) \leq \phi(r) < r,$$

which is a contradiction. Hence, r= 0, that is, (10) holds.

Next we prove that $\{y_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence. Because of (10) it is sufficient to verify that $\{y_{2n}\}_{n\in\mathbf{N}}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\in\mathbf{N}}$ is not a Cauchy sequence. It follows that there exist $\varepsilon > 0$ and two sub-sequences $\{y_{2m(k)}\}_{k\in\mathbf{N}}$ and $\{y_{2n(k)}\}_{k\in\mathbf{N}}$ of $\{y_{2n}\}_{n\in\mathbf{N}}$ such that

$$2n(k) > 2m(k) > 2k,\ d(y_{2m(k)}, y_{2n(k)}) \geq \varepsilon, \forall k \in \mathbf{N}, \tag{14}$$

where 2n(k) is the smallest index satisfying (14). It follows that

$$d(y_{2m(k)}, y_{2n(k)-1}) < \varepsilon, \forall k \in \mathbf{N}. \tag{15}$$

From conditions (14),(15) and using the b-metric triangular inequality, we have,

$$\begin{aligned} \varepsilon &\leq d(y_{2m(k)}, y_{2n(k)}) \\ &\leq s\left[d(y_{2m(k)}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)})\right] \\ &< s\left[\varepsilon + d(y_{2n(k)-1}, y_{2n(k)})\right] \end{aligned} \tag{16}$$

By taking the upper limit as $k \to \infty$ in (14) and using (16), we get

$$\varepsilon \leq \lim_{k\to\infty} \sup d(y_{2m(k)}, y_{2n(k)}) < s\varepsilon \tag{17}$$

From triangular inequality, we have

$$d(y_{2m(k)}, y_{2n(k)}) \leq s[d(y_{2m(k)}, y_{2m(k)+1}) + d(y_{2m(k)+1}, y_{2n(k)})] \tag{18}$$

and

$$d(y_{2m(k)+1}, y_{2n(k)}) \leq s[d(y_{2m(k)+1}, y_{2m(k)}) + d(y_{2m(k)}, y_{2n(k)})] \tag{19}$$

By taking the upper limit as $k \to \infty$ in (14) and applying (18), (19), we get

$$\begin{aligned} \varepsilon &\leq \lim_{k\to\infty} \sup d(y_{2m(k)}, y_{2n(k)}) \\ &\leq s\big(\lim_{k\to\infty} \sup d(y_{2m(k)+1}, y_{2n(k)})\big) \end{aligned}$$

Again by taking the upper limit as $k \to \infty$ in (19), we get

$$\begin{aligned}
& \lim_{k\to\infty} \sup d(y_{2m(k)+1}, y_{2n(k)}) \\
\leq \quad & s\big(\lim_{k\to\infty} \sup d(y_{2m(k)}, y_{2n(k)})\big) \\
\leq \quad & s(s\varepsilon) = s^2\varepsilon
\end{aligned}$$

Thus

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \sup d(y_{2m(k)+1}, y_{2n(k)}) \leq s^2\varepsilon \tag{20}$$

Note that (6) and (16) yield

$$\begin{aligned}
& \lim_{k\to\infty} \sup \Delta_1(x_{2m(k)}, x_{2n(k)-1}) \\
= \quad & \lim_{k\to\infty} \sup \max\big\{d(Ax_{2m(k)}, Bx_{2n(k)-1}), d(Ax_{2m(k)}, Tx_{2m(k)}), d(Bx_{2n(k)-1}, Sx_{2n(k)-1}), \\
& \frac{1}{2s}[d(Ax_{2m(k)}, Sx_{2n(k)-1}) + d(Tx_{2m(k)}, Bx_{2n(k)-1})], \\
& \frac{d(Ax_{2m(k)}, Sx_{2n(k)-1}) d(Tx_{2m(k)}, Bx_{2n(k)-1})}{1 + d(Ax_{2m(k)}, Bx_{2n(k)-1})}, \frac{d(Ax_{2m(k)}, Tx_{2m(k)}) d(Bx_{2n(k)-1}, Sx_{2n(k)-1})}{1 + d(Ax_{2m(k)}, Bx_{2n(k)-1})} \\
& \frac{1 + d(Ax_{2m(k)}, Sx_{2n(k)-1}) + d(Tx_{2m(k)}, Bx_{2n(k)-1})}{1 + s\big(d(Ax_{2m(k)}, Tx_{2m(k)}) + d(Bx_{2n(k)-1}, Sx_{2n(k)-1})\big)} d(Ax_{2m(k)}, Tx_{2m(k)})\big\}
\end{aligned}$$

$$\begin{aligned}
= \quad & \lim_{k\to\infty} \sup \max\big\{d(y_{2m(k)}, y_{2n(k)-1}), d(y_{2m(k)}, y_{2m(k)+1}), d(y_{2n(k)-1}, y_{2n(k)}), \\
& \frac{1}{2s}[d(y_{2m(k)}, y_{2n(k)}) + d(y_{2m(k)+1}, y_{2n(k)-1})], \\
& \frac{d(y_{2m(k)}, y_{2n(k)}) d(y_{2m(k)+1}, y_{2n(k)-1})}{1 + d(y_{2m(k)}, y_{2n(k)-1})}, \frac{d(y_{2m(k)}, y_{2m(k)+1}) d(y_{2n(k)-1}, y_{2n(k)})}{1 + d(y_{2m(k)}, y_{2n(k)-1})}, \\
& \frac{1 + d(y_{2m(k)}, y_{2n(k)}) + d(y_{2m(k)+1}, y_{2n(k)-1})}{1 + s\big(d(y_{2m(k)}, y_{2m(k)+1}) + d(y_{2n(k)-1}, y_{2n(k)})\big)} d(y_{2m(k)}, y_{2m(k)+1})\big\} \\
\to \quad & \max\big\{\varepsilon, 0, 0, \frac{1}{2s}(\varepsilon + \varepsilon), \frac{\varepsilon^2}{1 + \varepsilon}, 0, 0\big\} \\
= \quad & \varepsilon \; as \; k \to \infty. \qquad (21)
\end{aligned}$$

From condition (20), we have

$$\begin{aligned}
\varepsilon &\leq \lim_{k\to\infty}\sup d(y_{2m(k)+1}, y_{2n(k)}) \\
&= \lim_{k\to\infty}\sup d(Tx_{2m(k)}, Sx_{2n(k)-1}) \\
&\leq \lim_{k\to\infty}\phi\left(\Delta_1(x_{2m(k)}, x_{2n(k)-1})\right) \\
&\leq \phi(\varepsilon) < \varepsilon
\end{aligned}$$

which is a contradiction. Hence $\{y_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence.

Assume that $A(X)$ is complete. Observe that $\{y_{2n}\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $A(X)$. Consequently there exists $(z,v)\in A(X)\times X$ with $\lim\limits_{n\to\infty} y_{2n} = z = Av$. It is easy to see

$$z = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n-1} = \lim_{n\to\infty} Ax_{2n}. \tag{22}$$

Suppose that $Tv \neq z$. Note that (6) and(22) imply

$$\begin{aligned}
\Delta_1(v, x_{2n+1}) &= \max\Big\{d(Av, Bx_{2n+1}), d(Av, Tv), d(Bx_{2n+1}, Sx_{2n+1}), \\
&\quad \frac{1}{2s}[d(Av, Sx_{2n+1}) + d(Tv, Bx_{2n+1})], \\
&\quad \frac{d(Av, Sx_{2n+1})d(Tv, Bx_{2n+1})}{1+d(Av, Bx_{2n+1})}, \frac{d(Av, Tv)d(Bx_{2n+1}, Sx_{2n+1})}{1+d(Av, Bx_{2n+1})}, \\
&\quad \frac{1+d(Av, Sx_{2n+1})+d(Tv, Bx_{2n+1})}{1+s\big(d(Av, Tv)+d(Bx_{2n+1}, Sx_{2n+1})\big)} d(Av, Tv)\Big\} \\
&\to \max\{d(Av, z), d(Av, Tv), d(z,z), \frac{1}{2s}[d(Av, z)+d(Tv, z)], \\
&\quad \frac{d(Av, z)d(Tv, z)}{1+d(Av, z)}, \frac{d(Av, Tv)d(z,z)}{1+d(Av, z)}, \frac{1+d(Av, z)+d(Tv, z)}{1+s\big(d(Av, Tv)+d(z,z)\big)} d(Av, Tv)\} \\
&= \max\{0, d(z, Tv), 0, \frac{1}{2s} d(Tv, z), 0, 0, d(z, Tv)\} \\
&= d(Tv, z) \ as\ n\to\infty
\end{aligned}$$

which together with (iv), $\phi\in\Phi$, and lemma 1.1 gives

$$d(Tv, z) = \lim_{n\to\infty}\sup d(Tv, y_{2n+2}) = \lim_{n\to\infty}\sup d(Tv, Sx_{2n+1})$$

$$\leq \lim_{n\to\infty}\sup \phi\left(\Delta_1(v, x_{2n+1})\right) \leq \phi\left(d(Tv, z)\right) < d(Tv, z),$$

which is a contradiction. Hence $Tv = z$. It follows from(ii) that there exists a point $w \in X$ with $z = Bw = Tv$. Suppose that $Sw \neq z$. In light of (6) and (22), we deduce

$$\begin{aligned}
\Delta_1(x_{2n}, w) &= \max\Big\{d(Ax_{2n}, Bw), d(Ax_{2n}, Tx_{2n}), d(Bw, Sw), \frac{1}{2s}[d(Ax_{2n}, Sw) + d(Tx_{2n}, Bw)], \\
&\quad \frac{d(Ax_{2n}, Sw)d(Tx_{2n}, Bw)}{1 + d(Ax_{2n}, Bw)}, \frac{d(Ax_{2n}, Tx_{2n})d(Bw, Sw)}{1 + d(Ax_{2n}, Bw)}, \\
&\quad \frac{1 + d(Ax_{2n}, Sw) + d(Tx_{2n}, Bw)}{1 + s\big(d(Ax_{2n}, Tx_{2n}) + d(Bw, Sw)\big)} d(Ax_{2n}, Tx_{2n})\Big\}
\end{aligned}$$

$$\begin{aligned}
&\to \max\Big\{d(z, Bw), d(z, z), d(Bw, Sw), \frac{1}{2s}[d(z, Sw) + d(z, Bw)], \\
&\quad \frac{d(z, Sw)d(z, Bw)}{1 + d(z, Bw)}, \frac{d(z, z)d(Bw, Sw)}{1 + d(z, Bw)}, \frac{1 + d(z, Sw) + d(z, Bw)}{1 + s\big(d(z, z) + d(Bw, Sw)\big)} d(z, z)\Big\} \\
&= \max\Big\{0, 0, d(z, Sw), \frac{1}{2s} d(z, Sw), 0, 0, 0\Big\} \\
&= d(z, Sw) \ as\ n \to \infty
\end{aligned}$$

which together with (iv), $\phi \in \Phi$, and Lemma 1.1 yields

$$d(z, Sw) = \lim_{n\to\infty} \sup d(y_{2n+1}, Sw) = \lim_{n\to\infty} \sup d(Tx_{2n}, Sw)$$

$$\leq \lim_{n\to\infty} \sup \phi\big(\Delta_1(x_{2n}, w)\big) \leq \phi\big(d(z, Sw)\big) < d(z, Sw),$$

which is impossible, and hence $Sw = z$. Thus (i) means $Az = ATv = TAv = Tz$ and $Bz = BSw = SBw = Sz$. Suppose that $Tz \neq Sz$. It follows from (6), (iv), $\phi \in \Phi$ and Lemma 1.1 that

$$\begin{aligned}
\Delta_1(z, z) &= \max\Big\{d(Az, Sz), d(Az, Tz), d(Bz, Sz), \frac{1}{2s}[d(Az, Sz) + d(Tz, Bz)], \\
&\quad \frac{d(Az, Sz)d(Tz, Bz)}{1 + d(Az, Bz)}, \frac{d(Az, Tz)d(Bz, Sz)}{1 + d(Az, Bz)}, \\
&\quad \frac{1 + d(Az, Sz) + d(Tz, Bz)}{1 + s\big(d(Az, Tz) + d(Bz, Sz)\big)} d(Az, Tz)\Big\} \\
&= \max\Big\{d(Tz, Sz), 0, 0, \frac{1}{2s}[d(Tz, Sz) + d(Tz, Sz)], \frac{d^2(Tz, Sz)}{1 + d(Tz, Sz)}, 0, 0\Big\} \\
&= d(Tz, Sz)
\end{aligned}$$

and

$$d(Tz, Sz) \leq \phi\big(\Delta_1(z, z)\big) = \phi\big(d(Tz, Sz)\big) < d(Tz, Sz),$$

which is a contradiction, and hence $Tz = Sz$.

Suppose that $Tz \neq z$. It follows from (6) that

$$\begin{aligned}
\Delta_1(z,w) &= \max\Big\{d(Az,Bw), d(Az,Tz), d(Bw,Sw), \frac{1}{2s}[d(Az,Sw)+d(Tz,Bw)],\\
&\quad \frac{d(Az,Sw)d(Tz,Bw)}{1+d(Az,Bw)}, \frac{d(Az,Tz)d(Bw,Sw)}{1+d(Az,Bw)},\\
&\quad \frac{1+d(Az,Sw)+d(Tz,Bw)}{1+s\big(d(Az,Tz)+d(Bw,Sw)\big)}d(Az,Tz)\Big\}\\
&= \max\Big\{d(Tz,z),0,0,\frac{1}{2s}[d(Tz,z)+d(Tz,z)], \frac{d^2(Tz,z)}{1+d(Tz,z)},0,0\Big\}\\
&= d(Tz,z),
\end{aligned}$$

which together with (iv), $\phi \in \Phi$, and Lemma 1.1 implies

$$d(Tz,z) = d(Tz,Sw) \leq \phi\big(\Delta_1(z,w)\big) = \phi\big(d(Tz,z)\big) < d(Tz,z),$$

which is impossible and hence $Tz = z$, that is , z is a common fixed point of A,B,S and T. Suppose A,B,S and T have another common fixed point $u \in X \setminus \{z\}$. It follows from (6), (iv), $\phi \in \Phi$, and Lemma 1.1 that

$$\begin{aligned}
\Delta_1(u,z) &= \max\Big\{d(Au,Bz), d(Au,Tu), d(Bz,Sz), \frac{1}{2s}[d(Au,Sz)+d(Tu,Bz)],\\
&\quad \frac{d(Au,Sz)d(Tu,Bz)}{1+d(Au,Bz)}, \frac{d(Au,Tu)d(Bz,Sz)}{1+d(Au,Bz)},\\
&\quad \frac{1+d(Au,Sz)+d(Tu,Bz)}{1+s\big(d(Au,Tu)+d(Bz,Sz)\big)}d(Au,Tu)\Big\}\\
&= \max\Big\{d(u,z),0,0,\frac{1}{2s}[d(u,z)+d(u,z)], \frac{d^2(u,z)}{1+d(u,z)},0,0\Big\}\\
&= d(u,z)
\end{aligned}$$

and

$$d(u,z) = d(Tu,Sz) \leq \phi\big(\Delta_1(u,z)\big) = \phi\big(d(u,z)\big) < d(u,z),$$

which is a contradiction and hence z is a unique common fixed point of A,B,S and T in X.

Similarly, we conclude that A,B,S and T have a unique common fixed point in X if one of $B(X),S(X)$ and $T(X)$ is complete. This completes the proof. □

Theorem 2.2. *Let $\{A,B\}$ and $\{S,T\}$ be self mappings in a $b-$metric space (X,d) satisfying (i)-(iii) and*

$$d(Tx,Sy) \le \phi\big(\Delta_2(x,y)\big), \forall x,y \in X, \tag{23}$$

where $\phi \in \Phi$ and Δ_2 is defined by (7) and $s>1$ be a real number. Then, A,B,S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$. It follows from(ii) that there exist two sequences $\{y_n\}_{n\in\mathbf{N}}$ and $\{x_n\}_{n\in\mathbf{N}_0}$ in X satisfying (9). Put $d_n = d(y_n,y_{n+1}),\ \forall n \in \mathbf{N}$.

Now, we prove that (10) holds. In view of (7) and (23), we deduce

$$d_{2n} = d(Tx_{2n},Sx_{2n-1}) \le \phi\big(\Delta_2(x_{2n},x_{2n-1})\big), \forall n \in \mathbf{N} \tag{24}$$

and

$$\begin{aligned}
&\Delta_2(x_{2n},x_{2n-1}) = \max\big\{d(Ax_{2n},Bx_{2n-1}), d(Ax_{2n},Tx_{2n}), d(Bx_{2n-1},Sx_{2n-1}),\\
&\frac{1}{2s}[d(Ax_{2n},Sx_{2n-1})+d(Tx_{2n},Bx_{2n-1})],\\
&\frac{1+d(Ax_{2n},Tx_{2n})}{1+d(Ax_{2n},Bx_{2n-1})}d(Bx_{2n-1},Sx_{2n-1}), \frac{1+d(Bx_{2n-1},Sx_{2n-1})}{1+d(Ax_{2n},Bx_{2n-1})}d(Ax_{2n},Tx_{2n}),\\
&\frac{1+d(Ax_{2n},Sx_{2n-1})+d(Tx_{2n},Bx_{2n-1})}{1+s\big(d(Ax_{2n},Tx_{2n})+d(Bx_{2n-1},Sx_{2n-1})\big)}d(Bx_{2n-1},Sx_{2n-1})\big\}\\
=\ &\max\big\{d(y_{2n},y_{2n-1}), d(y_{2n},y_{2n+1}), d(y_{2n-1},y_{2n}),\\
&\frac{1}{2s}[d(y_{2n},y_{2n})+d(y_{2n+1},y_{2n-1})], \frac{1+d(y_{2n},y_{2n+1})}{1+d(y_{2n},y_{2n-1})}d(y_{2n-1},y_{2n}),\\
&\frac{1+d(y_{2n-1},y_{2n})}{1+d(y_{2n},y_{2n-1})}d(y_{2n},y_{2n+1}), \frac{1+d(y_{2n},y_{2n})+d(y_{2n+1},y_{2n-1})}{1+s\big(d(y_{2n},y_{2n+1})+d(y_{2n-1},y_{2n})\big)}d(y_{2n-1},y_{2n})\big\}
\end{aligned}$$

$$\begin{aligned} &= \max\left\{d_{2n-1}, d_{2n}, d_{2n-1}, \frac{1}{2s}d(y_{2n+1}, y_{2n-1}), \frac{1+d_{2n}}{1+d_{2n-1}}d_{2n-1}, d_{2n}, \right. \\ &\quad \left. \frac{1+d(y_{2n+1}, y_{2n-1})}{1+s\big(d_{2n}+d_{2n-1}\big)}d_{2n-1}\right\} \\ &= \max\left\{d_{2n-1}, d_{2n}, \frac{1+d_{2n}}{1+d_{2n-1}}d_{2n-1}\right\} \ \forall n \in \mathbf{N}. \end{aligned}$$

Suppose that $d_{2n_0-1} < d_{2n_0}$ for some $n_0 \in \mathbf{N}$. It follows that

$$d_{2n_0}(1+d_{2n_0-1}) = d_{2n_0} + d_{2n_0}d_{2n_0-1} > d_{2n_0-1} + d_{2n_0}d_{2n_0-1} = d_{2n_0-1}(1+d_{2n_0}),$$

that is ,

$$d_{2n_0} > \frac{1+d_{2n_0}}{1+d_{2n_0-1}}d_{2n_0-1},$$

which implies $\Delta_2(x_{2n_0}, x_{2n_0-1}) = d_{2n_0}$. By means of (24), $\phi \in \Phi$, and Lemma 1.1, we conclude

$$d_{2n_0} \le \phi\big(\Delta_2(x_{2n_0}, x_{2n_0-1})\big) = \phi(d_{2n_0}) < d_{2n_0},$$

which is a contradiction. Consequently, we deduce

$$d_{2n} \le d_{2n-1} = \Delta_2(x_{2n}, x_{2n-1}), \forall n \in \mathbf{N}. \tag{25}$$

Similarly, we have

$$d_{2n+1} \le d_{2n} = \Delta_2(x_{2n}, x_{2n+1}), \forall n \in \mathbf{N}. \tag{26}$$

It follows from (25) and (26) that

$d_{n+1} \le d_n, \forall n \in \mathbf{N}$,

which means that the sequence $\{d_n\}_{n\in\mathbf{N}}$ is non-increasing and bounded. Consequently, there exists $r \ge 0$ with $\lim\limits_{n\to\infty} d_n = r$. Suppose that $r > 0$. It follows from (24) and (25),$\phi \in \Phi$, and Lemma 1.1 that

$$\begin{aligned} r &= \limsup_{n\to\infty} d_{2n} \le \limsup_{n\to\infty} \phi\big(\Delta_2(x_{2n}, x_{2n-1})\big) \\ &= \limsup_{n\to\infty} \phi(d_{2n-1}) \le \phi(r) < r, \end{aligned}$$

which is a contradiction. Hence r=0, that is (10) holds.

In order to prove that $\{y_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence,we need to show that $\{y_{2n}\}_{n\in\mathbf{N}}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\in\mathbf{N}}$ is not a Cauchy sequence. It follows that there exist $\varepsilon > 0$ and two subsequences $\{y_{2m(k)}\}_{k\in\mathbf{N}}$ and$\{y_{2n(k)}\}_{n\in\mathbf{N}}$ of $\{y_{2n}\}_{n\in\mathbf{N}}$ satisfying (14) -(18) and

$$\begin{aligned}
& \Delta_2(x_{2m(k)},x_{2n(k)-1}) \\
= & \max\Big\{d(Ax_{2m(k)},Bx_{2n(k)-1}),d(Ax_{2m(k)},Tx_{2m(k)}),d(Bx_{2n(k)-1},Sx_{2n(k)-1}), \\
& \frac{1}{2s}[d(Ax_{2m(k)},Sx_{2n(k)-1})+d(Tx_{2m(k)},Bx_{2n(k)-1})], \\
& \frac{1+d(Ax_{2m(k)},Tx_{2m(k)})}{1+d(Ax_{2m(k)},Bx_{2n(k)-1})}d(Bx_{2n(k)-1},Sx_{2n(k)-1}), \\
& \frac{1+d(Bx_{2n(k)-1},Sx_{2n(k)-1})}{1+d(Ax_{2m(k)},Bx_{2n(k)-1})}d(Ax_{2m(k)},Tx_{2m(k)}), \\
& \frac{1+d(Ax_{2m(k)},Sx_{2n(k)-1})+d(Tx_{2m(k)},Bx_{2n(k)-1})}{1+s\big(d(Ax_{2m(k)},Tx_{2m(k)})+d(Bx_{2n(k)-1},Sx_{2n(k)-1})\big)}d(Bx_{2n(k)-1},Sx_{2n(k)-1})\Big\} \\
= & \max\Big\{d(y_{2m(k)},y_{2n(k)-1}),d(y_{2m(k)},y_{2m(k)+1}),d(y_{2n(k)-1},y_{2n(k)}), \\
& \frac{1}{2s}[d(y_{2m(k)},y_{2n(k)})+d(y_{2m(k)+1},y_{2n(k)-1})], \\
& \frac{1+d(y_{2m(k)},y_{2m(k)+1})}{1+d(y_{2m(k)},y_{2n(k)-1})}d(y_{2n(k)-1},y_{2n(k)}), \\
& \frac{1+d(y_{2n(k)-1},y_{2n(k)})}{1+d(y_{2m(k)},y_{2n(k)-1})}d(y_{2m(k)},y_{2m(k)+1}), \\
& \frac{1+d(y_{2m(k)},y_{2n(k)})+d(y_{2m(k)+1},y_{2n(k)-1})}{1+s\big(d(y_{2m(k)},y_{2m(k)+1})+d(y_{2n(k)-1},y_{2n(k)})\big)}d(y_{2n(k)-1},y_{2n(k)})\Big\} \\
\to & \max\Big\{\varepsilon,0,0,\frac{1}{2s}(\varepsilon+\varepsilon),0,0,0\Big\} \\
= & \varepsilon \; as \; k\to\infty. \qquad (27)
\end{aligned}$$

By virtue of (14),(23),(27),$\phi\in\Phi$, and Lemma 1.1, we infer

$$\begin{aligned}
\varepsilon &= \limsup_{k\to\infty} d(y_{2m(k)+1},y_{2n(k)}) = \limsup_{k\to\infty} d(Tx_{2m(k)},Sx_{2n(k)-1}) \\
&\leq \lim_{k\to\infty} sup\phi\big(\Delta_2(x_{2m(k)},x_{2n(k)-1})\big) \leq \phi(\varepsilon) < \varepsilon,
\end{aligned}$$

which is impossible. Hence,$\{y_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence.

Assume that $A(X)$ is complete. Observe that $\{y_n\}_{n\in\mathbf{N}} \subseteq A(X)$ is a Cauchy sequence. It follows that there exists $(z,v)\in A(X)\times X$ with $\lim_{n\to\infty} y_{2n} = z = Av$. It is easy to show that (22) holds.

Suppose that $Tv \neq z$. Note that (7),(22),(23), and $\phi \in \Phi$ imply

$$\begin{aligned}
\Delta_2(v,x_{2n+1}) &= \max\Big\{d(Av,Bx_{2n+1}),d(Av,Tv),d(Bx_{2n+1},Sx_{2n+1}),\\
&\quad \frac{1}{2s}[d(Av,Sx_{2n+1})+d(Tv,Bx_{2n+1})],\\
&\quad \frac{1+d(Av,Tv)}{1+d(Av,Bx_{2n+1})}d(Bx_{2n+1},Sx_{2n+1}),\\
&\quad \frac{1+d(Bx_{2n+1},Sx_{2n+1})}{1+d(Av,Bx_{2n+1})}d(Av,Tv),\\
&\quad \frac{1+d(Av,Sx_{2n+1})+d(Tv,Bx_{2n+1})}{1+s\big(d(Av,Tv)+d(Bx_{2n+1},Sx_{2n+1})\big)}d(Bx_{2n+1},Sx_{2n+1})\Big\}\\
&\to \max\Big\{d(Av,z),d(Av,Tv),d(z,z),\frac{1}{2s}[d(Av,z)+(Tv,z)],\frac{1+d(Av,Tv)}{1+d(Av,z)}d(z,z),\\
&\quad \frac{1+d(z,z)}{1+d(Av,z)}d(Av,Tv),\frac{1+d(Av,z)+d(Tv,z)}{1+s\big(d(Av,Tv)+d(z,z)\big)}d(z,z)\Big\}\\
&= \max\Big\{0,d(z,Tv),0,\frac{1}{2s}d(Tv,z),0,d(z,Tv),0\Big\}\\
&= d(Tv,z) \ as\ n\to\infty
\end{aligned}$$

which together with (23), $\phi \in \Phi$, and Lemma 1.1 gives

$$\begin{aligned}
d(Tv,z) &= \limsup_{n\to\infty} d(Tv,y_{2n+2}) = \limsup_{n\to\infty} d(Tv,Sx_{2n+1})\\
&\leq \limsup_{n\to\infty} \phi\big(\Delta_2(v,x_{2n+1})\big) \leq \phi\big(d(Tv,z)\big) < d(Tv,z),
\end{aligned}$$

which is a contradiction. Hence $Tv = z$.

Since $T(X) \subseteq B(X)$, it follows that there exists a point $w \in X$ such that $z = Bw = Tv$.

Suppose that $Sw \neq z$. In light of (7) and (22), we obtain

$$\begin{aligned}
\Delta_2(x_{2n},w) &= \max\Big\{d(Ax_{2n},Bw),d(Ax_{2n},Tx_{2n}),d(Bw,Sw),\frac{1}{2s}[d(Ax_{2n},Sw)+(Tx_{2n},Bw)],\\
&\quad \frac{1+d(Ax_{2n},Tx_{2n})}{1+d(Ax_{2n},Bw)}d(Bw,Sw),\frac{1+d(Bw,Sw)}{1+d(Ax_{2n},Bw)}d(Ax_{2n},Tx_{2n}),\\
&\quad \frac{1+d(Ax_{2n},Sw)+d(Tx_{2n},Bw)}{1+s\big(d(Ax_{2n},Tx_{2n})+(Bw,Sw)\big)}d(Bw,Sw)\Big\}
\end{aligned}$$

$$\begin{aligned}
&\to \max\big\{d(z,z),d(z,z),d(z,Sw),\frac{1}{2s}[d(z,Sw)+d(z,Bw)],\\
&\quad \frac{1+d(z,z)}{1+d(z,z)}d(z,Sw),\frac{1+d(z,Sw)}{1+d(z,z)}d(z,z),\\
&\quad \frac{1+d(z,Sw)+d(z,z)}{1+s\big(d(z,z)+d(z,Sw)\big)}d(z,Sw)\big\}\\
&= \max\big\{0,0,d(z,Sw),\frac{1}{2s}d(z,Sw),d(z,Sw),0,d(z,Sw)\big\}\\
&= d(z,Sw)\ as\ n\to\infty
\end{aligned}$$

which together with (23), $\phi\in\Phi$, and Lemma 1.1 yields

$$\begin{aligned}
d(z,Sw) &= \lim_{n\to\infty}\sup d(y_{2n+1},Sw)=\lim_{n\to\infty}\sup d(Tx_{2n},Sw)\\
&\le \lim_{n\to\infty}\sup\phi\big(\Delta_2(x_{2n},w)\big)\le\phi\big(d(z,Sw)\big)<d(z,Sw),
\end{aligned}$$

which is impossible, and hence $Sw=z$. Clearly, (i) yields $Az=ATv=TAv=Tz$ and $Bz=BSw=SBw=Sz$. Suppose that $Tz\neq Sz$. It follows from (7) that

$$\begin{aligned}
\Delta_2(z,z) &= \max\big\{d(Az,Bz),d(Az,Tz),d(Bz,Sz),\frac{1}{2s}[d(Az,Sz)+d(Tz,Bz)],\\
&\quad \frac{1+d(Az,Tz)}{1+d(Az,Bz)}d(Bz,Sz),\frac{1+d(Bz,Sz)}{1+d(Az,Bz)}d(Az,Tz),\\
&\quad \frac{1+d(Az,Sz)+(Tz,Bz)}{1+s\big(d(Az,Tz)+d(Bz,Sz)\big)}d(Bz,Sz)\big\}\\
&= \max\big\{d(Tz,Sz),0,0,\frac{1}{2s}[d(Tz,Sz)+d(Tz,Sz)],0,0,0]\big\}\\
&= d(Tz,Sz).
\end{aligned}$$

Taking account of (23), $\phi\in\Phi$, and Lemma 1.1 , we conclude

$$d(Tz,Sz)\le\phi\big(\Delta_2(z,z)\big)=\phi\big(d(Tz,Sz)\big)<d(Tz,Sz),$$

which is a contradiction, and hence $Tz = Sz$.

Suppose that $Tz \neq z$.It follows from (7) that

$$\begin{aligned}
\Delta_2(z,w) &= \max\Big\{d(Az,Bw), d(Az,Tz), d(Bw,Sw), \frac{1}{2s}[d(Az,Sw)+d(Tz,Bw)],\\
&\quad \frac{1+d(Az,Tz)}{1+d(Az,Bw)}d(Bw,Sw), \frac{1+d(Bw,Sw)}{1+d(Az,Bw)}d(Az,Tz),\\
&\quad \frac{1+d(Az,Sw)+d(Tz,Bw)}{1+s\big(d(Az,Tz)+d(Bw,Sw)\big)}d(Bw,Sw)\Big\}\\
&= \max\Big\{d(Tz,z),0,0,\frac{1}{2s}[d(Tz,z)+d(Tz,z)],0,0,0\Big\}\\
&= d(Tz,z),
\end{aligned}$$

which together with (23), $\phi \in \Phi$, and Lemma 1.1 means

$$d(Tz,z) = d(Tz,Sw) \leq \phi\big(\Delta_2(z,w)\big) = \phi\big(d(Tz,z)\big) < d(Tz,z),$$

which is impossible, and hence $Tz = z$, that is, z is a common fixed point of A, B, S and T.

Suppose that A, B, S and T have another common fixed point $u \in X \setminus \{z\}$. It follows from (7) that

$$\begin{aligned}
\Delta_2(u,z) &= \max\Big\{d(Au,Bz), d(Au,Tu), d(Bz,Sz), \frac{1}{2s}[d(Au,Sz)+d(Tu,Bz)],\\
&\quad \frac{1+d(Au,Tu)}{1+d(Au,Bz)}d(Bz,Sz), \frac{1+d(Bz,Sz)}{1+d(Au,Bz)}d(Au,Tu),\\
&\quad \frac{1+d(Au,Sz)+d(Tu,Bz)}{1+s\big(d(Au,Tu)+d(Bz,Sz)\big)}d(Bz,Sz)\Big\}\\
&= \max\Big\{d(u,z),0,0,\frac{1}{2s}[d(u,z)+d(u,z)],0,0,0\Big\}\\
&= d(u,z)
\end{aligned}$$

which together with (23), $\phi \in \Phi$, and Lemma 1.1 ensures

$$d(u,z) = d(Tu,Sz) \leq \phi\big(\Delta_2(u,z)\big) = \phi\big(d(u,z)\big) < d(u,z),$$

which is a contradiction, and hence z is a unique common fixed point of A, B, S and T in X. Similarly we conclude that A, B, S and T have a unique common fixed point in X if one of $B(X), S(X)$, and $T(X)$ is complete. This completes the proof. $\square$

Similar to the proofs of Theorems 2.1 and 2.2, we have the following result and omit its proof.

Theorem 2.3. *Let $\{A,B\}$ and $\{S,T\}$ be self mappings in a $b-$metric (X,d) satisfying (i)-(iii) and*

$$d(Tx,Sy) \le \phi\big(\Delta_3(x,y)\big), \forall x,y \in X, \tag{28}$$

where $\phi \in \Phi$ and Δ_3 is defined by (8) and $s>1$ is a real number. Then A,B,S and T have a unique common fixed point in X.

Example 2.1. *Let $X=[0,1]$ be endowed with the Euclidean metric $d(x,y)=|x-y|^2, \forall x,y \in X$ and $s=2$. Let $A,B,S,T: X \to X$ and $\phi: \mathbf{R}^+ \to \mathbf{R}^+$ be defined by*

$$Ax = x^2,\ Bx = \frac{1}{2}x^2,\ Sx = 0,\ \forall\ x \in X,\ Tx = \begin{cases} 0,\ \forall x \in [0,1), \\ \frac{1}{4},\ x=1 \end{cases}$$

$$\phi(t) = \begin{cases} 16t^2,\ \forall t \in [0,\frac{1}{4}), \\ 8t-1,\ \forall t \in [\frac{1}{4},+\infty), \end{cases}$$

It is easy to see that (i) -(iii) hold, $\phi \in \Phi$ and $\phi(\mathbf{R}^+) \subset [0,\frac{1}{4})$. Let $x,y \in X$. In order to verify (iv), we have to consider two possible cases as follows:

Case 1: $x \in X \setminus \{1\}$. It is clear that

$$d(Tx,Sy) = 0 \le \phi\big(\Delta_1(x,y)\big);$$

Case 2: $x=1$. It follows that

$$\begin{aligned}\Delta_1(1,y) &= \max\Big\{\big|1-\frac{y^2}{2}\big|^2, \frac{9}{16}, \frac{y^4}{4}, \frac{1}{4}(1+|\frac{1}{4}-\frac{y^2}{2}|)^2, \frac{|\frac{1}{4}-\frac{y^2}{2}|^2}{1+|1-\frac{y^2}{2}|^2}, \\ &\qquad \frac{(\frac{3}{4}.\frac{y^2}{2})^2}{1+|1-\frac{y^2}{2}|^2}, \frac{1+1+|\frac{1}{4}-\frac{y^2}{2}|^2}{1+2\big((\frac{3}{4})^2+(\frac{y^2}{2})^2\big)}.\frac{9}{16}\Big\} \ge \frac{9}{16}\end{aligned}$$

and

$$d(T1,Sy) = d(\frac{1}{4},0) = \frac{1}{16} \le \phi(\frac{9}{16}) \le \phi\big(\Delta_1(1,y)\big).$$

That is (iv) holds. It follows from Theorem 2.1 that the mappings A,B,S and T have a unique common fixed point $0 \in X$.

Example 2.2. *Let $X = [-1,1]$ be endowed with the Euclidean metric $d(x,y) = \left|x-y\right|^2, \forall x,y \in X$ Let $A,B,S,T : X \to X$ and $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ be defined by*

$$Ax = \frac{x^2}{2},\ Tx = 0,\ \forall x \in X,\ Bx = \begin{cases} 0,\ \forall x \in [-1,1), \\ \frac{1}{2},\ x = 1, \end{cases} ,Sx = \begin{cases} 0,\ \forall x \in [-1,1), \\ \frac{1}{8},\ x = 1, \end{cases}$$

and

$$\phi(t) = \begin{cases} 64t^3,\ \forall t \in [0,\frac{1}{4}), \\ 32t^2 - 1,\ \forall t \in [\frac{1}{4},\infty), \end{cases}$$

Clearly, (i) -(iii) holds and $\phi \in \Phi$. In order to verify (23), we have to consider two possible cases as follows:

Case 1: $y \in X \setminus \{1\}$.Obviously

$$d(Tx,Sy) = d(0,Sy) = 0 \le \phi\big(\Delta_2(x,y)\big);$$

Case 2: y=1. It follows that

$$\begin{aligned} \Delta_2(x,1) \quad = \quad & \max\Big\{\big|\frac{1-x^2}{2}\big|^2, \frac{x^4}{4}, \frac{9}{64}, \frac{1}{2s}\big(\big|\frac{x^2}{2} - \frac{1}{8}\big|^2 + \frac{1}{4}\big), \\ & \frac{1+\frac{x^4}{4}}{1+|\frac{1-x^2}{2}|^2}.\frac{9}{64}, \frac{1+\frac{9}{64}}{1+|\frac{1-x^2}{2}|^2}.\frac{x^4}{4}, \frac{1+|\frac{x^2}{2}-\frac{1}{8}|^2+\frac{1}{4}}{1+s\big(\frac{x^4}{4}+\frac{9}{64}\big)}.\frac{9}{64}\Big\} \ge \frac{9}{64} \end{aligned}$$

and

$$d(Tx,S1) = d(0,\frac{1}{8}) = \frac{1}{64} < \frac{9}{64}$$

$$d(Tx,S1) \le \phi\big(\Delta_2(x,1)\big) = \phi(\frac{9}{64}) = 64\big(\frac{9}{64}\big)^3$$

That is, (23) holds. Consequently, Theorem 2.2 guarantees that the mappings A,B,S and T have a unique common fixed point $0 \in X$.

Example 2.3. *Let $X = \mathbf{R}^+$ be endowed with the Euclidean metric $d(x,y) = \left|x-y\right|^2, \forall x,y \in X$. Let $A,B,S,T : X \to X$ be defined by*

$$Ax = x^3,\ Sx = 1,\ \forall x \in X.$$

$$Bx = x^2,\ \forall x \in X \text{ and } Tx = \begin{cases} 1,\ \forall x \in \mathbf{R}^+ - \{\frac{1}{32}\}, \\ \frac{15}{16},\ x = \frac{1}{32} \end{cases}$$

$$\phi(t) = \begin{cases} 16t,\ \forall t \in [0, \frac{1}{16}) \\ 512t^2 - 1,\ \forall t \in [\frac{1}{16}, \infty) \end{cases}$$

Clearly, (i) - (iii) holds and $\phi \in \Phi$. In order to verify (28), we have to consider two possible cases as follows :

Case (1) : $x \in X \setminus \{\frac{1}{32}\}$.

$$d(Tx, Sy) = d(1,1) = 0 \leq \phi\big(\Delta_3(x,y)\big).$$

Case (2) : $x = \frac{1}{32}$. *It follows that*

$$\begin{aligned} \Delta_3\big(\frac{1}{32}, y\big) &= \max\Big\{\big|\frac{1}{32^3} - y^2\big|^2, \big|\frac{1}{32^3} - \frac{15}{16}\big|^2, \big|y^2 - 1\big|^2, \frac{1}{2s}\big[\big|\frac{1}{32^3} - 1\big|^2 + \big|\frac{15}{16} - y^2\big|^2\big]\Big\} \\ &\geq \big|\frac{15}{16} - \frac{1}{32^3}\big|^2 > \big(\frac{1}{16}\big)^2 = \frac{1}{256} \end{aligned}$$

$$d(T\frac{1}{32}, Sy) = d(\frac{15}{16}, 1) = \big|\frac{15}{16} - 1\big|^2 = \big(\frac{1}{16}\big)^2 = \frac{1}{256}$$

and

$$d\big(T\frac{1}{32}, Sy\big) \leq \phi\big(\Delta_3(\frac{1}{32}, y)\big) = \phi(\frac{1}{256}) = 16 \times \frac{1}{256} = \frac{1}{16}$$

That is, (28) holds. Thus, the conditions of Theorem 2.3 are satisfied. It follows from Theorem 2.3 that the mappings A, B, S and T have a unique common fixed point $1 \in X$.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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