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COMPUTATIONAL RESULTS OF DIFFERENTIAL DIFFERENCE EQUATIONS WITH MIXED SHIFTS HAVING LAYER STRUCTURE USING CUBIC NON-POLYNOMIAL SPLINE

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Abstract: A computational scheme using non − polynomial spline is suggested for solving differential−difference equations having delay and advance terms, solutions with layer structure at the left−end of the interval. First, the small shifts are tackled with Taylor's expansion and accordingly the problem is transformed to a second order singular perturbation problem. The domain is decomposed into inner and outer regions using a terminal boundary point and the problem is treated as inner region and outer region problems. Terminal boundary condition has been determined by using the reduced problem of the singular perturbation problem. In order to solve the inner and exterior region problems, a fourth order method was suggested using cubic non−polynomial spline. The method is repeated for numerous terminal point choices, until the solution profiles do not differ greatly from iteration to iteration. To illustrates the process, numerical examples were solved for specific values of the parameters of perturbation, delay and advance. The results of the computations are tabulated and compared to exact solutions. Convergence of the scheme was also studied.

Keywords: differential-difference equations; inner region; outer region; non-polynomial spline; terminal condition.

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1. INTRODUCTION

The differential-difference equation models arise in practical phenomenon of bio- science, engineering, variational problems in control theory [1], physiology [4], neural networks [6] as well as many others. The authors in [8] provided literature on approximate solutions to differential − difference equations with layer structure using method of matched asymptotic expansions and Laplace transforms. In [9], the authors explored the problems having transition regions between rapid oscillations and exponential behaviour. In [5], Shishkin mesh method was constructed for the study of these problem by Kadalbajoo and Sharma [5]. In [7] authors.

developed a scheme focused on B−spline collocation, to estimate the solution of these boundary value problems. Authors in [10] developed fitted operator methods utilizing non−standard differences for the solution of differential − difference equations with mixed shifts. Phaneendra et al. [11] suggested a computational approach of seventh order to solve differential−difference equations with negative shift.

Sharma and Pratima [12] explained a difference scheme focused on Il'in − Allen − Southwell fitting, for solving a class of differential − difference equations. Chakravarthy and Rao [14] examined an initial value technique for solving these problems. Authors in [15] developed a technique utilizing cubic non polynomial spline for a class of boundary value problems on a standard mesh.

The paper is organized in the following manner. Section 2 provides description of the problem. Section 3 demonstrates domain decomposition of the problem with terminal condition into inner and outer region problems. In section 4, non − polynomial spline difference method is explained. The method is analysed for convergence in section 5. Lower bound for the terminal point is discussed in section 6. In section 7, Numerical examples with results are given to support the method. Conclusion and discussions are given in last section

2. DESCRIPTION OF THE PROBLEM

Preliminaries Consider differential − difference equation having layer behaviour, with negative and positive shifts, of the form:

$$\varepsilon y''(s) + a(s)y'(s) + b(s)y(s-\delta) + c(s)y(s) + d(s)y(s+\eta) = f(s),\ 0 < s < 1 \tag{1}$$

subject to the conditions

$$y(s) = \varphi(s), \quad -\delta \le s \le 0 \tag{2}$$

$$y(s) = \gamma(s), \quad 1 \le s \le 1+\eta \tag{3}$$

where $a(s)$, $b(s)$, $c(s)$, $d(s)$, $f(s)$, $\phi(s)$, and $\gamma(s)$ are continuously differentiable functions on (0, 1), ε is the perturbation parameter ($0 < \varepsilon << 1$), δ ($0 < \delta = o(\varepsilon)$) is the delay and η ($0 < \eta = o(\varepsilon)$) is the advance parameter.

Using Taylor series for the retarded terms, we have

$$y(s-\delta) \approx y(s) - \delta y'(s) \tag{4}$$

$$y(s+\eta) \approx y(s) + \eta y'(s) \tag{5}$$

With Eq. (4) and Eq. (5), Eq. (1) is transformed to an equivalent singular perturbation problem

$$\varepsilon y''(s) + p(s)y'(s) + q(s)y(s) = f(s) \tag{6}$$

with conditions

$$y(0) = \varphi(0) \tag{7}$$

$$y(1) = \gamma(1) \tag{8}$$

where $p(s) = a(s) - \delta b(s) + \eta d(s)$ and $q(s) = b(s) + c(s) + d(s)$.

Since $0 < \delta << 1$ and $0 < \eta << 1$, the alteration from Eq. (1) to Eq. (6) is acknowledged. Under the assumptions $q(s) \leq 0$, $p(s) > 0$ throughout the region [0, 1], solution of Eq. (1) has the layer on the left $-$ end of the interval. Divide the problem into the inner and the outer region problems. Let terminal point be s_p ($0 < s_p << 1$), gives the thickness of the boundary layer, then the inner region problem is defined in $0 \leq s \leq s_p$ and outer region problem in $s_p \leq s \leq 1$ respectively.

3. Domain Decomposition with Terminal Boundary Condition

Main results At terminal point s_p, the boundary condition is achieved by solving the reduced problem with suitable boundary condition, that is,

$$p(s)y'(s) + q(s)y(s) = f(s) \tag{9}$$

with

$$y(1) = \gamma(1) \tag{10}$$

At the terminal point $s = s_p$, let $y(s_p) = k$ be the solution of the reduced problem. Then, the outer region problem is:

$$\varepsilon y''(s) + p(s)y'(s) + q(s)y(s) = f(s) \;, \quad s_p \leq s \leq 1 \tag{11}$$

with conditions

$$y(s_p) = k \tag{12}$$

$$y(1) = \gamma(1) \tag{13}$$

By solving Eq. (11) with the conditions Eq. (12) – Eq. (13), the solution $y(s)$ over the region $[s_p, 1]$ is obtained. Since the terminal point s_p is joint to both the inner and outer regions, the problem for inner region is:

$$\varepsilon y''(s) + p(s)y'(s) + q(s)y(s) = f(s) \;, \quad 0 \leq s \leq s_p \tag{14}$$

with $$y(0)=\varphi(0) \tag{15}$$

$$y(s_p)=k \tag{16}$$

To solve the inner region problem, Eqs. (14) - (16), choose the transformation

$$t=\frac{s}{\varepsilon} \tag{17}$$

Then $$y(s)=y(t\varepsilon)=Y(t) \tag{18}$$

$$y'(s)=\frac{y'(t\varepsilon)}{\varepsilon}=\frac{Y'(t)}{\varepsilon} \tag{19}$$

$$y''(s)=\frac{y''(t\varepsilon)}{\varepsilon^2}=\frac{Y''(t)}{\varepsilon^2} \tag{20}$$

$$p(s)=p(t\varepsilon)=P(t) \tag{21}$$

$$q(s)=q(t\varepsilon)=Q(t) \tag{22}$$

$$f(s)=f(t\varepsilon)=F(t) \tag{23}$$

Subsequently, Eq. (14) with Eqs. (18) - (23) transforms into a new inner region problem of the form: $$Y''(t)+P(t)Y'(t)+\varepsilon Q(t)Y(t)=\varepsilon F(t),\ 0\le t\le t_p \tag{24}$$

with $$Y(0)=\varphi(0) \tag{25}$$

$$Y(t_p)=y(s_p)=k \tag{26}$$

where $t_p=\frac{s_p}{\varepsilon}$.

By solving the new problem Eqs. (24) $-$ (26), solutions over the interval $0\le t\le t_p$ are obtained. Cubic non - polynomial spline is implemented to solve the problem in outer region given by Eqs. (11) - (13) and the problem in inner region given by Eq. (24) $-$ Eq. (26). Finally, the solutions of both regions are combined to estimate the solution of the problem.

For various values of s_p, the numerical method is replicated until the solution profiles are not significantly affected by the iteration. For computational point of view, an absolute error criterion is taken as

$$|y^{m+1}(s)-y^m(s)|\le\sigma\ ,\ 0\le s\le s_p \tag{27}$$

where $y^m(s)$ = the solution at the m^{th} iteration and σ = the tolerance bound.

4. NON - POLYNOMIAL SPLINE

To construct the difference scheme for Eqs. (11) - (13), divide $[s_p,1]$ into N equal parts $s_p=s_0<s_1<s_2<\ldots<s_N=1$, each of length h. Then, we have $s_i=s_p+ih$ for $i=0,1,2,\ldots,N$. For simplicity, let $p(s_i)=p_i$, $q(s)=q_i$, $f(s_i)=f_i$, $y(s_p)=y_0$, $y(s_i)=y_i$, $y(s_i+h)=$

$y_{i+1},\ y(s_i - h) = y_{i-1},\ y'(s_i) = y_i',\ y''(s_i) = y_i''$, etc.

The cubic non $-$ polynomial spline $S_i(s)$ for each i^{th}segment is

$$S_i(s) = \tilde{a}_i + \tilde{b}_i(s - s_i) + \tilde{c}_i \sin\tau(s - s_i) + \tilde{d}_i \cos\tau(s - s_i),\ i = 0, 1, \ldots, N - 1 \tag{28}$$

where $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i$ and $\tilde{d}_i$ are the constants and τ is a free parameter.

Let exact solution be $y(s)$ and the approximation be y_i to $y(s_i)$ obtained by the non$-$ polynomial cubic spline $S_i(s)$ passing via (s_i, y_i) and (s_{i+1}, y_{i+1}). The spline $S_i(s)$ fulfills the interpolatory conditions at s_i and s_{i+1}, also the continuity at common nodes (s_i, y_i) for the first derivative.

The non-polynomial function $S(s)$ interpolates $y(s)$ at the grids s_i for $i = 0,1,2, \ldots, N$, and reduces to an cubic polynomial spline $S(s)$ in $[a, b]$ as $\tau \to 0$. To derive the expressions for the coefficients of Eq. (28) in terms of $y_i,\ y_{i+1},\ M_i$ and M_{i+1}, we define

$$S_i(s_i) = y_i,\ S_i(s_{i+1}) = y_{i+1}$$
$$S_i''(s_i) = M_i,\ S_i''(s_{i+1}) = M_{i+1}$$

Using these, we get

$$\tilde{a}_i = y_i + \frac{M_i}{\tau^2},\ \ \tilde{b}_i = \frac{y_{i+1} - y_i}{h} + \frac{M_{i+1} - M_i}{\tau\theta},$$
$$\tilde{c}_i = \frac{M_i \cos\theta - M_{i+1}}{\tau^2 \sin\theta},\ \tilde{d}_i = -\frac{M_i}{\tau^2}$$

where $\theta = \tau h$ for $i = 0,1,2, \ldots, N - 1$.

Using the first derivative continuity at (s_i, y_i), that is $S_{i-1}'(s_i) = S_i'(s_i)$, we get the following relation,

$$\alpha M_{i+1} + 2\beta M_i + \alpha M_{i-1} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \text{ for } i = 1,2, \ldots, N - 1 \tag{29}$$

where $\alpha = \frac{-1}{\theta^2} + \frac{1}{\theta \sin\theta},\ \ \beta = \frac{1}{\theta^2} - \frac{\cos\theta}{\theta \sin\theta},\ \ M_j = y''(x_j),\ j = i, i \pm 1$ and $\theta = \tau h$

For our convenience, at the grid points x_i, rearranging Eq. (1) we have:

$$\varepsilon y_i'' = \tilde{p}(s_i) y_i' + \tilde{q}(s_i) y_i + f_i$$

where $\tilde{p}(s) = -p(s),\ \tilde{q}(s) = -q(s)$

By using the second derivatives of spline, we have

$$\varepsilon M_j = \tilde{p}(s_j) y_j'(s) + \tilde{q}(s_j) y(s_j) + f(s_j) \text{ for } j = i, i \pm 1$$

Substituting the above equations in Eq. (28) and using the first order derivative approximations of *y* at the grid points $s_1, s_2, \ldots, s_{N-1}$ given in [13],

$$y_{i+1}^{'} \cong \frac{y_{i-1}-4y_i+3y_{i+1}}{2h} \tag{30a}$$

$$y_{i-1}^{'} \cong \frac{-3y_{i-1}+4y_i-y_{i+1}}{2h} \tag{30b}$$

$$y_i^{'} \cong \left(\frac{1+2\omega h^2\tilde{q}_{i+1}+\omega h[3\tilde{p}_{i+1}+\tilde{p}_{i-1}]}{2h}\right) y_{i+1} - 2\omega\,[\tilde{p}_{i+1}+\tilde{p}_{i-1}]y_i$$

$$-\left(\frac{1+2\omega h^2\tilde{q}_{i-1}-\omega h[\,\tilde{p}_{i+1}+3\tilde{p}_{i-1}]}{2h}\right) y_{i-1} + \omega\,h[f_{i+1}-f_{i-1}] \tag{30c}$$

we get the following three term recurrence relation given as follows:

$$E_i y_{i-1} + F_i\,y_i + G_i y_{i+1} = H_i \quad \text{for } i = 1,2,\dots,N-1 \tag{31}$$

where

$$E_i = -\,\varepsilon - \frac{3}{2}\alpha\,\tilde{p}_{i-1}\,h\; + \beta\,\tilde{p}_i h^2\omega[\tilde{p}_{i+1}+3\tilde{p}_{i-1}] - 2\omega\,\tilde{p}_i\beta\,h^3\tilde{q}_{i-1}$$

$$+\frac{\alpha}{2}\tilde{p}_{i+1}h + \alpha\,\tilde{q}_{i-1}h^2 - h\beta\,\tilde{p}_i$$

$$F_i = 2\varepsilon + 2\alpha\,\tilde{p}_{i-1}\,h - 4\beta\,\tilde{p}_i h^2\omega[\tilde{p}_{i+1}+\tilde{p}_{i-1}]\; - 2\alpha\,\tilde{p}_{i+1}h + 2\beta\,\tilde{q}_i h^2$$

$$G_i = -\varepsilon - \frac{\alpha}{2}\,\tilde{p}_{i-1}\,h\; + \beta\,\tilde{p}_i h^2\omega[3\tilde{p}_{i+1}+\tilde{p}_{i-1}] + 2\omega h^3\beta\,\tilde{p}_i\tilde{q}_{i+1} + \frac{3}{2}\alpha\,\tilde{p}_{i+1}h$$

$$+\,\alpha\,\tilde{q}_{i+1}h^2 + h\beta\,\tilde{p}_i$$

$$H_i = \; -h^2\big[(\alpha - 2\omega\beta\,\tilde{p}_i h)\tilde{f}_{i-1} + 2\beta f_i + (\alpha + 2\omega\beta\,\tilde{p}_i h)f_{i+1}\big]$$

The tri − diagonal system of Eq. (31) is solved by Thomas algorithm, to get the approximations $y_{1,}y_2,\dots\dots,y_{N-1}$ of the solution $y(x)$ at $s_1, s_2,\dots\dots s_{N-1}$.

To determine the difference scheme in the inner region, divide the region $[0,t_p]$ into N equal parts $0 = t_0 < t_1 < \cdots < t_N = t_p$, each of length $h = t_p/N$. Then we have $t_i = ih$, $i = 0,1,2,\dots,N$.

Applying cubic non - polynomial spline to Eqs. (24)-(26) as in the outer region problem we obtain the three - term relation as follows:

$$E_i y_{i-1} + F_i\,y_i + G_i y_{i+1} = H_i \quad \text{for } i = 1,2,\dots,N-1 \tag{32}$$

where

$$E_i = -\,1 - \frac{3}{2}\alpha\,\tilde{P}_{i-1}\,h\; + \beta\,\tilde{P}_i h^2\omega[\tilde{P}_{i+1}+3\tilde{P}_{i-1}] - 2\omega\tilde{P}_i\beta\,h^3\tilde{Q}_{i-1}$$

$$+\frac{\alpha}{2}\tilde{P}_{i+1}h + \alpha\varepsilon\tilde{Q}_{i-1}h^2 - h\beta\,\tilde{P}_i$$

$$F_i = 2 + 2\alpha\,\tilde{P}_{i-1}\,h - 4\beta\tilde{P}_i h^2\omega[\tilde{P}_{i+1}+\tilde{P}_{i-1}]\; - 2\alpha\,\tilde{P}_{i+1}h + 2\beta\varepsilon\tilde{Q}_i h^2$$

$$G_i = -1 - \frac{\alpha}{2}\ \tilde{P}_{i-1}\ h\ + \beta\ \tilde{P}_i h^2 \omega[3\tilde{P}_{i+1} + \tilde{P}_{i-1}] + 2\omega h^2 \beta \tilde{P}_i \tilde{Q}_{i+1} + \frac{3}{2}\alpha\ \tilde{P}_{i+1}\ + \alpha\varepsilon \tilde{Q}_{i+1} h^2 + h\beta\ \tilde{P}_i$$

$$H_i = -h^2 \varepsilon\left[\left(\alpha - 2\omega\beta\ \tilde{P}_i h\right)\tilde{F}_{i-1} + 2\beta \tilde{F}_i + \left(\alpha + 2\omega\beta\ \tilde{P}_i h\right)\tilde{F}_{i+1}\right]$$

The tri-diagonal system given by Eq. (32) is solved by Thomas algorithm, for $y_{1,}y_2, \cdots, y_{N-1}$ at $x_1, x_2, \cdots, x_{N-1}$.

5. CONVERGENCE ANALYSIS

Including the boundary conditions, the system of equations Eq. (32) in the matrix form is

$$(D + J)Y + \tilde{Q} + T(h) = O \qquad (33)$$

where $$D = [-\varepsilon,\ 2\varepsilon,\ -\varepsilon] = \begin{bmatrix} 2\varepsilon & -\varepsilon & 0 & 0 & \cdots\cdots & 0 \\ -\varepsilon & 2\varepsilon & -\varepsilon & 0 & \cdots\cdots & 0 \\ 0 & -\varepsilon & 2\varepsilon & -\varepsilon & \cdots\cdots & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & 0 & -\varepsilon & 2\varepsilon \end{bmatrix}$$

$$J = [z_i, v_i, w_i] = \begin{bmatrix} v_1 & w_1 & 0 & 0 & \cdots\cdots & 0 \\ z_2 & v_2 & w_2 & 0 & \cdots\cdots & 0 \\ 0 & z_3 & v_3 & w_3 & \cdots\cdots & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & 0 & z_{N-1} & v_{N-1} \end{bmatrix}$$

$$z_i = -\tfrac{3}{2}\alpha\ \tilde{p}_{i-1}\ h + \beta\ \tilde{p}_i h^2 \omega[\tilde{p}_{i+1} + 3\tilde{p}_{i-1}] - 2\omega\ \tilde{p}_i \beta\ h^3 \tilde{q}_{i-1} + \tfrac{\alpha}{2}\tilde{p}_{i+1} h\alpha\ \tilde{q}_{i-1} h^2 - h\beta\ \tilde{p}_i$$

$$v_i = 2\alpha\ \tilde{p}_{i-1}\ h - 4\beta\ \tilde{p}_i h^2 \omega[\tilde{p}_{i+1} + \tilde{p}_{i-1}]\ - 2\alpha\ \tilde{p}_{i+1} h + 2\beta\ \tilde{q}_i h^2$$

$$w_i = -\frac{\alpha}{2}\ \tilde{p}_{i-1}\ h\ + \beta\ \tilde{p}_i h^2 \omega[3\tilde{p}_{i+1} + \tilde{p}_{i-1}] + 2\omega h^3 \beta\ \tilde{p}_i \tilde{q}_{i+1} + \frac{3}{2}\alpha\ \tilde{p}_{i+1} h + \alpha\ \tilde{q}_{i+1} h^2 + h\beta\ \tilde{p}_i, \ \text{for}\ \ i = 1,2,\ldots,N-1$$

$$\tilde{Q} = [\tilde{q}_1 + (-\varepsilon + z_1)\gamma_0, \tilde{q}_2, \tilde{q}_3, \ldots., \tilde{q}_{N-1} + (-\varepsilon + w_{N-1})\gamma_1]$$

$$\tilde{q}_i = h^2[(\alpha - 2\omega\beta\ \tilde{p}_i h) f_{i-1} + 2\beta\ f_i + (\alpha + 2\omega\beta\ \tilde{p}_i h) f_{i+1}], \quad i = 1,2,\ldots., N-1$$

and $Y = [Y_1, Y_2, Y_3, \ldots., Y_{N-1}]^T$, $T(h) = [T_1, T_2, \ldots., T_{N-1}]^T, O = [0,0,\ldots.,0]^T$ are the associated vectors with Eq. (33).

The local truncation error associated with the scheme developed in Eq. (31) is

$$T(h) = [-1 + 2(\alpha + \beta)]\varepsilon h^2 y''(s_i) + \left\{\left[\left(4\omega\varepsilon + \frac{1}{3}\right)\beta - \frac{2\alpha}{3}\right]\tilde{p}(s_i) y'''(s_i) + (-1 + 12\alpha)\frac{\varepsilon}{12} y^{(4)}(s_i)\right\} h^4 + O(h^6)$$

$i.e.,\ T(h) = O(h^6)$ for $\alpha = \frac{1}{12}, \beta = \frac{5}{12}, \omega = -\frac{1}{20\varepsilon}$

Let $y = [y_1, y_2, \ldots, y_{N-1}]^T \cong Y$ which satisfies the equation

$$(D + J)y + \tilde{Q} = 0 \tag{34}$$

Let $e_i = y_i - Y_i, i = 1,2,\ldots,N-1$ be the discretization error so that $E = [e_1, e_2, \ldots, e_{N-1}]^T = y - Y$.

Subtracting Eq.(33) from Eq. (34), we obtain the error equation

$$(D + J)E = T(h) \tag{35}$$

Let $|\tilde{p}(x)| \le C_1$ and $|\tilde{q}(x)| \le C_2$ where C_1, C_2 are positive constants. If $J_{i,j}$ be the $(i,j)^{th}$element of J, then

$$|J_{i,i+1}| = |w_i| \le \left(h(\alpha+\beta)C_1 + h^2\alpha C_2 + 4\beta\omega h^2 C_1^2 + 2h^3\beta\omega C_1 C_2\right), \quad i = 1,2,3,\ldots,N-2$$

$$|J_{i,i-1}| = |z_i| \le \left(h(\alpha+\beta)C_1 + h^2\alpha C_2 + 4\beta\omega h^2 C_1^2 + 2h^3\beta\omega C_1 C_2\right), \quad i = 2,3,\ldots,N-1$$

Thus for small h , $$|J_{i,i+1}| < \varepsilon, \qquad i = 1,2,\ldots,.N-2 \tag{36a}$$

and $$|J_{i,i-1}| < \varepsilon, \qquad i = 2,3,\ldots,N-1 \tag{36b}$$

Hence (D+J) is irreducible. Let S_i be the sum of the elements of the ith row of the matrix (D+J), then we have

$$S_i = \varepsilon + \frac{\alpha h}{2}(3\tilde{p}_{i-1} - \tilde{p}_{i+1}) - h\beta\tilde{p}_i + h^2(\alpha\,\tilde{q}_{i+1} + 2\beta\,\tilde{q}_i) - h^2\beta\omega\,\tilde{p}_i(\tilde{p}_{i+1} + 3\tilde{p}_{i-1}) + 2h^3\beta\omega\tilde{p}_i\tilde{q}_{i+1} \ \text{ for } i = 1$$

$$S_i = h^2(\alpha\tilde{q}_{i-1} + 2\beta\tilde{q}_i + \alpha\tilde{q}_{i+1}) + 2h^3\beta\tilde{p}_i\omega(\tilde{q}_{i+1} - \tilde{q}_{i-1}) \quad \text{for } i = 2,3,\ldots,N-2$$

$$S_i = \varepsilon + \frac{\alpha h}{2}(\tilde{p}_{i-1} - 3\tilde{p}_{i+1}) - h\beta\tilde{p}_i + h^2(\alpha\tilde{q}_{i-1} + 2\beta\tilde{q}_i) - h^2\beta\omega\tilde{p}_i(\tilde{p}_{i+1} + \tilde{p}_{i-1}) - 2h^3\beta\omega\tilde{p}_i\tilde{q}_{i-1} \ \text{ for } i = N-1$$

Let $C_{1^*} = \underset{1\le i\le N}{min}|\tilde{p}(s)|$ and $C_1^* = \underset{1\le i\le N}{max}|\tilde{p}(s)|$, $C_{2^*} = \underset{1\le i\le N}{min}|\tilde{q}(s)|$ and $C_2^* = \underset{1\le i\le N}{max}|\tilde{q}(s)|$.

Since $0 < \varepsilon << 1$ and $\varepsilon \propto O(h), (D+J)$ is monotone small h. Hence $(D+J)^{-1}$ exists and $(D+J)^{-1} \ge 0$.

Thus from Eq. (34), we have $$\|E\| \le \|(D+J)^{-1}\|.\ \|T\| \tag{37}$$

Let $(D+J)^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $(D+J)^{-1}$ and we define

$$\|(D+J)^{-1}\| = \underset{1\le i\le N-1}{max}\sum_{k=1}^{N-1}(D+J)^{-1}_{i,k} \ \text{ and } \ \|T(h)\| = \underset{1\le i\le N-1}{max}|T(h)|, \tag{38a}$$

Since $(D+J)^{-1}_{i,k} \geq 0$ and $\sum_{k=1}^{N-1}(D+J)^{-1}_{i,k}.S_k = 1$ for $i = 1,2,3,\ldots,N-1$.

Hence, $$(D+J)^{-1}_{i,k} \leq \frac{1}{S_i} < \frac{1}{h^2[(\alpha+2\beta)C_{2^*}-4\beta\omega C_{1^*}^2]},\ i=1 \tag{38b}$$

$$(D+J)^{-1}_{i,k} \leq \frac{1}{S_i} < \frac{1}{h^2[(\alpha+2\beta)C_{2^*}-4\beta\omega C_{1^*}^2]},\ i=N-1 \tag{38c}$$

Furthermore, $$\sum_{k=1}^{N-1}(D+J)^{-1}_{i,k} \leq \frac{1}{\underset{2\leq i\leq N-2}{min} S_i} < \frac{1}{h^2(2(\alpha+\beta)C_{2^*})}. \tag{38d}$$

By the help of Eqs. (38a) - (38d), from Eq. (35), we obtain

$$\|E\| \leq O(h^4).$$

Hence, the method is fourth order convergent for $\alpha = \frac{1}{12},\ \beta = \frac{5}{12},\ \omega = -\frac{1}{20\varepsilon}$ on uniform mesh.

6. NUMERICAL EXAMPLES

The exact solution of problems of the type given by Eqs. (1) - (3) with constant coefficients is given by [82] : $$y(s) = c_1 e^{m_1 s} + c_2 e^{m_2 s} + f/c$$

where $c_1 = \frac{[-f+\gamma c_3+e^{m_2}(f-\phi c_3)]}{[(e^{m_1}-e^{m_2}\quad)c_3]}$, $c_2 = \frac{[f-\gamma c_3+e^{m_1}(-f+\phi c_3)]}{[(e^{m_1}-e^{m_2}\quad)c_3]}$

$m_1 = \frac{\left[-(a-b\delta+d\eta)+\sqrt{(a-b\delta+d\eta)^2-4\varepsilon c_3}\right]}{2\varepsilon}$, $m_2 = \frac{\left[-(a-b\delta+d\eta)-\sqrt{(a-b\delta+d\eta)^2-4\varepsilon c_3}\right]}{2\varepsilon}$, $c_3 = (b+c+d)$.

Example 1. $\varepsilon y''(s) + y'(s) + 2y(s-\delta) - 3y(s) = 0$

The computed results are tabulated in Tables 1 and 2 for $\varepsilon = 10^{-3}, 10^{-4}$ with various parameter values of the shifts. Graph 1 revealed the effect of the shifts on the layer.

Example 2. $\varepsilon y''(s) + y'(s) - 3y(s) + 2y(s+\eta) = 0$

Tables 3 and 4 shows the numerical results for $\varepsilon = 10^{-3}, 10^{-4}$ with distinct values of shift parameters. The consequence of the shifts on the layer solutions has been exposed in Graph 2.

Example 3. $\varepsilon y''(s) + y'(s) - 2y(s-\delta) - 5y(s) + y(s+\eta) = 0$

The computational results are tabled in Tables 5 and 6 for $\varepsilon = 10^{-3}, 10^{-4}$ with different values of shift parameters. Graph 3 shows the effect of the shifts on the boundary layer.

7. DISCUSSIONS AND CONCLUSION

A finite difference method of fourth order has been illustrated for solving the differential– difference equations with small shifts, solutions that display layer at the left – end of the interval. The classical numerical schemes was considered to be insufficient to estimate the solution in the inner region due to the existence of perturbation, delay and advanced parameters.

To address such problem, we insert a terminal point and decompose the region into inner and outer regions to handle the original problem as inner region and outer region problems. The difference scheme of fourth order using cubic non − polynomial spline is suggested for differential-difference equations. Both the inner and outer regions are solved using the proposed fourth order finite difference scheme. The procedure is repeated for various terminal point values, until both the inner and outer regions solutions profile stabilizes. Theoretical convergence of the method was also analyzed. The numerical results were tabulated and compared with the exact solutions (Tables 1-6) and with Gemechis and Reddy results [3]. Although the solutions are measured at all the points, only few values were tabulated. In addition, the inner layer solutions presented in graphs for different values of the terminal points and ε (Figures 1-3). From the tables and graphs, it can be observed that the present approach very well approximates the exact solution. This method is simple, easy and efficient to solve differential-difference equations, especially in the area of boundary layer, where the conventional methods do not achieve good results.

Table 1: Numerical results of Example 1 for $\varepsilon = 10^{-3}$ with $\delta = 0.1\varepsilon$.

s	t_p =10		t_p =20		t_p =30		Exact Sol.
	Result by present Method	Result in [3]	Result by present method	Result in [3]	Result by present method	Result in [3]	
0.0000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.0002	0.8853975	0.8891662	0.8854292	0.8863916	0.8854489	0.8856926	0.8854601
0.0004	0.7915958	0.7984523	0.7916487	0.7934058	0.7916810	0.7921346	0.7917112
0.0006	0.7148218	0.7242065	0.7148882	0.7173000	0.7149280	0.7155601	0.7149817
0.0008	0.6519869	0.6634415	0.6520612	0.6550117	0.6521048	0.6528881	0.6521843
0.0100	<u>0.3715031</u>	<u>0.3925705</u>	0.3715353	0.3771362	0.3715372	0.3732482	0.3718990
0.0200			<u>0.3752375</u>	<u>0.3819092</u>	0.3752394	0.3779721	0.3756049
0.0300					<u>0.3790094</u>	<u>0.3828013</u>	0.3793767
0.1000	0.4068583	0.4072314	0.4068583	0.4072314	0.4068583	0.4072314	0.4068583
0.2000	0.4496075	0.4500288	0.4496075	0.4500288	0.4496075	0.4500288	0.4496075
0.4000	0.5490532	0.5495898	0.5490532	0.5495898	0.5490532	0.5495898	0.5490532
0.6000	0.6705609	0.6711769	0.6705609	0.6711769	0.6705609	0.6711769	0.6705609
0.8000	0.8188778	0.8196632	0.8188778	0.8196632	0.8188778	0.8196632	0.8188778
0.9000	0.9049187	0.9058052	0.9049187	0.9058052	0.9049187	0.9058052	0.9049187
1.0000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000

Table 2: Numerical results of Example 1 for $\varepsilon = 10^{-4}$ with $\delta = 0.1\varepsilon = \eta$.

s	t_p =10		t_p =20		t_p =30		Exact Sol.
	Result by present Method	Result in [3]	Result by present method	Result in [3]	Result by present method	Result in [3]	
0.0000	1.000000	1.00000	1.000000	1.00000	1.00000	1.00000	1.000000
0.00002	0.885409	0.889053	0.885417	0.886319	0.885419	0.885664	0.8854203
0.00004	0.791593	0.798220	0.7916069	0.793246	0.791610	0.792055	0.7916133
0.00006	0.714785	0.723852	0.7148036	0.717045	0.714807	0.715414	0.7148132
0.00008	0.651902	0.662965	0.6519243	0.654657	0.651928	0.652666	0.6519369
0.00100	0.368240	0.388117	0.3682688	0.373027	0.368268	0.369410	0.3683056
0.00200			0.3686085	0.374089	0.368608	0.370467	0.3686453
0.00300					0.368977	0.371570	0.3690141
0.10000	0.406595	0.406657	0.4065952	0.406657	0.406595	0.406657	0.4065952
0.20000	0.449349	0.449415	0.4493491	0.449415	0.449349	0.449415	0.4493491
0.40000	0.548816	0.548890	0.5488165	0.548890	0.548816	0.548890	0.5488165
0.60000	0.670368	0.670384	0.6703683	0.670384	0.670368	0.670384	0.6703683
0.80000	0.818760	0.740871	0.8187602	0.818770	0.818760	0.818770	0.8187602
0.90000	0.904853	0.818770	0.9048537	0.904859	0.904853	0.904859	0.9048537
1.00000	1.000000	1.000000	1.0000000	1.000000	1.000000	1.000000	1.0000000

Table 3: Numerical results of Example 2 for $\varepsilon = 10^{-3}$ with $\eta = 0.1\varepsilon$.

s	t_p =10		t_p =20		t_p =30		Exact Sol.
	Result by present method	Result in [3]	Result by present method	Result in [3]	Result by present method	Result in [3]	
0.000	1.00000	1.00000	1.00000	1.000000	1.0000000	1.000000	1.00000
0.0002	0.885382	0.88987	0.885414	0.887745	0.885434	0.887702	0.885452
0.0004	0.791576	0.799711	0.791629	0.795840	0.791661	0.795761	0.791701
0.0006	0.714805	0.725893	0.714871	0.720595	0.7149114	0.720487	0.714973
0.0008	0.651977	0.665456	0.652051	0.6589903	0.65209523	0.658859	0.652179
0.0100	0.371650	0.392650	0.371682	0.3809045	0.37168431	0.380665	0.371972
0.0200			0.375384	0.3819941	0.37538652	0.381754	0.375678
0.0300					0.37915657	0.382887	0.379523
0.1000	0.407004	0.407318	0.407004	0.4073182	0.40700436	0.407318	0.407004
0.2000	0.449751	0.450114	0.449751	0.4501143	0.44975104	0.450114	0.449751
0.4000	0.549184	0.549668	0.549184	0.549668	0.549184	0.549668	0.549184
0.6000	0.670667	0.671240	0.670667	0.671240	0.6706679	0.671240	0.670667
0.8000	0.818943	0.819702	0.818943	0.819702	0.8189431	0.8197021	0.818943
0.9000	0.904954	0.905826	0.904954	0.905826	0.904954	0.905826	0.904954
1.000	1.00000	1.00000	1.00000	1.000000	1.0000000	1.000000	1.000000

Table 4: Numerical results of Example 2 for $\varepsilon = 10^{-4}$ with $\eta = 0.1\varepsilon$.

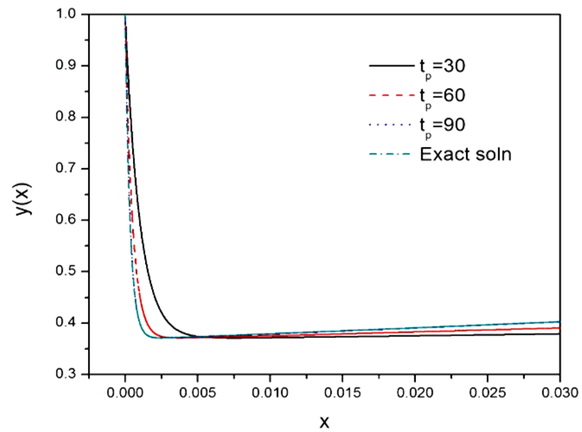
s	t_p =10		t_p =20		t_p =30		Exact Sol.
	Result by present method	Result in [3]	Result by present method	Result in [3]	Result by present method	Result in [3]	
0.0000	1.000000	1.00000	1.000000	1.00000	1.000000	1.00000	1.000000
0.00002	0.885409	0.889136	0.885417	0.886401	0.885418	0.885745	0.885419
0.00004	0.791594	0.7983701	0.791607	0.793395	0.791609	0.792202	0.791612
0.00006	0.714787	0.7240578	0.714805	0.717249	0.714808	0.715617	0.714812
0.00008	0.651905	0.6632162	0.651927	0.654906	0.651929	0.652914	0.651936
0.00100	<u>0.368254</u>	<u>0.3885801</u>	0.368283	0.373485	0.368283	0.369866	0.368313
0.00200			<u>0.368623</u>	<u>0.374547</u>	0.368623	0.370925	0.368652
0.00300					<u>0.368992</u>	<u>0.372027</u>	0.369028
0.10000	0.406609	0.4071122	0.406609	0.407112	0.406609	0.407112	0.406609
0.20000	0.449368	0.449861	0.449368	0.449861	0.449368	0.449861	0.449368
0.40000	0.548846	0.549299	0.548846	0.549299	0.548846	0.549299	0.548846
0.60000	0.670345	0.670717	0.670345	0.670717	0.670345	0.670717	0.670345
0.80000	0.818742	0.818973	0.818742	0.818973	0.818742	0.818973	0.818742
0.900000	0.904839	0.904971	0.904839	0.904971	0.904839	0.904971	0.904839
1.00000	1.000000	1.00000	1.000000	1.00000	1.000000	1.00000	1.00000

Table 5: Numerical results of Example 3 for $\varepsilon = 10^{-3}$ with $\delta = 0.1\varepsilon = \eta$.

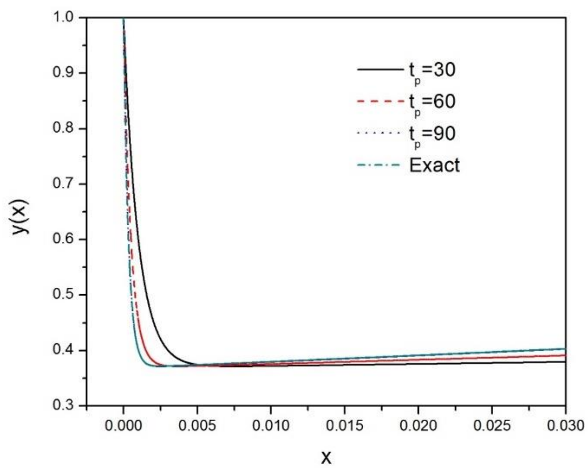
s	t_p =10		t_p =20		t_p =30		Exact Sol.
	Result by present method	Result in [3]	Result by present method	Result in [3]	Result by present method	Result in [3]	
0.0000	1.0000	1.0000	1.0000	1.00000	1.00000	1.00000	1.00000
0.0002	0.818152	0.820883	0.818200	0.819029	0.81823	0.818669	0.818162
0.0004	0.669454	0.674386	0.669535	0.671013	0.669586	0.670358	0.669475
0.0006	0.547865	0.554569	0.547966	0.549952	0.548028	0.549056	0.547896
0.0008	0.448441	0.456574	0.448554	0.450938	0.448621	0.449844	0.448441
0.0100	<u>0.002638</u>	<u>0.017168</u>	0.002681	0.006531	0.002682	0.004466	0.002775
0.0200			<u>0.0028013</u>	<u>0.006830</u>	0.002801	0.004657	0.002900
0.0300					<u>0.002974</u>	<u>0.004903</u>	0.003078
0.1000	0.004673	0.004696	0.004673	0.004696	0.004673	0.004696	0.004673
0.2000	0.008482	0.008526	0.008482	0.008526	0.008482	0.008526	0.008482
0.4000	0.027946	0.028101	0.027946	0.028101	0.027946	0.028101	0.027946
0.6000	0.092126	0.092615	0.092126	0.092615	0.092126	0.092615	0.092126
0.8000	0.303524	0.305236	0.3035240	0.305236	0.303524	0.305236	0.303524
0.9000	0.550930	0.554131	0.550930	0.554131	0.550930	0.554131	0.550930
1.0000	1.00000	1.00000	1.000000	1.00000	1.000000	1.00000	1.00000

Table 6: Numerical results of Example 3 for$\varepsilon = 10^{-4}$ with $\delta = 0.1\varepsilon = \eta$.

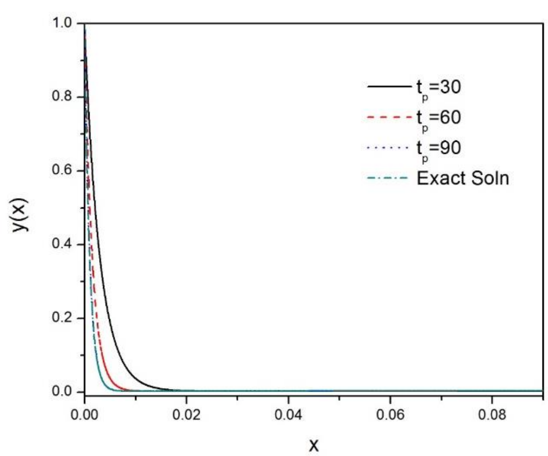
s	t_p =10		t_p =20		t_p =30		Exact Sol.
	Result by present method	Result in [3]	Result by present method	Result in [3]	Result by present method	Result in [3]	
0.0000	1.000000	1.00000	1.000000	1.000000	1.00000	1.00000	1.000000
0.00002	0.819069	0.821789	0.819081	0.819885	0.819084	0.819503	0.819077
0.00004	0.670954	0.675887	0.670975	0.672422	0.670980	0.671728	0.670969
0.00006	0.549703	0.556429	0.549732	0.551691	0.549738	0.550741	0.549725
0.00008	0.450444	0.458627	0.450478	0.452845	0.450484	0.451685	0.450471
0.00100	<u>0.002494</u>	<u>0.016947</u>	0.002539	0.006443	0.002539	0.004336	0.002548
0.00200			<u>0.002509</u>	<u>0.006416</u>	0.002509	0.004303	0.002518
0.00300					<u>0.002524</u>	<u>0.004316</u>	0.002533
0.10000	0.004532	0.004528	0.004532	0.004528	0.004532	0.004528	0.004532
0.20000	0.008255	0.008248	0.008255	0.008248	0.008255	0.008248	0.008255
0.40000	0.027386	0.027369	0.027386	0.027369	0.027386	0.027369	0.027386
0.60000	0.090857	0.090819	0.090874	0.090819	0.090857	0.090819	0.090857
0.80000	0.301425	0.301362	0.301425	0.301362	0.301425	0.301362	0.301425
0.90000	0.549022	0.548965	0.549022	0.548965	0.549022	0.548965	0.549022
1.0000	1.00000	1.00000	1.000000	1.000000	1.000000	1.000000	1.000000



Graph 1. Numerical results of Example 1 for $\varepsilon = 10^{-3}$, $\delta = 0.5\varepsilon = \eta$ with different t_p.



Graph 2. Numerical results of Example 2 for $\varepsilon = 10^{-3}$, $\delta = 0.5\varepsilon = \eta$ with different t_p.



Graph 3. Numerical results of Example 3 for $\varepsilon = 10^{-3}$, $\delta = 0.5\varepsilon = \eta$ with different t_p.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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