Available online at http://scik.org

J. Math. Comput. Sci. 10 (2020), No. 5, 1399-1408

https://doi.org/10.28919/jmcs/4596

ISSN: 1927-5307

SOLUTION OF CONVECTION-DIFFUSION PROBLEMS WITH SINGULARITY USING VARIABLE MESH SPLINE OF THIRD ORDER

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Abstract: The paper deals with a third order finite difference approach with variable mesh using the non−polynomial spline for the solution of a problems with singularity in convection-diffusion equation. The problem's discretization equation is constructed using the continuity condition at the inner nodes for the derivatives of first order of the non−polynomial spline, which is not valid at singularity. At the singularity, the problem is modified in order to have a three-term relationship. The method's tridiagonal scheme is interpreted by means of discrete invariant imbedding algorithm. Error analysis of the method is analyzed and the maximum absolute error in the solution is tabulated. Layer behaviour is picturized in graphs.

Keywords: singularly perturbed singular boundary value problem; variable mesh, interior nodes; singular point; non-polynomial spline; boundary layer.

2010 AMS Subject Classification: 65L10, 65L11, 65L12.

1. INTRODUCTION

Consider the problem with singularity in convection - diffusion equation

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Received March 28, 2020

$$\epsilon y''(x) = \frac{k}{x} y'(x) + q(x) + r(x) \tag{1}$$

with $$y(0) = \gamma_0, \quad y(1) = \gamma_1$$

where $0 < \epsilon \ll 1$, $q(x), r(x)$ are continuous functions in (0, 1), and $\gamma_{0,} \gamma_1$ are finite constants. Let $p(x) = \frac{k}{x}$. If $p(x) \geq \bar{M} > 0$ all over the domain, $\bar{M}$ is a positive constant, then the layer exists in the neighbourhood of $x = 1$. If $p(x) \leq \bar{N} < 0$ all over the domain, $\bar{N}$ is a negative constant, then the layer will be in the neighborhood of $x = 0$.

In many areas of the applied mathematics such as quantum mechanics, elasticity, optimal control, chemical-reactor theory, aerodynamics, fluid mechanics, geophysics, and many other fields, this class of problems also occurs. Equations of this type show layer solutions; that is, the problem-solving domain includes narrow areas with extremely large solution derivatives. Due to the presence of interior or boundary layers, the numerical treatment of these problems gives significant computational difficulties. A wide range of books and papers have been published, including [2-8], [11], [13] detailing different methods for solving singularly perturbed boundary value problems. In [5], authors proposed a B-spline-fitted mesh scheme to solve Eq. (1). Variety of schemes based on tension spline and spline compression developed by Mohanty et al. [9, 10, 11] for the solution of Eq. (1). Cubic spline solution is used by Rashidinia [13] on a uniform mesh for the solution of Eq. (1).

In the present paper, a variable mesh non polynomial spline is used to develop a numerical method for the smooth approximation to the solution for Eq. (1). Parameter ω is introduced in the difference scheme of first order derivative term to achieve third order convergence. The paper is organized as follows: In section 2, we develop the nonpolynomial spline method for solving Eq. (1). In section 3, description of the method is given. The error analysis of the method is considered in section 4. Finally, Maximum absolute errors of the solutions of the considered examples are given in section 5.

2. NON – POLYNOMIAL SPLINE

Let $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ be a sub - division of the region [0, 1], where $h_i = x_i - x_{i-1}$, $i = 1,2,\ldots,n$ and $h_{i+1} = \sigma h_i$. Let the exact solution be $y(x)$ and y_i be an approximation to $y(x_i)$ achieved by the non $-$ polynomial spline $S_i(x)$ passing through (x_i, y_i) and (x_{i+1}, y_{i+1}). The spline satisfies interpolatory conditions at x_i and x_{i+1} and also first derivative continuity at the common nodes $(x_i\ ,y_i\ )$.

We write $S_i(x\ )$ in the form

$$S_i(x\ ) = a_i + b_i\ (x - x_i) + c_i\ sin\tau\ (x - x_i) + d_i\ cos\tau\ (x - x_i), i = 0,1,2, \dots, n \tag{2}$$

where $a_i, b_{i,}c_i$ and d_i are constants and τ is a free parameter.

The function in Eq. (2) interpolates $y(x)$ at the grid points x_i, $i = 1,2, \dots, n$, having the parameter τ and as $\tau \to 0$ reduces to cubic spline.

To get the values for the coefficients of Eq. (1) in term of $y_i, y_{i+1,}M_i$ and M_{i+1} define

$S_i(x\ ) = y_i\ ,\ S_i(x_{i+1}\ ) = y_{i+1},\quad S''(x_i\ ) = M_i, S''(x_{i+1}\ ) = M_{i+1}$.

From algebraic calculations, we get

$$a_i = y_i + \frac{M_i}{\tau^2}, b_i = \frac{y_{i+1} - y_i}{h_i} + \frac{M_{i+1} - M_i}{\theta\tau},\quad c_i = \frac{M_i cos\theta - M_{i-1}}{\tau^2 sin\theta}, d_i = -\frac{M_i}{\tau^2} \tag{3}$$

where $\theta = \tau h_{i+1}$ for $i\ =\ 0, 1, 2, \dots, n$.

Using the first order derivative continuity at $(x_i\ ,y_i\ )$, that is $S'_{i-1}(x_i\ ) = S'_i(x_i\ )$, we get the following relations for $i\ = 1, 2, \dots, n-1$.

$$\sigma y_{i-1} - (1+\sigma)y_i + y_{i+1} = h_{i+1}^2[\alpha_i M_{i-1} + \beta M_i + \alpha_2 M_{i+1}] \tag{4}$$

where $\alpha_i = \frac{-1}{\tau^2 h_i^2} + \frac{1}{\tau h_i \sin(\tau h_i)}, \beta_i = \frac{-1}{\tau^2 h_i^2} + \frac{1}{\tau h_i \sin(\tau h_i)}, \alpha_2 = \frac{-1}{\theta^2} + \frac{1}{\theta \sin\theta},$ $M_j = y''(x_j),$

$j = i, i \pm 1\ and\ \theta = \tau h_{i+1}$.

3. NUMERICAL SCHEME

At the mesh points x_i, the Eq. (1) may be written as

$$\epsilon y_i''(x) = p(x_i)y_i' + q(x_i)y_i + r_i \text{ where } p(x_i) = \frac{k}{x_i}$$

and using spline's second derivatives, we have

$$\epsilon M_j = p(x_j)y_j'(x) + q(x_j)y_i(x_j) + r_i(x_j) \text{ for } j = i-1, i+1 \tag{5}$$

Using Eq. (5) in Eq. (4) with the following approximations for the first derivative

$$y'_{i+1} \cong \frac{1}{h_{i+1}}\left[\frac{2\sigma+1}{\sigma+1}\right] y_{i+1} - (\sigma+1)y_i + \frac{\sigma^2}{\sigma+1}\ y_{i-1}$$

$$y'_{i-1} \cong \frac{1}{h_i}\left[\frac{-1}{\sigma(\sigma+1)}\right] y_{i+1} + \frac{(\sigma+1)}{\sigma} y_i - \frac{2+\sigma}{\sigma+1} y_{i-1}$$

$$y_i' \cong \frac{1}{h_i}\left(\frac{1+\omega h_i^2\sigma(\sigma+1)q_{i+1}+\omega h_i\,[(2\sigma+1)p_{i+1}+p_{i-1}]}{\sigma(\sigma+1)h_i}\right)y_{i+1}$$
$$+\left(\left(\frac{\sigma-1}{\sigma h_i}\right)-\omega\left(\frac{1+\sigma}{\sigma}\right)[P_{i+1}+P_{i-1}]\right)y_i$$
$$-\left(\frac{\sigma+\omega(1+\sigma)h_i^2q_{i-1}-\omega h_i\,[\sigma p_{i+1}+(2+\sigma)p_{i-1}]}{(\sigma+1)h_i}\right)y_{i-1}$$
$$+\,\omega h_i\,[\mathrm{r_{i+1}}-\mathrm{r_{i-1}}]\,, \tag{6}$$

we get the tri-diagonal system

$$E_{i-1}y_{i-1}+F_i\,y_i+G_{i+1}y_{i+1}=H_i \quad i=2,3,\ldots,n-1 \tag{7}$$

where

$E_{i-1}=-\varepsilon\sigma-\alpha_1\,p_{i-1}\,h_i\,\left(\frac{2\sigma^2+\sigma^3}{1+\sigma}\right)+\beta\,p_ih_i^2\left(\frac{-\sigma^3}{(\sigma+1)h_i}+\frac{\omega\sigma^2}{1+\sigma}[\sigma p_{i+1}+(2\sigma+1)p_{i-1}]-\right.$

$\left.\omega\sigma^2h_iq_{i-1}\right)+\alpha_2p_{i+1}h_i\left(\frac{\sigma^3}{(\sigma+1)}\right)+\alpha_1q_{i-1}h_i^2(\sigma^2)$

$F_i=\varepsilon(1+\sigma)+\alpha_1\,p_{i-1}\,h_i(\sigma+\sigma^2)+\beta\,p_ih_i^2\left(\frac{(\sigma^2-\sigma)}{h_i}-\omega(\sigma^2+\sigma)[p_{i+1}+p_{i-1}]\right)$

$\quad -\alpha_2p_{i+1}h_i(\sigma+\sigma^2)+\beta\,q_ih_i^2(\sigma^2)$

$G_{i+1}=-\varepsilon-\alpha_1\,p_{i-1}\,h_i\,\left(\frac{\sigma}{1+\sigma}\right)+\beta\,p_ih_i^2\left(\frac{\sigma}{(\sigma+1)h_i}+\frac{\omega\sigma}{(1+\sigma)}[(2\sigma+1)p_{i+1}+p_{i-1}]+\omega h_iq_{i+1}\sigma^2\right)$

$+\alpha_2p_{i+1}h_i\left(\frac{2\sigma^2+\sigma}{\sigma+1}\right)+\alpha_2q_{i+1}h_i^2(\sigma^2)$

$$H_i=\,h_{i+1}^2[(\alpha_1-\omega\beta\,p_ih_i)r_{i-1}+\beta\,r_i+(\alpha_2+\omega\beta\,p_ih_i)r_{i+1}] \tag{8}$$

where $h_{i+1}=\sigma\,h_i$.

For i = 1, the coefficients y_{i-1}, y_i and y_{i+1} are not defined in Eq. (7), thus we need to develop an equation for this case. By using L-Hospital rule, from Eq. (5), we get

$$y_i''=\frac{q_iy_i+r_i}{\varepsilon-k} \quad \text{gives} \quad M_i=\frac{q_iy_i+r_i}{\varepsilon-k} \tag{9}$$

Again using Eq. (4), we get the following boundary formula for i = 1

$$\left[-\sigma+\frac{\alpha_1\,h_2^2q_0}{\varepsilon-k}\right]y_0+\left[(1+\sigma)+\frac{\beta\,h_2^2q_1}{\varepsilon-k}\right]y_1+\left[-\sigma+\frac{\alpha_2\,h_2^2q_2}{\varepsilon-k}\right]y_2$$
$$=\frac{-h_2^2}{\varepsilon-k}[k_1\,r_0+k_2\,r_1+k_3\,r_2] \tag{10}$$

We solve the tri-diagonal system Eq. (7) together with the Eq. (10) for

$i = 2,3,\dots,n-1$ to get the approximations $y_{1,}y_2,,\dots,y_{n-1}$ of the solution $y(x)$ at $x_1,x_2,\dots,x_{n-1}$.

4. ERROR ANALYSIS

Using the approximations for first order derivative of y in Eq. (6), we get

$$e'_{i-1} = y'(x_{i-1}) - y'_{i-1}$$

$$= \left(\frac{\sigma^2+2\sigma+1}{6(1+\sigma)}\right) h_i^2 y^{(3)}(x_i) + \left(\frac{\sigma^3-3\sigma-2}{24(1+\sigma)}\right) h_i^3 y^{(4)}(x_i) + \left(\frac{\sigma^4+4\sigma+3}{120(1+\sigma)}\right) h_i^4 y^{(5)}\left(\xi_1^{(i)}\right) \qquad (11)$$

$$e'_{i+1} = y'(x_{i+1}) - y'_{i+1}$$

$$= \left(\frac{\sigma^3+2\sigma^2+\sigma}{6(1+\sigma)}\right) h_i^2 y^{(3)}(x_i) + \left(\frac{2\sigma^4+3\sigma^3-\sigma}{24(1+\sigma)}\right) h_i^3 y^{(4)}(x_i) + \left(\frac{3\sigma^5+4\sigma^4+\sigma}{120(1+\sigma)}\right) h_i^4 y^{(5)}\left(\xi_2^{(i)}\right) \qquad (12)$$

$$e'_i = y'(x_i) - y'_i = -h_i^2\left(\frac{\sigma}{6} + (1+\sigma)\varepsilon\omega\right) y^{(3)}(x_i) + h_i^3\left(\frac{\omega\varepsilon(1-\sigma^2)}{2} + \frac{(\sigma-\sigma^2)}{24}\right) y^{(4)}(x_i)$$

$$-h_i^4\left[\frac{\omega\varepsilon(1+\sigma^3)}{6} y^{(5)}\left(\epsilon_3^{(i)}\right) + \frac{1}{120} y^{(5)}\left(\epsilon_4^{(i)}\right) - \frac{\omega(1+\sigma^3)}{12} p_i\left(\epsilon_5^{(i)}\right) y^{(4)}\left(\epsilon_5^{(i)}\right) - \right.$$

$$\left.\frac{\omega(\sigma^3+\sigma^2+\sigma+1)}{6} p'_i\left(\epsilon_6^{(i)}\right) y_i^{(3)}\left(\xi_6^{(i)}\right)\right] \qquad (13)$$

where $x_i < \xi_1^{(i)}, \xi_2^{(i)}, \xi_3^{(i)}, \xi_4^{(i)}, \xi_5^{(i)}, \xi_6^{(i)} < x_i$

Substituting $\varepsilon M_j = p(x_j)y'_j(\mathrm{x}) + \mathrm{q}(x_j)\mathrm{y}(x_j) + \mathrm{r}(x_j),\ j = i, i \pm 1, \text{in Eq. (4)},$ we get

$$\varepsilon\sigma y_{i-1} - \varepsilon(1+\sigma)y_i + \varepsilon y_{i+1} = h_{i+1}^2[\alpha_1(p_{i-1}y'_{i-1} + q_{i-1}y_{i-1} + r_{i-1}) + \beta(p_i y'_i + q_i y_i + r_i) + \alpha(p_{i+1}y'_{i+1} + q_{i+1}y_{i+1} + r_{i+1})] \qquad (14)$$

Putting exact solution in (14), we get

$$\varepsilon\sigma y(x_{i-1}) - \varepsilon(1+\sigma)y(x_i) + \varepsilon y(x_{i+1}) = h_{i+1}^2[\alpha_1(p_{i-1}y'(x_{i-1}) + q_{i-1}y(x_{i-1}) + r_{i-1}) + \beta(p_i y'(x_i) + q_i y(x_i) + r_i) + \alpha_2(p_{i+1}y'(x_{i+1}) + q_{i+1}y(x_{i+1}) + r_{i+1})] + T_i(h_i) \qquad (15)$$

where $T_i(h_i) = \left[-\frac{\sigma}{2}(1+\sigma) + \sigma^2(\alpha_1+\beta+\alpha_2)\right] y_i^{(2)} h_i^2 + \left[-\frac{\sigma}{6}(-1+\sigma^2) - \sigma^2(\alpha_1 - \sigma\alpha_2)\right] y_i^{(3)} h_i^3 + \left[-\frac{\sigma}{24}(1+\sigma^3) + \frac{\sigma^2}{2}(\alpha_1 + \sigma^2\alpha_2)\right] y_i^{(4)} h_i^4 + \left[-\frac{\sigma}{120}(-1+\sigma^4) - \frac{\sigma^2}{6}(\alpha_1 - \sigma^3\alpha_2)\right] y_i^{(5)} h_i^5 + 0\left(h^6\right)$

Subtracting Eq. (14) from Eq. (15) and substituting $e_j = y(x_j) - y_j, j = i, i \pm 1$ with the help of Eq. (11) - Eq. (13) we get

$$(\varepsilon\sigma - \sigma^2 h_i^2 \alpha_1 q_{i-1})e_{i-1} + (-(1+\sigma)\varepsilon - \sigma^2 h_i^2 \beta q_i)e_i + (\varepsilon - \sigma^2 h_i^2 \alpha_2 q_{i+1})e_{i+1} =$$

$$h_i^4\left[\alpha_1\left(\frac{\sigma^4+2\sigma^3+\sigma^2}{6(1+\sigma)}\right)p_{i-1}-\beta\left(\frac{\sigma^3}{6}+(\sigma^2+\sigma^3)\varepsilon\omega\right)p_i+\alpha_2\left(\frac{\sigma^5+2\sigma^4+\sigma^3}{6(1+\sigma)}\right)p_{i+1}\right]y^{(3)}(x_i)$$
$$+h_i^3\left[\alpha_1\left(\frac{\sigma^5-3\sigma^3-2\sigma^2}{24(1+\sigma)}\right)p_{i-1}-\beta\left(\frac{\omega\varepsilon}{2}(\sigma^2-\sigma^4)+\frac{\sigma^3-\sigma^4}{24}\right)p_i+\alpha_2\left(\frac{2\sigma^6+3\sigma^5-\sigma^3}{24(1+\sigma)}\right)p_{i+1}\right]y^{(4)}(x_i)+$$
$$O(h_i^6)+T_i(h) \tag{16}$$

Let $p_{i+1}=p_i+h_{i+1}p_i'+\frac{h_{i+1}^2}{2}p_i^{(2)}\left(\eta_1^{(i)}\right)$, $p_{i-1}=p_i-h_ip_i'+\frac{h_i^2}{2}p_i^{(2)}\left(\eta_2^{(i)}\right)$ where $x_{i-1}<\eta_1^{(i)},\eta_2^{(i)}<x_i$.

Substituting these expressions in Eq. (16) and simplifying, we get

$$\left(\varepsilon\sigma-\sigma^2h_i^2\alpha_1q_{i-1}\right)e_{i-1}-\left((1+\sigma)\varepsilon+\sigma^2h^2\beta q_i\right)e_i+\left(\varepsilon-\sigma^2h_i^2\alpha_2q_{i+1}\right)e_{i+1}=T_i(h) \tag{17}$$

Where

$$T_i(h)=h_i^4\left[\alpha_1\left(\frac{\sigma^4+2\sigma^3+\sigma^2}{6(1+\sigma)}\right)-\beta\left(\frac{\sigma^3}{6}+(\sigma^2+\sigma^3)\omega\varepsilon\right)+\alpha_2\left(\frac{\sigma^5+2\sigma^4+\sigma^3}{6(1+\sigma)}\right)\right]p_iy^{(4)}(x_i)+$$
$$h_i^3\left\{\left[-\alpha_1\left(\frac{\sigma^4+2\sigma^3+\sigma^2}{6(1+\sigma)}\right)+\alpha_2\left(\frac{\sigma^6+2\sigma^5+\sigma^4}{6(1+\sigma)}\right)\right]p_i'y_i^{(3)}+\left[-\alpha_1\left(\frac{\sigma^5-3\sigma^3-2\sigma^2}{24(1+\sigma)}\right)+\beta\left(\frac{\omega\varepsilon}{2}(\sigma^2-\sigma^4)+\frac{\sigma^3-\sigma^4}{24}\right)+\right.\right.$$
$$\left.\left.\alpha_2\left(\frac{2\sigma^6+3\sigma^5-\sigma^3}{24(1+\sigma)}\right)\right]p_iy_i^{(4)}+\frac{\varepsilon}{360}[\sigma(\sigma^2-1)(\sigma+2)(2\sigma+1)]y_i^{(5)}\right\}+O(h_i^6)$$

It can be noticed easily, that

(i) $T_i(h_i)=O\left(h_i^4\right)$ for the choice of

$$\alpha_1=\frac{1+\sigma-\sigma^2}{12\sigma},\alpha_2=\frac{\sigma^2+\sigma-1}{12\sigma^2},\alpha_3=\frac{\sigma^3+4\sigma^2+4\sigma+1}{12\sigma^2} \text{ and any value } \omega$$

(ii) $T_i(h_i)=O\left(h_i^5\right)$ for the choice of

$$\alpha_1=\frac{1+\sigma-\sigma^2}{12\sigma},\alpha_2=\frac{\sigma^2+\sigma-1}{12\sigma^2},\alpha_3=\frac{\sigma^3+4\sigma^2+4\sigma+1}{12\sigma^2} \quad \text{and} \quad \omega=-\frac{1}{6\varepsilon}\left[\frac{(\sigma^3+\sigma^2+\sigma)}{(1+\sigma)(\sigma^2+3\sigma+1)}\right]$$

Let J = trid $[\sigma\varepsilon \;\; -(1+\sigma)\varepsilon \;\; \varepsilon]$, D = trid$[\sigma^2\alpha_1 \;\; \sigma^2\beta \;\; \sigma^2\alpha_2]$, are $(n-1)\times(n-1)$ tri-diagonal matrices and $Q=(q_1,q_2,\ldots,q_{n-1})^t$ and $E=\left(e_1,e_2,\ldots,e_{n-1}\right)^t$ are $(n-1)$ component vectors.

Hence, Eq. (17) can be written in matrix vector form as

$$\left(J-h^2DQ\right)E=T_i(h) \tag{18}$$

Following [3], it can be shown that, for sufficiently small h,

$$\left\|\left(J-h^2DQ\right)^{-1}\right\|\le\left\|J^{-1}\right\|\le\frac{1}{6}\left(\frac{1}{8h^2}+\frac{1}{2}\right)$$

Hence, $\|E\|\le\|(J-h^2DQ)^{-1}\|\|T_i(h)\|.$

Therefore,

$\|E\| = O(h_i^2)$ for the choice of $\alpha_1, \beta, \alpha_2$ (mentioned above (i)) and any value of ω, gives a second - order method and

$\|E\| = O(h_i^3)$ for the choice of $\alpha_1, \beta, \alpha_2$ (mentioned above (ii)), gives a third - order method.

5. NUMERICAL ILLUSTRATIONS

In order to demonstrate the proposed method on a computational basis, we consider three problems of type Eq. (1). The mesh ratio σ is chosen based on the location of the boundary layer. We choose the starting value of the step length given by:

$h_1 = \frac{\sigma-1}{\sigma^N-1}$ for $\sigma > 1$ gives more mesh points near the left end $x = 0$ and

$h_1 = \frac{1-\sigma}{1-\sigma^N}$ for $\sigma < 1$ gives more mesh points near right end $x = 1$.

Example 1. $-\varepsilon y'' + \left(\frac{1}{x}\right) y' + (1 + x^2) y = f(x),\ \ 0 < x < 1.$

The exact solution is $y(x) = e^{x^2}$. Maximum errors in the solution are shown in Table 1 for different values ϵ and *h*.

Example 2. $-\epsilon y'' + \left(\frac{1}{x}\right) y' = f(x),\ \ 0 < x < 1.$

The exact solution to this is $y(x) = x\ sinhx$.

Maximum errors are shown in Table 2 for different values ϵ and *h*.

Example 3. $\epsilon y'' + \left(\frac{1}{x}\right) y' + y = 0,\ \ 0 < x < 1.$

with boundary conditions $y(0) = 0, y(1) = e^{\left(\frac{-1}{2}\right)}$

whose exact solution is not known. The numerical results are shown in Table 4 for different values ϵ and *h* using double mesh principle.

6. DISCUSSIONS AND CONCLUSION

In this paper, variable mesh non - polynomial spline scheme is suggested for a class of singularly perturbed two-point singular boundary value problems. The discretization equation is developed with the continuity condition especially for the first order derivatives of the non - polynomial spline at the internal nodes. A three-term relationship is achieved by modifying the boundary value problem at the singularity zero. Using this, the problem's discretization equation is solved with discrete invariant imbedding algorithm. A parameter ω is introduced in this method to achieve the third order convergence. Maximum errors in the solution of the standard examples selected from the literature are tabulated in order to demonstrate the method. It is observed based on the numerical results and graphs, that the suggested scheme yield good results for smaller values of ε.

Table 1. Maximum absolute errors for Example 1

$\varepsilon \downarrow$ $N \rightarrow$	2^4	2^5	2^6	2^7	2^8
2^{-4}	1.96e-004	1.45e-005	1.05e-006	7.58e-008	5.35e-009
2^{-5}	2.80e-004	2.17e-005	1.64e-006	1.21e-007	8.73e-009
2^{-6}	4.16e-004	3.40e-005	2.66e-006	2.02e-007	1.49e-008
2^{-7}	6.25e-004	5.45e-005	4.44e-006	3.45e-007	2.60e-008
2^{-8}	8.89e-004	8.75e-005	7.45e-006	5.98e-007	4.60e-008

Table 2. Maximum absolute errors for Example 2

$\varepsilon \downarrow$ $N \rightarrow$	2^4	2^5	2^6	2^7	2^8
2^{-4}	7.98e-005	5.92e-006	4.27e-007	3.03e-008	2.12e-009
2^{-5}	1.20e-004	9.33e-006	6.95e-007	5.05e-008	3.59e-009
2^{-6}	1.83e-004	1.52e-005	1.17e-006	8.73e-008	6.33e-009
2^{-7}	2.65e-004	2.50e-005	2.01e-006	1.53e-007	1.13e-008
2^{-8}	3.50e-004	3.97e-005	3.45e-006	2.72e-007	2.04e-008

Table 3. Maximum absolute errors for Example 3

$\varepsilon \downarrow$ $N \rightarrow$	2^4	2^5	2^6	2^7	2^8
2^{-4}	1.06e-004	8.36e-006	5.60e-007	9.61e-008	1.08e-007
2^{-5}	2.11e-004	1.72e-005	1.25e-006	8.35e-008	1.08e-007
2^{-6}	3.82e-004	3.21e-005	2.50e-006	1.01e-007	1.05e-007
2^{-7}	6.34e-004	5.76e-005	4.67e-006	2.71e-007	9.88e-008
2^{-8}	9.38e-004	9.77e-005	8.39e-006	5.80e-007	8.88e-008

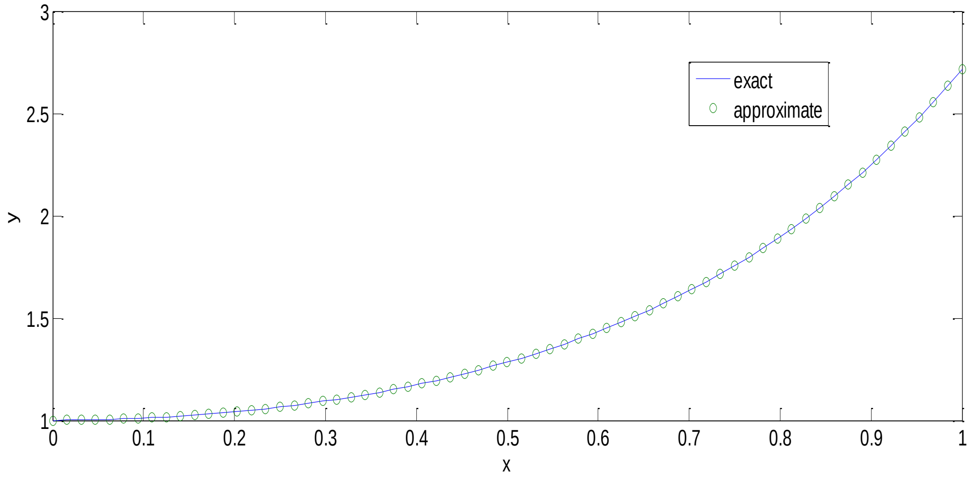

Fig1. Example 1 with $n = 64$ and $\epsilon = 2^{-10}$

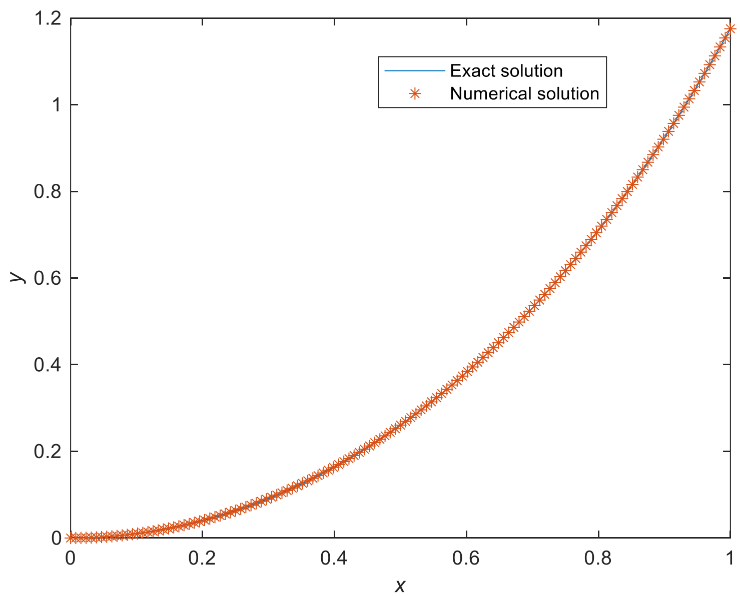

Fig 2. Example 2 with $n = 64$ and $\epsilon = 10^{-3}$

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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