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EXISTENCE AND UNIQUENESS OF SOLUTION OF FIRST ORDER NONLINEAR DIFFERENCE EQUATION

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Abstract. This paper is devoted to obtaining the existence and uniqueness result of nonhomogeneous first order nonlinear difference equation with nonlocal condition in cone metric space. Banach contraction principle is used to prove the results. Finally an application of the established result is demonstrated.

Keywords: difference equation; existence of solution; cone metric space; Banach contraction principle.

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1. INTRODUCTION

The aim of this paper is to study the existence and uniqueness of solution of nonhomogeneous first order nonlinear difference equation with nonlocal condition in cone metric space of the form:

$$\Delta x(t) = A(t)x(t) + f(t, x(t)), \qquad t \in J = [0, b] \tag{1.1}$$

$$x(0) + g(x) = x_0, \tag{1.2}$$

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where $A(t)$ is a bounded linear operator on a Banach space X with domain $D(A(t))$, the unknown $x(\cdot)$ takes values in the Banach space $X; f : J \times X \to X,\ g : C(J,X) \to X$ are appropriate continuous functions and x_0 is given element of X.

Many authors have been studied the problems of existence and uniqueness of solutions of differential equations, integral equations and Integro-differential equations by using different techniques studied in [2, 9, 12, 15, 16]. Later K.L. Bondar etal[3, 4, 5, 6, 7, 8], studied existence and uniqueness of some difference equations and summation equations .The objective of the present paper is to study the existence and uniqueness of solution of the Differnce equation $(1.1)-(1.2)$ under the conditions in respect of the cone metric space and fixed point theory.

In Section 2, we discuss the preliminaries. Section 3, deals with study of existence and uniqueness of solution of nonhomogeneous first order nonlinear difference equation with nonlocal condition in cone metric space. Finally in Section 4, we give example to illustrate the application of our result.

2. Definitions and Preliminaries

Let us recall the concepts of the cone metric space and we refer the reader to [1, 10, 11, 13, 14, 17] for the more details.

Definition 2.1. Let E be a real Banach space and P is a subset of E. Then P is called a cone if and only if,

1.P is closed, nonempty and $P \neq 0$.

2.$a,b \in \mathbb{R}, a,b \geq 0, x,y \in P \Rightarrow ax+by \in P$.

3.$x \in P$ and $-x \in P \Rightarrow x = 0$.

For a given cone $P \in E$, we define a partial ordering relation $\leq$ with respect to P by $x \leq y$ if and only if $y-x \in P$.We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y-x \in intP$. Where int P denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $\leq x \leq y$ implies $\|x\| \leq k\|y\|$, for every $x,y \in E$. The least positive number satisfying above is called the normal constant of P.

In the following way, we always suppose E is a real Banach space, P is cone in E with int $P \neq \phi$, and $\leq$ is partial ordering with respect to P.

Definition 2.2. Let X a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:
(d_1) $0 \leq d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$
(d_2) $d(x,y) = d(y,x)$, for all $x,y \in X$;
(d_3) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x,y \in X$.
Then d is called a cone metric on X and (X,d) is called a cone metric space. The concept of cone metric space is more general than that of metric space. The following example is a cone metric space.

Example 2.1. Let $E = \mathbb{R}^2, p = \{(x,y) \in E : x,y \geq 0\}, x = \mathbb{R}$, and $d : X \times X \to E$ such that $d(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constat and then (X,d) is cone metric space.

Definition 2.3. Let X be an ordered space. A function $\Phi : X \to X$ is said to a comparison function if every $x,y \in X, x \leq y$, implies that $\Phi(x) \leq \Phi(y), \Phi(x) \leq x$ and $\lim_{n\to\infty} \|\Phi^n(x)\| = 0$, for every $x \in X$.

Example 2.2. Let $E = \mathbb{R}^2, p = \{(x,y) \in E : x,y \geq 0\}$, it is easy to check that $\Phi : E \to E$ with $\Phi(x,y) = (ax, ay)$, for some $a \in (0,1)$ is a comparison function. also if Φ_1, Φ_2 are two comparison function over $\mathbb{R}$. then
$\Phi(x,y) = (\Phi_1(x), \Phi_2(y))$ is also a comparison function over E.

3. Main Result

Let X is a Banach space with norm $\|.\|$. Let $B = C(J,X)$ be the Banach space of all continuous function from J into X endowed with supremum norm

$$\|x\|_\infty = sup\{\|x(t)\| : t \in [0,b]\}.$$

Let $P = (x,y) : x,y \geq 0 \subset E = \mathbb{R}^2$, and define

$$d(f,g) = (\|f-g\|_\infty, \alpha\|f-g\|_\infty)$$

for every $f,g \in B$, then it is easily seen that (B,d) is a cone metric space.

Definition 3.1. The function $x \in B$ satisfies the summation equation

$$x(t) = x_0 - g(x) + \sum_{s=0}^{t-1} A(s) f(s, x(s)), \quad t \in J = [0, b] \tag{3.1}$$

is called the solution of the Difference equation $(1.1) - (1.2)$.

We need the following theorem for further discussion:

Lemma 3.1.[14] Let (X, d) be a complete cone metric space,where P is a normal cone with normal constant K. Let $f : X \to X$ be a function such that there exists a comparison function $\Phi : P \to P$ such that

$$d(f(x), f(y)) \leq \Phi(d(x,y)),$$

for very $x, y \in X$.Then f has unique fixed point.

We list the following hypothesis for our convenience:

(H_1) A(t) is a bounded linear operator on X for each $t \in J$, the function $t \to A(t)$ is continuous in the uniform operator topology and hence there exists a constant K such that

$$K = \sup_{t \in J} \|A(t)\|.$$

(H_2) Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a comparison function

(i) There exist continuous function $p : J \to \mathbb{R}^+$ such that

$$(\|f(t,x) - f(t,y)\|, \alpha \|f(t,x) - f(t,y)\|) \leq p(t)\Phi(d(x,y)),$$

for every $t \in J$ and $x, y \in X$

(ii) There exists a positive constant G such that

$$(\|g(x) - g(y)\|, \alpha \|g(x) - g(y)\|) \leq G\Phi(d(x,y)),$$

for every $x, y \in X$

(H_3)
$$\sup_{t \in J} \left\{ G + K \sum_{s=0}^{t-1} p(s) \right\} = 1.$$

Our main result is given in the following theorem:

Theorem 3.1 Assume that hypotheses $(H_1)-(H_3)$ hold.Then the Difference equation $(1.1)-(1.2)$ has a unique solution x on J

Proof: The operartor $F : B \to B$ is defined by

$$Fx(t) = x_0 - g(x) + \sum_{s=0}^{t-1} A(s)f(s,x(s)), \qquad t \in J = [0,b] \tag{3.2}$$

By using the hypothesis $(H_1)-(H_3)$, We have

$$\begin{aligned}
&(\|Fx(t)-Fy(t)\|, \alpha\|Fx(t)-Fy(t)\|) \\
&\qquad \le \Bigg(\|g(x)-g(y)\| + \sum_{s=0}^{t-1} \|A(s)\| \Big[\|f(s,x(s)) - f(s,y(s))\| \Big], \\
&\qquad\qquad \alpha\|g(x)-g(y)\| + \alpha \sum_{s=0}^{t-1} \|A(s)\| \Big[\|f(s,x(s)) - f(s,y(s))\| \Big] \Bigg) \\
&\qquad \le (\|g(x)-g(y)\|, \alpha\|g(x)-g(y)\|) \\
&\qquad\qquad + \sum_{s=0}^{t-1} K(\|f(s,x(s)) - f(s,y(s))\|, \alpha\|f(s,x(s)) - f(s,y(s))\|) \\
&\qquad \le G\Phi(\|x-y\|, \alpha\|x-y\|) + \sum_{s=0}^{t-1} Kp(s)\Phi(\|x(s)-y(s)\|, \alpha\|x(s)-y(s)\|) \\
&\qquad \le G\Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty) + \Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty) \sum_{s=0}^{t-1} Kp(s) \\
&\qquad \le G\Phi(d(x,y)) + \Phi(d(x,y)) \sum_{s=0}^{t-1} Kp(s) \\
&\qquad \le \Phi(d(x,y)) \left\{ G + K \sum_{s=0}^{t-1} p(s) \right\} \\
&\qquad \le \Phi(d(x,y)),
\end{aligned} \tag{3.3}$$

for every $x,y \in B$. This implies that $d(Fx,Fy) \le \Phi(d(x,y))$, for every $x,y \in B$. Now an application of Lemma 3.1, the operator has a unique point in B. This means that the equation (1.1)-(1.2) has unique solution.

4. Application

In this section, we give an example to illustrate the usefulness of our result discussed in previous section. Let us consider the following difference equation:

$$\Delta x(t) = \frac{140}{16}e^{-t}x(t) + \frac{te^{-t}x(t)}{(9+e^t)(1+x(t))}, \qquad t \in J = [0,2], \ x \in X \tag{4.1}$$

$$x(0) + \frac{x}{8+x} = x_0, \tag{4.2}$$

Therefore, we have

$$A(t) = \tfrac{140}{16}e^{-t}, \ t \in J$$

$$f(x,x(t)) = \tfrac{te^{-t}x(t)}{(9+e^t)(1+x(t))}, \ (t,x) \in J \times X$$

$$g(x) = \tfrac{x}{8+x}, \ x \in X.$$

Now for $x, y \in C(J,X)$ and $t \in J$, we have

$$\begin{aligned}
(\|f(t,x) - f(t,y)\|, \alpha\|f(t,x) - f(t,y)\|) &= \tfrac{te^{-t}}{9+e^t}(\|\tfrac{x(t)}{1+x(t)} - \tfrac{y(t)}{1+y(t)}\|, \alpha\|\tfrac{x(t)}{1+x(t)} - \tfrac{y(t)}{1+y(t)}\|) \\
&= \tfrac{te^{-t}}{9+e^t}(\|\tfrac{x(t)-y(t)}{(1+x(t))(1+y(t))}\|, \alpha\|\tfrac{x(t)-y(t)}{(1+x(t))(1+y(t))}\|) \\
&\le \tfrac{te^{-t}}{9+e^t}(\|x(t) - y(t)\|, \alpha\|x(t) - y(t)\|) \\
&\le \tfrac{te^{-t}}{9+e^t}(\|x-y\|_\infty, \alpha\|x-y\|_\infty) \\
&\le \tfrac{te^{-t}}{9+e^t}d(x,y) \\
&\le \tfrac{t}{10}\Phi(d(x,y)),
\end{aligned}$$

where $p(t) = \frac{t}{10}$, which is continuous function of J into $\mathbb{R}^+$ and a comparison function $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\Phi(d(x,y)) = d(x,y)$.

Similarly, we can have

$$
\begin{aligned}
(\|g(x)-g(y)\|, \alpha\|g(x)-g(y)\|) &= 8\left(\tfrac{\|x-y\|}{(8+\|x\|)(8+\|y\|)}, \alpha\tfrac{\|x-y\|}{(8+\|x\|)(8+\|y\|)}\right) \\
&\leq \tfrac{8}{64}(\|x-y\|, \alpha\|x-y\|) \\
&\leq \tfrac{1}{8}(\|x-y\|_\infty, \alpha\|x-y\|_\infty) \\
&\leq \tfrac{1}{8}\Phi(d(x,y)),
\end{aligned}
$$

where $G = \frac{1}{8}$, and the comparison function Φ defined as above. Hence the condition (H_1) holds with $K = \frac{140}{16}$.

Moreover,

$$
\begin{aligned}
\sup_{t\in J}\left\{G + K\sum_{s=0}^{t-1} p(s)\right\} &= \sup_{t\in J}\left\{\frac{1}{8} + \frac{140}{16}\sum_{s=0}^{t-1}\frac{s}{10}\right\} \\
&= \sup_{t\in J}\left\{\frac{1}{8} + \frac{140}{16}\left[\frac{s^{\underline{2}}}{20}\right]_0^t\right\} \\
&= \sup_{t\in J}\left\{\frac{1}{8} + \frac{140}{16}\left[\frac{t^{\underline{2}}}{20}\right]\right\} \\
&= \sup_{t\in J}\left\{\frac{1}{8} + \frac{140}{16}\left[\frac{t(t-1)}{20}\right]\right\} \\
&= \left[\frac{1}{8} + \frac{140}{16} \times \frac{1}{10}\right] \\
&= \left[\frac{1}{8} + \frac{7}{8}\right] = 1.
\end{aligned}
$$

Since all the conditions of Theorem 3.1 are satisfied, the problem (4.1)-(4.2) has a unique solution x on J.

5. Conclusion

In this paper, we studied the existence for nonhomogeneous first order nonlinear difference type equation in cone metric spaces and proved that solution of this result is unique. Moreover we also gave application of above result.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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