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A RESULT ON MULTIPLICATIVE METRIC SPACE

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Abstract: The focus of our paper is to establish the existence of common fixed point theorem in multiplicative metric space by using the conditions compatible mappings of type –E and weak reciprocally continuous mappings and further provide some examples to substantiate our result.

Keywords: fixed point; multiplicative metric space; weakly compatible mappings; weak reciprocally continuous; compatible type-E mappings.

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1. INTRODUCTION

In the recent years the concept of multiplicative metric space (MMS) was introduced by Bashirove et al.[1]. Ozavsar and cevikel [7] proved some fixed point theorems of multiplicative contraction mappings in MMS. Also M.R singh and Y.M singh [5] proposed the notion of compatible mappings of type-E in 2007. Further this mapping is divided into G-compatible of

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type –E and H-compatible of type –E. Moreover weak reciprocally continuous (WRC) mappings are weaker than reciprocally continuous (RC) mappings and hence weaker than continuity. The fixed point theorems on multiplicative contraction mappings and generalizations are seen in [2], [4] and [8]. In this paper we split WRC mapping into G-WRC and H-WRC mappings and generate a common fixed point theorem on a multiplicative metric space and also justified our result with a suitable example.

1.1 Definition:

Let X ≠ϕ, a multiplicative metric is a mapping $\delta: X \times X \rightarrow \mathbb{R}_{+}$ holding the following conditions:

(i) $\delta(\alpha,\beta) \geq 1$, ∀ α, β∈ X and $\delta(\alpha,\beta) = 1 \Leftrightarrow \alpha = \beta$;

(ii) $\delta(\alpha,\beta) = \delta(\beta,\alpha)$, ∀ α, β∈ X;

(iii) $\delta(\alpha,\beta) \leq \delta(\alpha,\gamma).\delta(\gamma,\beta)$ ∀ α, β, γ∈ X.

Mapping δ together with X, (X,δ) is called (MMS).

1.2 Definition:

Let (X,δ) be aMMS, and then a sequence $\{\alpha_k\}$ is assumed as

i. A multiplicative convergent if for any multiplicative open ball $B_{\in}$ (α) = $\{\beta / \delta(\alpha,\beta) < \in\}, \in > 1$, there exists N∈ℕ such that $\alpha_k \epsilon B_{\in}$ (x) $\forall\ k \geq \mathbb{N}\ holds$. That is $\delta(\alpha_n,\alpha) \rightarrow 1$ as $k \rightarrow \infty$.

ii. A multiplicative Cauchy sequence if $\forall \epsilon > 1$, N∈ ℕ such that $\delta(\alpha_k,\alpha_l) < \epsilon\ \forall$ k,l≥ $\mathbb{N}\ holds$. That is $\delta(\alpha_k,\alpha_l) \rightarrow 1 \quad as \quad k,l \rightarrow \infty$.

iii. In MMS**,** if every multiplicative Cauchy sequence is convergent, then it is called complete.

1.3 Definition:

Let f be a mapping of MMS and if the existence of a number $\lambda\epsilon[0, 1)$ such that $\delta(f\alpha,f\beta) \leq \delta^{\lambda}(\alpha,\beta)\ \forall$ α,β ϵ X holds, then f is known as multiplicative contraction.

1.4. Definition:

We define mappings G and I of a MMS as compatible if $\delta(GI\alpha_k, IG\alpha_k) = 1$ as $k \rightarrow \infty$, whenever $\{\alpha_k\}$ is a sequence in X such that $G\alpha_k = I\alpha_k = \mu$ as $k \rightarrow \infty$ for some $\mu \in X$.

1.5. Definition:

We define mappings G and I of a MMS in which if $G\mu = I\mu$ for some $\mu \epsilon X$ such that $GI\mu = IG\mu$ holds then G and I are known as weakly compatible mappings.

1.6. Definition:

We mean two mappings G and I of a MMS are G-WRC**,** if we can find a sequence $\{\alpha_k\}$ in X such that for some $\mu \in X$ with $GI\alpha_k = G\mu$, and I-WRC if $IG\alpha_k = I\mu$ as $k \to \infty$.

Now we give an example to discuss about G-WRC and I-WRC mappings.

Example-1: Let X=[0,1] and $\delta(\alpha,\beta) = e^{|\alpha-\beta|}$ *where* $\alpha,\beta \in X$ defined as

$$G(\alpha) = \begin{cases} 1-\alpha & \text{if } 0 \le \alpha < \frac{1}{2} \\ \frac{1}{2} & \text{if } \frac{1}{2} < \alpha \le 1 \end{cases} \quad and \quad I(\alpha) = \begin{cases} \alpha & \text{if } 0 \le \alpha < \frac{1}{2} \\ \frac{3}{4} & \text{if } \frac{1}{2} \le \alpha \le 1 \end{cases}$$

Define a sequence $\{\alpha_k\} = \frac{1}{2} - \frac{1}{n}$ *for* $k \ge 0$.

Then

$$G\alpha_k = G(\frac{1}{2} - \frac{1}{k}) = (1 - \frac{1}{2} + \frac{1}{k}) = \frac{1}{2} \text{ as } k \to \infty.$$

and $I\alpha_k = I(\frac{1}{2} - \frac{1}{k}) = (\frac{1}{2} - \frac{1}{k}) = \frac{1}{2}$ as $k \to \infty$.

This gives $G\alpha_k = I\alpha_k = \frac{1}{2}$ as $k \to \infty$

also we find

$$GI\alpha_k = GI(\frac{1}{2} - \frac{1}{k}) = G(\frac{1}{2} - \frac{1}{k}) = 1 - (\frac{1}{2} - \frac{1}{k}) = \frac{1}{2} \text{ as } k \to \infty$$

and $IG\alpha_k = IG(\frac{1}{2} - \frac{1}{k}) = I(\frac{1}{2}) = \frac{3}{4}$ as $k \to \infty$.

which gives $G(\frac{1}{2}) = \frac{1}{2}$ and $I(\frac{1}{2}) = \frac{3}{4}$.

$$GI\alpha_k = G(\alpha_k) = \frac{1}{2} \text{ as } k \to \infty$$

proving the pair (G, I) as G - WRC.

$IG\alpha_k = I(\alpha_k) = \frac{3}{4}$ as $k \to \infty$

proving the pair (G, I) as $I-W$RC.

1.7. Definition:

If $\{\alpha_k\}$ is a sequence in MMS such that $G\alpha_k = I\alpha_k = \mu$ for some $\mu \in X$ and if $GG\alpha_k = GI\alpha_k = I\mu$ holds, as $k \to \infty$, then G and I are known as G-Compatible of type **(E)** and I-Compatible of type (E) if $II\alpha_k = IG\alpha_k = G\mu$ as $k \to \infty$.

Now we give an example to discuss about G-compatible of type –E and I-compatible of type–E mappings.

Example-2: Let X= [2, 6] and $\delta(\alpha, \beta) = e^{|\alpha - \beta|}$ where $\alpha, \beta \in$ X defined as

$$G(\alpha) = \begin{cases} \alpha & \text{if } \alpha = 2 \\ \frac{2\alpha}{3} & \text{if } 3 \le \alpha \le 6 \end{cases} \quad \textit{and} \quad \mathrm{I}(\alpha) = \begin{cases} 2 & \text{if } 2 \le \alpha \le 3 \\ \frac{\alpha + 21}{12} & \text{if } 3 \le \alpha \le 6 \end{cases}$$

Define a sequence $\{\alpha_k\} = 3 + \frac{1}{k}$ *for* $k \ge 0$.

Now

$$G\alpha_k = \left(\frac{2(3+\frac{1}{k})}{3}\right) = 2 \quad \text{as} \quad k \to \infty \text{ and}$$

$$I\alpha_k = \left(\frac{3+\frac{1}{k}+21}{12}\right) = 2 \quad \text{as} \quad k \to \infty.$$

This proves $G\alpha_k = I\alpha_k = 2$ as $k \to \infty$.

Also

$$IG\alpha_k = IG(3+\frac{1}{k}) = I(\frac{2}{3}(3+\frac{1}{k})) = I(2+\frac{2}{3k}) = 2 \quad \text{as} \quad k \to \infty$$

$$II\alpha_k = II(3+\frac{1}{k}) = I\,\frac{(3+\frac{1}{k})+21)}{12} = I(2+\frac{1}{12k}) = 2 \quad \text{as} \quad k \to \infty.$$

Hence $II\alpha_k = IG\alpha_k = G\alpha_k = 2$ as $k \to \infty$

This proves the pair (I, G) as I - Compatible of type - E.

Further

$$GG\alpha_k = GG(3+\frac{1}{k}) = G(\frac{2}{3}(3+\frac{1}{k})) = G(2+\frac{1}{3k}) = (2+\frac{1}{3k}) = 2 \text{ as } k \to \infty.$$

$$\text{and } GI\alpha_k = GI(3+\frac{1}{k}) = G(\frac{(3+\frac{1}{k})+21}{4}) = G(2+\frac{1}{4k}) = 2 \text{ as } k \to \infty.$$

Hence $GG\alpha_k = GI\alpha_k = I\alpha_k = 2$ as $k \to \infty$.

This proves the pair (I,G) as G-Compatible of type-E.

Now we prove a theorem on MMS.

2. THEOREM

Suppose in a complete MMS (X,d), there are four mappings G, H, I and J holding the conditions

(C1) $G(X) \subseteq I(X)$ *and* $H(X) \subseteq J(X)$

(C2) $$\delta(G\alpha, H\beta) \le \left[\max\left\{\frac{\delta(G\alpha, I\alpha)\delta(H\beta, J\beta)}{1+\delta(I\alpha, J\beta)}, \frac{\delta(G\alpha, J\beta)\delta(I\alpha, H\beta)}{1+\delta(J\beta, I\alpha)}, \frac{\delta(G\alpha, J\beta)\delta(H\beta, J\beta)}{1+\delta(I\alpha, J\beta)}, \frac{\delta(G\alpha, I\alpha)\delta(H\beta, I\alpha)}{1+\delta(I\alpha, J\beta)}\right\}\right]^{\lambda}$$

$$\text{forall } \alpha, \beta \in X, \text{where } \lambda \in \left(0, \frac{1}{2}\right)$$

(C3) (G, I) will be G-Compatible of type-E and G-WRC

(C4) (H, J) will be H-Compatible of type-E and H-WRC.

Then the above mappings have a unique common fixed point.

Proof:

Begin with using the condition (C1), there is a point $\alpha_0 \in X$ such that $G\alpha_0 = I\alpha_1 = \beta_0$.

For this point α_1 then there exists $\alpha_2 \in X$ such that $H\alpha_1 = J\alpha_2 = \beta_1$.

Continuing this process it is possible to construct a Sequence $\{\beta_k\}$ in X

Such that $\beta_{2k} = G\alpha_{2k} = I\alpha_{2k+1}, \beta_{2k+1} = H\alpha_{2k+1} = J\alpha_{2k+2}$ for $k \ge 0$.

We now prove $\{\beta_k\}$ is a Cauchy sequence.

Consider

$\delta(\beta_{2k},\beta_{2k+1.}) =$

$$\delta(G\alpha_{2k},H\alpha_{2k+1}) \leq \left[\max,\left\{\begin{array}{l}\dfrac{\delta(G\alpha_{2k},I\alpha_{2k+1})\delta(H\alpha_{2k+1},J\alpha_{2k+1})}{1+\delta(I\alpha_{2k},J\alpha_{2k+1})},\dfrac{\delta(G\alpha_{2k},J\alpha_{2k+1})\delta(I\alpha_{2k},H\alpha_{2k+1})}{1+\delta(J\alpha_{2k+1},I\alpha_{2k})},\\ \dfrac{\delta(G\alpha_{2k},J\alpha_{2k+1})\delta(H\alpha_{2k+1},J\alpha_{2k+1})}{1+\delta(I\alpha_{2k},J\alpha_{2k+1})},\dfrac{\delta(G\alpha_{2k},I\alpha_{2k})\delta(H\alpha_{2k+1},I\alpha_{2k})}{1+\delta(I\alpha_{2k},J\alpha_{2k+1})}\end{array}\right\}\right]^{\lambda}$$

$$\delta(\beta_{2k},\beta_{2k+1}) \leq \left[\max,\left\{\begin{array}{l}\dfrac{\delta(\beta_{2k},\beta_{2k})\delta(\beta_{2k+1},\beta_{2k-1})}{1+\delta(\beta_{2k},\beta_{2k-1})},\dfrac{\delta(\beta_{2k},\beta_{2k-1})\delta(\beta_{2k},\beta_{2k+1})}{1+\delta(\beta_{2k-1},\beta_{2k})},\\ \dfrac{\delta(\beta_{2k},\beta_{2k-1})\delta(\beta_{2k+1},\beta_{2k-1})}{1+\delta(\beta_{2k},\beta_{2k-1})},\dfrac{\delta(\beta_{2k},\beta_{2k})\delta(\beta_{2k+1},\beta_{2k})}{1+\delta(\beta_{2k},\beta_{2k-1})}\end{array}\right\}\right]^{\lambda}$$

$$\delta(\beta_{2k},\beta_{2k+1}) \leq \left[\max,\left\{\begin{array}{l}\dfrac{\delta(\beta_{2k+1},\beta_{2k})\delta(\beta_{2k},\beta_{2k-1})}{1+\delta(\beta_{2k},\beta_{2k-1})},\dfrac{\delta(\beta_{2k},\beta_{2k-1})\delta(\beta_{2k},\beta_{2k+1})}{1+\delta(\beta_{2k-1},\beta_{2k})},\\ \dfrac{\delta(\beta_{2k},\beta_{2k-1})\delta(\beta_{2k+1},\beta_{2k-1})}{1+\delta(\beta_{2k},\beta_{2k-1})},\dfrac{\delta(\beta_{2k+1},\beta_{2k-1})\delta(\beta_{2k-1},\beta_{2k})}{1+\delta(\beta_{2k},\beta_{2k-1})}\end{array}\right\}\right]^{\lambda}$$

$$\delta(\beta_{2k},\beta_{2k+1}) \leq \left[\max,\left\{\delta(\beta_{2k},\beta_{2k+1}),\delta(\beta_{2k},\beta_{2k+1}),\delta(\beta_{2k+1},\beta_{2k-1}),\delta(\beta_{2k-1},\beta_{2k+1})\right\}\right]^{\lambda}$$

On simplification

$$\delta(\beta_{2k},\beta_{2k+1}) \leq \left[\delta(\beta_{2k-1},\beta_{2k+1})\right]^{\lambda}$$

$$\delta(\beta_{2k},\beta_{2k+1}) \leq \left[\delta(\beta_{2k-1},\beta_{2k}),\delta(\beta_{2k},\beta_{2k+1})\right]^{\lambda}$$

$$\delta^{1-\lambda}(\beta_{2k},\beta_{2k+1.}) \leq \delta^{\lambda}\left(\beta_{2k-1},\beta_{2k}\right)$$

$$\delta(\beta_{2k},\beta_{2k+1.}) \leq \delta^{\frac{\lambda}{1-\lambda}}\left(\beta_{2k-1},\beta_{2k}\right)$$

$$\delta(\beta_{2k},\beta_{2k+1.}) \leq \delta^{h}\left(\beta_{2k-1},\beta_{2n}\right) \text{ where } \mathrm{h} = \frac{\lambda}{1-\lambda} \in (0,1) -------(1)$$

Now it gives

$$\left[\delta(\beta_k,\beta_{k+1})\right] \le \delta^{h}(\beta_{k-1},\beta_k) \le \delta^{h^2}(\beta_{k-2},\beta_{k-1}) \le --- \le \delta^{h^k}(\beta_0,\beta_1).$$

Hence for k<l, on using the multiplicative triangle inequality we get

$$\left[\delta(\beta_k,\beta_l)\right] \le \left[\delta^{h^k}(\beta_0,\beta_1)\right]\left[\delta^{h^{k+1}}(\beta_0,\beta_1)\right] ----- \left[\delta^{h^{l-1}}(\beta_0,\beta_1)\right]$$

$$\left[\delta(\beta_k,\beta_l)\right] \le \left[\delta^{\frac{1}{1-h}(h^k)}(\beta_0,\beta_1)\right].$$

Hence $\{\beta_k\}$ is obtained as a Cauchy sequence in MMS.

Now, X being complete, there exists a point μ ϵ X such that $\beta_k \to \mu$ as $k \to \infty$. Consequently, the sub sequences $\{G\alpha_{2k}\}, \{I\alpha_{2k}\}, \{J\alpha_{2k+1}\}$ and $\{H\alpha_{2k+1}\}$ of $\{\beta_k\}$ also converges to the same point μϵ X.

Since, on using the condition G-WRC, we have $GI\alpha_k = G\mu$ as $k \to \infty$.

Also using the fact that (G,I) is G-Compatible mapping of type –E, we have $GG\alpha_k = GI\alpha_k = I\mu$ as $k \to \infty$ for $\mu \in X$. This gives $I\mu = G\mu$.

Now in the inequality (C2), by putting $\alpha = \mu$ and $\beta = \alpha_{2k+1}$

$$\delta(G\mu, H\beta_{2k+1}) \le \left[\max,\left\{\begin{array}{l}\frac{\delta(G\mu,I\mu)\delta(H\alpha_{2k+1},J\alpha_{2k+1})}{1+\delta(I\mu,J\beta_{2k+1})}, \frac{\delta(G\mu,J\alpha_{2k+1})\delta(I\mu,H\alpha_{2k+1})}{1+\delta(J\beta_{2k+1},I\mu)}, \\ \frac{\delta(G\mu,J\alpha_{2k+1})\delta(H\alpha_{2k+1},J\alpha_{2k+1})}{1+\delta(I\mu,J\alpha_{2k+1})}, \frac{\delta(G\mu,I\mu)\delta(H\alpha_{2k+1},I\mu)}{1+\delta(I\mu,J\alpha_{2k+1})}\end{array}\right\}\right]^{\lambda}$$

$$\delta(G\mu,\mu) \le \left[\max,\left\{\begin{array}{l}\frac{\delta(G\mu,I\mu)\delta(\mu,\mu)}{1+\delta(I\mu,\mu)}, \frac{\delta(G\mu,\mu)\delta(I\mu,\mu)}{1+\delta(\mu,I\mu)}, \\ \frac{\delta(G\mu,\mu)\delta(\mu,\mu)}{1+\delta(I\mu,\mu)}, \frac{\delta(G\mu,I\mu)\delta(\mu,I\mu)}{1+\delta(I\mu,\mu)}\end{array}\right\}\right]^{\lambda}$$

$$\delta(G\mu,\mu)\le\left[\max,\left\{\begin{array}{l}\dfrac{\delta(G\mu,G\mu)\delta(\mu,\mu)}{1+\delta(G\mu,\mu)},\dfrac{\delta(G\mu,\mu)\delta(G\mu,\mu)}{1+\delta(\mu,G\mu)},\\ \dfrac{\delta(G\mu,\mu)\delta(\mu,\mu)}{1+\delta(G\mu,\mu)},\dfrac{\delta(G\mu,G\mu)\delta(\mu,G\mu)}{1+\delta(G\mu,\mu)}\end{array}\right\}\right]^{\lambda}\quad[\because G\mu=I\mu]$$

$$\delta(G\mu,\mu)\le\left[\max,\left\{\frac{1}{\delta(G\mu,\mu)},\delta(G\mu,\mu),1,1\right\}\right]^{\lambda}$$

$$\delta(G\mu,\mu)\le\left[\delta^{\lambda}(G\mu,\mu)\right]$$

this implies that $G\mu=\mu$.

Therefore $G\mu=I\mu=\mu$.

Now using the condition $H(X)\subseteq J(X)$ there is $\omega\in X$ such that $H\alpha_{2k}=J\omega$, letting $k\to\infty$ we get

$J\omega=\mu$.

We claim that $H\omega=\mu$ for this

Substitute $\alpha=\alpha_{2k}$ and $\beta=\omega$ in the inequality (C2)

$$\delta(G\alpha_{2k},H\omega)\le\left[\max,\left\{\begin{array}{l}\dfrac{\delta(G\alpha_{2k},I\alpha_{2k})\delta(H\omega,J\omega)}{1+\delta(I\alpha_{2k},J\omega)},\dfrac{\delta(G\alpha_{2k},J\omega)\delta(I\alpha_{2k},H\omega)}{1+\delta(J\omega,I\alpha_{2k})},\\ \dfrac{\delta(G\alpha_{2k},J\omega)\delta(H\omega,J\omega)}{1+\delta(I\alpha_{2k},J\omega)},\dfrac{\delta(G\alpha_{2k},I\alpha_{2k})\delta(H\omega,I\alpha_{2k})}{1+\delta(I\alpha_{2k},J\omega)}\end{array}\right\}\right]^{\lambda}$$

$$\delta(\mu,H\omega)\le\left[\max,\left\{\begin{array}{l}\dfrac{\delta(\mu,\mu)\delta(H\omega,\mu)}{1+\delta(\mu,\mu)},\dfrac{\delta(\mu,\mu)\delta(\mu,H\omega)}{1+\delta(\mu,\mu)},\\ \dfrac{\delta(\mu,\mu)\delta(H\omega,\mu)}{1+\delta(\mu,\mu)},\dfrac{\delta(\mu,\mu)\delta(H\omega,\mu)}{1+\delta(\mu,\mu)}\end{array}\right\}\right]^{\lambda}\quad[\because J\omega=\mu]$$

$$\delta(\mu,H\omega)\le\left[\max,\left\{\delta(H\omega,\mu),\delta(H\omega,\mu),\delta(H\omega,\mu),\delta(H\omega,\mu)\right\}\right]^{\lambda}$$

$$\delta(\mu,H\omega)\le\left[\delta(\mu,H\omega)\right]^{\lambda}$$

this implies that $H\omega = \mu$.

Hence $H\omega = J\omega = \mu$.

Now by the H-WRC of the pair (H, J) we get $HJ\alpha_{2k} = H\mu$ as $k \to \infty$.

Again by the H-Compatible mapping of Type- E, we have $HH\alpha_{2k}=HJ\alpha_{2k}=J\mu$ as $k\to\infty$.

This implies that $H\mu=J\mu$.

To claim $H\mu=\mu$, substitute $\alpha=\alpha_{2k}$ and $\beta=\mu$ in the inequality (C2)

$$\delta(G\alpha_{2k}, H\mu) \le \left[\max\left\{ \begin{array}{l} \dfrac{\delta(G\alpha_{2k}, I\alpha_{2k})\delta(H\mu, J\mu)}{1+\delta(I\alpha_{2k}, J\mu)}, \dfrac{\delta(G\alpha_{2k}, J\mu)\delta(I\alpha_{2k}, H\mu)}{1+\delta(J\mu, I\alpha_{2k})}, \\ \dfrac{\delta(G\alpha_{2k}, J\mu)\delta(H\mu, J\mu)}{1+\delta(I\alpha_{2k}, J\mu)}, \dfrac{\delta(G\alpha_{2k}, I\alpha_{2k})\delta(H\mu, I\alpha_{2k})}{1+\delta(I\alpha_{2k}, J\mu)} \end{array} \right\} \right]^{\lambda}$$

$$\delta(\mu, H\mu) \le \left[\max\left\{ \begin{array}{l} \dfrac{\delta(\mu,\mu)\delta(H\mu, H\mu)}{1+\delta(\mu, H\mu)}, \dfrac{\delta(\mu, H\mu)\delta(\mu, H\mu)}{1+\delta(H\mu, \mu)}, \\ \dfrac{\delta(\mu, H\mu)\delta(H\mu, H\mu)}{1+\delta(\mu, H\mu)}, \dfrac{\delta(\mu,\mu)\delta(H\mu, \mu)}{1+\delta(\mu, H\mu)} \end{array} \right\} \right]^{\lambda} \quad [\because J\mu = H\mu]$$

$$\delta(\mu, H\mu) \le \left[\max\left\{ \frac{1}{\delta(\mu, H\mu)}, \delta(\mu, H\mu), 1, 1 \right\} \right]^{\lambda}$$

$$\delta(\mu, H\mu) \le \left[\delta^{\lambda}(\mu, H\mu)\right]$$

this implies that $H\mu = \mu$.

Therefore $J\mu = H\mu = G\mu = I\mu = \mu$.

Proving the existence of common fixed point for the four mappings G,H,I and J.

For Uniqueness

Suppose μ and ϕ ($\mu \ne \phi$) are common fixed points of G,H,I and J.

Substitute $\alpha=\mu$ and $\beta=\phi$ in the inequality (C2)

$$\delta(G\mu, H\phi) \le \left[\max, \left\{ \begin{array}{l} \dfrac{\delta(G\mu, I\mu)\delta(H\phi, J\phi)}{1+\delta(I\mu, J\phi)}, \dfrac{\delta(G\mu, J\phi)\delta(I\mu, H\phi)}{1+\delta(I\mu, J\phi)}, \\ \dfrac{\delta(G\mu, J\phi)\delta(H\phi, J\phi)}{1+\delta(I\mu, J\phi)}, \dfrac{\delta(G\mu, Iv)\delta(H\phi, I\mu)}{1+\delta(I\mu, J\phi)} \end{array} \right\} \right]^{\lambda}$$

$$\delta(\mu, \phi) \le \left[\max, \left\{ \begin{array}{l} \dfrac{\delta(\mu, \mu)\delta(\phi, \phi)}{1+\delta(\mu, \phi)}, \dfrac{\delta(\mu, \phi)\delta(\mu, \phi)}{1+\delta(\mu, \phi)}, \\ \dfrac{\delta(\mu, \phi)\delta(\phi, \phi)}{1+\delta(\mu, \phi)}, \dfrac{\delta(\mu, \phi)\delta(\phi, \mu)}{1+\delta(\mu, \phi)} \end{array} \right\} \right]^{\lambda}$$

$$\delta(\mu, \phi) \le \left[\max, \left\{ \frac{1}{\delta(\mu, \phi)}, \delta(\mu, \phi), 1, \delta(\mu, \phi) \right\} \right]^{\lambda}$$

$\delta(\mu, \phi) \le [\{\delta(\mu, \phi)\}]^{\lambda}$ which implies $\mu = \phi$

where $\lambda \in \left[0, \frac{1}{2}\right]$

This assures the uniqueness of the fixed point.

Now we substantiate our result with an example.

3. EXAMPLE

Let $X = (0,1)$ with $\delta(\alpha, \beta) = e^{|\alpha-\beta|}$ $\forall\ \alpha, \beta \in X$.

Define $G(\alpha)=H(\alpha)= \begin{cases} \dfrac{2\alpha+1}{4} & if\ \alpha \in (0, \frac{1}{2}) \\ \alpha & if\ \alpha \ge \frac{1}{2} \end{cases}$

and $I(\alpha) = J(\alpha) = \begin{cases} 1-\alpha & if\ \alpha \in (0, \frac{1}{2}] \\ 3\alpha - 1 & \text{If } \alpha > \frac{1}{2} \end{cases}$

$G(X) = H(X) = (\frac{1}{4}, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ while $I(X) = J(X) = (\frac{1}{2}, 1) \cup (\frac{1}{4})$.

$G(X) \subseteq I(X)$ & $H(X) \subseteq J(X)$, so that the condition (C1) is satisfied.

Take a sequence $\{\alpha_k\}$ *as* $\alpha_k = \frac{1}{2} - \frac{1}{k}$ *for* $k \ge 0$.

Now

$$G\alpha_k = G\left(\frac{1}{2}-\frac{1}{k}\right) = \frac{2\left(\frac{1}{2}-\frac{1}{k}\right)+1}{4} = \frac{\left(2-\frac{2}{k}\right)}{4} = \left(\frac{2}{4}-\frac{1}{2k}\right) = \frac{1}{2} \text{ as k} \to \infty \textit{ and}$$

$$I\alpha_k = I\left(\frac{1}{2}-\frac{1}{k}\right) = (1-(\frac{1}{2}-\frac{1}{k}) = (\frac{1}{2}+\frac{1}{k}) = \frac{1}{2} \quad as \;\; k \to \infty.$$

This gives $G\alpha_k = I\alpha_k = \frac{1}{2}$ as $k \to \infty$.

Also $\quad G\left(\frac{1}{2}\right) = \frac{1}{2}$ and $I\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right) - 1 = \frac{1}{2}$

similarly $H\left(\frac{1}{2}\right) = \frac{1}{2}$ and $J\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right) - 1 = \frac{1}{2}$

Now $GI\alpha_k = GI\left(\frac{1}{2}-\frac{1}{k}\right) = G(1-(\frac{1}{2}-\frac{1}{k}) = G(\frac{1}{2}+\frac{1}{k}) = (\frac{1}{2}+\frac{1}{k}) = \frac{1}{2}$ as $k \to \infty$.

$$Also \;\; IG\alpha_k = IG\left(\frac{1}{2}-\frac{1}{k}\right) = \mathrm{I}\left[\frac{2\left(\frac{1}{2}-\frac{1}{k}\right)+1}{4}\right] = \mathrm{I}\left(\frac{1}{2}-\frac{1}{2k}\right) = 1-\left(\frac{1}{2}-\frac{1}{2k}\right) = \frac{1}{2} \text{ as k} \to \infty$$

$$GG\alpha_k = G\left[G\left(\frac{1}{2}-\frac{1}{k}\right)\right] = G\left(\frac{2\left(\frac{1}{2}-\frac{1}{k}\right)+1}{4}\right) = G\left(\frac{1}{2}-\frac{1}{2k}\right) = \left[\frac{2\left(\frac{1}{2}-\frac{1}{2k}\right)+1}{4}\right] = \left(\frac{1}{2}-\frac{1}{2k}\right) = \frac{1}{2} \quad as \;\; k \to \infty.$$

Therefore $GI\alpha_k = GG\alpha_k = I\mu$ as $k \to \infty$. This establishes G-Compatible mapping of type –E.

Similarly $HJ\alpha_k = HH\alpha_k = J\mu$ as $k \to \infty$. This establishes H-Compatible mapping of type –E.

Now $II\alpha_k = I\left[I\left(\frac{1}{2}-\frac{1}{k}\right)\right] = I[1-\left(\frac{1}{2}-\frac{1}{k}\right)] = I\left[\frac{1}{2}+\frac{1}{k}\right] = 3\left[\frac{1}{2}+\frac{1}{k}\right] - 1 = \frac{1}{2}$ as $k \to \infty$.

Also since $IG\alpha_k = II\alpha_k = G\mu$ as $k \to \infty$ this proves that the pair (G,I) is G-WRC.

Similarly $JH\alpha_k = JJ\alpha_k = H\mu$ as $k \to \infty$ this proves that the pair (H,J) is H-WRC.

Also since $G\alpha_k = I\alpha_k = \frac{1}{2}$ as $k \to \infty$.

Similarly $H\alpha_k = J\alpha_k = \frac{1}{2}$ as $k \to \infty$

this gives $\lim_{k\to\infty}\delta\left(GI\alpha_k, IG\alpha_k\right)=\delta(\frac{3}{8},\frac{1}{2})\neq 1$ similarly $\lim_{k\to\infty}\delta\left(HJ\alpha_k, JH\alpha_k\right)=\delta(\frac{3}{8},\frac{1}{2})\neq 1$.

Showing that the compatibility condition is not fulfilled.

We now establish that the mappings G,H,I and J satisfy the Condition(C2) .

Case 1:

If $\alpha,\beta\in(0,\frac{1}{2})$, then we have $\delta(G\alpha,H\beta)=e^{|G\alpha-H\beta|}$

Putting $\alpha=\frac{1}{3},\beta=\frac{1}{5}$ in the equality (C2) gives

$$\delta(G\alpha,H\beta)\leq\left[\max,\left\{\begin{array}{l}\frac{\delta(G\alpha,I\alpha)\delta(H\beta,J\beta)}{1+\delta(I\alpha,J\beta)},\frac{\delta(G\alpha,J\beta)\delta(I\alpha,H\beta)}{1+\delta(J\beta,I\alpha)},\\ \frac{\delta(G\alpha,J\beta)\delta(H\beta,J\beta)}{1+\delta(I\alpha,J\beta)},\frac{\delta(G\alpha,I\alpha)\delta(H\beta,I\alpha)}{1+\delta(I\alpha,J\beta)}\end{array}\right\}\right]^{\lambda}$$

$$d(\frac{5}{12},\frac{7}{20})\leq\left[\max\left\{\begin{array}{l}\frac{d(\frac{5}{12},\frac{2}{3})d(\frac{7}{20},\frac{4}{5})}{1+d(\frac{2}{3},\frac{4}{5})},\frac{d(\frac{5}{12},\frac{4}{5})d(\frac{2}{3},\frac{7}{20})}{1+d(\frac{2}{3},\frac{4}{5})},\\ \frac{d(\frac{5}{12},\frac{4}{5})d(\frac{7}{20},\frac{4}{5})}{1+d(\frac{2}{3},\frac{4}{5})},\frac{d(\frac{5}{12},\frac{2}{3})d(\frac{7}{20},\frac{2}{3})}{1+d(\frac{2}{3},\frac{4}{5})}\end{array}\right\}\right]^{\lambda}$$

$$e^{0.066}\leq\left[\max\left\{\frac{e^{0.25}e^{0.45}}{1+e^{0.133}},\frac{e^{0.38}e^{0.31}}{1+e^{0.133}},\frac{e^{0.38}e^{0.45}}{1+e^{0.133}},\frac{e^{0.25}e^{0.31}}{1+e^{0.133}}\right\}\right]^{\lambda}$$

$$e^{0.066}\leq\left[\max\left\{\frac{e^{0.7}}{1+e^{0.133}},\frac{e^{0.69}}{1+e^{0.133}},\frac{e^{0.83}}{1+e^{0.133}},\frac{e^{0.56}}{1+e^{0.133}}\right\}\right]^{\lambda}$$

$$e^{0.066}\leq\left[\max\left\{e^{0.567},e^{0.557},e^{0.697},e^{0.427}\right\}\right]^{\lambda}$$

$$e^{0.066}\leq e^{0.697\lambda}$$

Thus we have

$e^{0.066} \leq e^{0.69\lambda} \Rightarrow \lambda = 0.009 \ where \lambda \in \left[0, \frac{1}{2}\right]$.

Hence the condition (C2) is satisfied.

Case 2:

If $\alpha, \beta \in \left[\frac{1}{2}, 1\right)$, then we have $\delta(G\alpha, H\beta) = e^{|G\alpha - H\beta|}$

Putting $\alpha = \frac{2}{3}, \beta = 1$ in the equality (C2) gives

$$\delta(G\alpha, H\beta) \leq \left[\max, \left\{ \begin{array}{l} \frac{\delta(G\alpha, I\alpha)\delta(H\beta, J\beta)}{1+\delta(I\alpha, J\beta)}, \frac{\delta(G\alpha, J\beta)\delta(I\alpha, H\beta)}{1+\delta(J\beta, I\alpha)}, \\ \frac{\delta(G\alpha, J\beta)\delta(H\beta, J\beta)}{1+\delta(I\alpha, J\beta)}, \frac{\delta(G\alpha, I\alpha)\delta(H\beta, I\alpha)}{1+\delta(I\alpha, J\beta)} \end{array} \right\} \right]^{\lambda}$$

$$\delta(\frac{2}{3},1) \leq \left[\max\left\{ \frac{\delta(\frac{2}{3},1)\delta(1,2)}{1+\delta(1,2)}, \frac{\delta(\frac{2}{3},2)\delta(1,1)}{1+\delta(1,2)}, \frac{\delta(\frac{2}{3},2)\delta(1,2)}{1+\delta(2,1)}, \frac{\delta(\frac{2}{3},1)\delta(1,1)}{1+\delta(2,1)} \right\} \right]^{\lambda}$$

$$e^{0.33} \leq \left[\max\left\{ \frac{e^{0.33}e^{1}}{1+e}, \frac{e^{1.33}e^{0}}{1+e}, \frac{e^{1.33}e^{1}}{1+e}, \frac{e^{0.33}e^{0}}{1+e} \right\} \right]^{\lambda}$$

$$e^{0.33} \leq \left[\max\left\{ \frac{e^{1.33}}{1+e}, \frac{e^{1.33}}{1+e}, \frac{e^{2.33}}{1+e}, \frac{e^{0.33}}{1+e} \right\} \right]^{\lambda}$$

$$e^{0.33} \leq \left[\max\left\{ e^{0.33}, e^{0.33}, e^{1.33}, e^{-0.7} \right\} \right]^{\lambda}$$

$$e^{0.33} \leq e^{1.33\lambda}.$$

Therefore $e^{0.033} \leq e^{1.3\lambda} \Rightarrow \lambda = 0.24 \ where \ \lambda \in \left[0, \frac{1}{2}\right]$.

Hence the condition (C2) is satisfied.

Case 3:

If $\alpha \in \left(0, \frac{1}{2}\right]$ and $\beta \in \left[\frac{1}{2}, 1\right), \delta(G\alpha, H\beta) = e^{|G\alpha - H\beta|}$

Putting $\alpha = \frac{1}{3}, \beta = \frac{2}{3}$ in the equality (C2) gives

$$\delta(G\alpha, H\beta) \leq \left[\max, \left\{ \begin{array}{l} \dfrac{\delta(G\alpha, I\alpha)\delta(H\beta, J\beta)}{1+\delta(I\alpha, J\beta)}, \dfrac{\delta(G\alpha, J\beta)\delta(I\alpha, H\beta)}{1+\delta(J\beta, I\alpha)}, \\ \dfrac{\delta(G\alpha, J\beta)\delta(H\beta, J\beta)}{1+\delta(I\alpha, J\beta)}, \dfrac{\delta(G\alpha, I\alpha)\delta(H\beta, I\alpha)}{1+\delta(I\alpha, J\beta)} \end{array} \right\} \right]^{\lambda}$$

$$\delta(\tfrac{5}{12}, \tfrac{4}{5}) \leq \left[\max\left\{ \frac{\delta(\frac{5}{12}, \frac{2}{3})\delta(\frac{4}{5}, \frac{7}{5})}{1+\delta(\frac{2}{3}, \frac{7}{5})}, \frac{\delta(\frac{5}{12}, \frac{7}{5})\delta(\frac{2}{3}, \frac{4}{5})}{1+\delta(\frac{2}{3}, \frac{7}{5})}, \frac{\delta(\frac{5}{12}, \frac{7}{5})\delta(\frac{4}{5}, \frac{7}{5})}{1+\delta(\frac{2}{3}, \frac{7}{5})}, \frac{\delta(\frac{5}{12}, \frac{2}{3})\delta(\frac{4}{5}, \frac{2}{3})}{1+\delta(\frac{2}{3}, \frac{7}{5})} \right\} \right]^{\lambda}$$

$$e^{0.38} \leq \left[\max\left\{ \frac{e^{0.25}e^{0.6}}{1+e^{0.73}}, \frac{e^{0.98}e^{0.13}}{1+e^{0.73}}, \frac{e^{0.98}e^{0.6}}{1+e^{0.73}}, \frac{e^{0.25}e^{0.13}}{1+e^{0.73}} \right\} \right]^{\lambda}$$

$$e^{0.38} \leq \left[\max\left\{ \frac{e^{0.85}}{1+e^{0.73}}, \frac{e^{1.11}}{1+e^{0.73}}, \frac{e^{1.58}}{1+e^{0.73}}, \frac{e^{0.38}}{1+e^{0.73}} \right\} \right]^{\lambda}$$

$$e^{0.38} \leq \left[\max\left\{ e^{0.12}, e^{0.38}, e^{0.85}, e^{-0.35} \right\} \right]^{\lambda}$$

where $\lambda \in \left[0, \frac{1}{2}\right]$. $\quad e^{0.38} \leq e^{0.85\lambda} \Rightarrow \lambda = 0.44$.

Hence the condition (C2) is satisfied. It can be observed that $\frac{1}{2}$ is the unique common fixed point for the four mappings.

4. CONCLUSION

In this paper we established a common fixed point theorem in multiplicative metric space using the conditions compatible mapping of type –E and weak reciprocally continuous mappings and an example is given to justify our theorem.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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